

## CHARACTERIZATION OF ORTHONORMAL HIGH-ORDER BALANCED MULTIWAVELETS IN TERMS OF MOMENTS

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ABSTRACT. In this paper, we derive a characterization of orthonormal balanced multiwavelets of order  $p$  in terms of the continuous moments of the multiscaling function  $\phi$ . As a result, the continuous moments satisfy the discrete polynomial preserving properties of order  $p$  (or degree  $p - 1$ ) for orthonormal balanced multiwavelets. We derive polynomial reproduction formula of degree  $p - 1$  in terms of continuous moments for orthonormal balanced multiwavelets of order  $p$ . Balancing of order  $p$  implies that the series of scaling functions with the discrete-time monomials as expansion coefficients is a polynomial of degree  $p - 1$ . We derive an algorithm for computing the polynomial of degree  $p - 1$ .

### 1. Introduction and motivation

One of the difficulties in multiwavelets was that the prefiltering step was necessary when implementing a multiwavelet transform. To avoid the prefiltering step, balanced was considered. For balanced multiwavelets, the prefiltering step is not necessary when implementing a multiwavelet transform. This is a great advantage both in terms of computational cost and quality of results in many applications.

Orthonormal balanced multiwavelets were introduced by Selesnick in [9] and by Lebrun and Vetterli in [5] with a stronger condition. High-order balanced multiwavelets were also introduced by Lebrun and Vetterli in [6, 7] and by Selesnick in [10, 11]. Biorthogonal balanced multiwavelets via lifting were constructed by Bacchelli *et al.* in [1]. Characterizations of biorthogonal balanced multiwavelets on  $\mathbb{R}^s$ ,  $s \geq 1$ , were introduced by Chui and Jiang in [2].

Balanced is a condition for the construction of orthonormal multiscaling function and multiwavelets to ensure the property of preservation/annihilation of scalar-valued discrete polynomial data of some degree, when implementing a multiwavelet transform. The definition of balanced or high-order balanced multiwavelets in the form of integrals is related to the continuous moments. It is well-known that the continuous moments of the multiscaling function  $\phi$  and

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the multiwavelet function  $\psi$  can be computed if the recurrence coefficients of  $\phi$  and  $\psi$  are given, for example, see [3]. This is a motivation for us to investigate a characterization of the orthonormal balanced multiwavelets of order  $p$  in terms of the continuous moments in this paper.

The main objective of this paper is to derive a characterization of the orthonormal balanced multiwavelets of order  $p$  in terms of the continuous moments of the multiscaling function  $\phi$ . As a result, the continuous moments satisfy the discrete polynomial preserving properties of degree  $p - 1$  for orthonormal balanced multiwavelets of order  $p$ . We derive polynomial reproduction formula of degree  $p - 1$  in terms of the continuous moments for orthonormal balanced multiwavelets of order  $p$ . Balancing of order  $p$  implies that the series of scaling functions with the discrete-time monomials as expansion coefficients is a polynomial of degree  $p - 1$ . We derive an algorithm for computing the above polynomial of degree  $p - 1$ .

This paper is organized as follows. Preliminaries and computation of the continuous moments of the multiscaling function  $\phi$  are introduced in Section 2. The main result is stated in Section 3. Polynomial reproducing formula is stated in Section 4. Preservation of discrete polynomials is stated in Section 5. Finally, examples are given in Section 6.

## 2. Preliminaries

A *multiscaling function* of multiplicity  $r$  and dilation factor  $d$  is a vector of  $r$  real or complex-valued functions

$$\phi(x) = [\phi_1(x), \phi_2(x), \dots, \phi_r(x)]^T, \quad x \in \mathbb{R},$$

which satisfies a recursion relation

$$(2.1) \quad \phi(x) = \sqrt{d} \sum_{k \in \mathbb{Z}} h_k \phi(dx - k)$$

and generates a multiresolution approximation of  $L^2(\mathbb{R})$ . The corresponding *multiwavelet function*  $\psi$  satisfies

$$\psi(x) = \sqrt{d} \sum_{k \in \mathbb{Z}} g_k \phi(dx - k).$$

The recursion coefficients  $h_k$  and  $g_k$  are  $r \times r$  matrices.

Multiwavelets  $\phi$  and  $\psi$  are *orthonormal* if

$$\begin{aligned} \langle \phi(x - j), \phi(x - k) \rangle &= \delta_{jk} I, \\ \langle \psi(x - j), \psi(x - k) \rangle &= \delta_{jk} I, \\ \langle \phi(x - j), \psi(x - k) \rangle &= 0, \end{aligned}$$

where  $I$  is the  $r \times r$  identity matrix. Here the inner product is defined by

$$\langle \phi, \psi \rangle = \int \phi(x) \psi^*(x) dx,$$

where  $*$  denotes the complex conjugate transpose. This inner product is an  $r \times r$  matrix.

The multiscaling function approximation to a function  $f$  at resolution  $d^{-j}$  is given by the series

$$(2.2) \quad P_j f = \sum_{k \in \mathbb{Z}} \langle f, \phi_{jk} \rangle \phi_{jk},$$

where

$$\phi_{jk}(x) = d^{-j/2} \phi(d^j x - k).$$

A multiscaling function  $\phi$  has *approximation order*  $p$  if all polynomials up to degree  $p - 1$  can be represented exactly as a series

$$(2.3) \quad x^j = \sum_{k \in \mathbb{Z}} (\mathbf{y}_k^j)^* \phi(x - k), \quad j = 0, \dots, p - 1,$$

for some coefficient vectors  $\mathbf{y}_k^j$ .

In [9, p. 2902] and [7, Theorem 10], the orthonormal balanced multiwavelets of order  $p$  are defined with integrals. An orthonormal multiwavelet is *balanced of order*  $p$  if

$$(2.4) \quad \int_{-\infty}^{\infty} x^n \phi_1(x) dx = \int_{-\infty}^{\infty} \left(x - \frac{1}{r}\right)^n \phi_2(x) dx = \dots = \int_{-\infty}^{\infty} \left(x - \frac{r-1}{r}\right)^n \phi_r(x) dx$$

for  $n = 0, 1, \dots, p - 1$ . A stronger condition than (2.4) was introduced in [5].

Throughout this paper we assume that the multiscaling function  $\phi$  is orthonormal, has compact support, and is continuous (which implies approximation order at least 1), and satisfies condition E. Condition E means that the matrix

$$M_0 = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} h_k$$

has a unique eigenvalue of 1, and all other eigenvalues are less than 1 in absolute value. Condition E is necessary for the stability of the multiresolution approximation produced by  $\phi$  [8].

We denote the vector of *discrete-time monomials*  $\mathbf{u}_{n,k}$  by

$$(2.5) \quad \mathbf{u}_{n,k} := \left[ k^n, \left(k + \frac{1}{r}\right)^n, \dots, \left(k + \frac{r-1}{r}\right)^n \right]^T.$$

### 2.1. Computation of multiwavelet moments

It is well-known that the continuous moments of the multiscaling function  $\phi$  and the multiwavelet function  $\psi$  can be computed if the recurrence coefficients of  $\phi$  and  $\psi$  are given, for example, see [3].

In this section we present how to compute the integral  $\mathbf{m}_0$  (the zeroth continuous moment) and higher moments of the multiscaling function  $\phi$ . It is similar for the computation of moments of the multiwavelet function  $\psi$ .

We begin by defining some terms. The *symbol* of the multiscaling function  $\phi$  is

$$H(\xi) = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} h_k e^{-ik\xi}.$$

The  $j$ th *discrete moment* of the multiscaling function  $\phi$  is

$$M_j = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} k^j h_k.$$

The  $j$ th *continuous moment* of the multiscaling function  $\phi$  is

$$(2.6) \quad \mathbf{m}_j = \int_{-\infty}^{\infty} x^j \phi(x) dx.$$

The symbol and the discrete moments are uniquely defined and easy to calculate. They are related by

$$(2.7) \quad M_j = i^j H^{(j)}(0),$$

where the superscript  $(j)$  denotes the  $j$ th derivative.

The continuous moments can be computed as follows. By substituting the recursion formula (2.1) into the integral in (2.6), we find after simplification

$$(2.8) \quad \mathbf{m}_j = d^{-j} \sum_{s=0}^j \binom{j}{s} M_{j-s} \mathbf{m}_s,$$

where  $\binom{j}{s} = \frac{j!}{s!(j-s)!}$  stands for the combination.

For  $j = 0$ , we get

$$\mathbf{m}_0 = M_0 \mathbf{m}_0.$$

By condition E,  $\mathbf{m}_0$  is uniquely determined up to a constant factor. The normalizing condition for  $\mathbf{m}_0$  is

$$(2.9) \quad \mathbf{m}_0^* \mathbf{m}_0 = 1.$$

This follows from expanding the constant 1 in a multiscaling function series

$$(2.10) \quad 1 = \sum_{k \in \mathbb{Z}} \langle 1, \phi(x-k) \rangle \phi(x-k) = \mathbf{m}_0^* \sum_{k \in \mathbb{Z}} \phi(x-k),$$

and integrating:

$$(2.11) \quad \begin{aligned} 1 &= \int_0^1 1 dx = \mathbf{m}_0^* \sum_{k \in \mathbb{Z}} \int_0^1 \phi(x-k) dx \\ &= \mathbf{m}_0^* \int_{-\infty}^{\infty} \phi(x) dx = \mathbf{m}_0^* \mathbf{m}_0. \end{aligned}$$

For  $j \geq 1$ , (2.8) leads to

$$(2.12) \quad (d^j I - M_0) \mathbf{m}_j = \sum_{s=0}^{j-1} \binom{j}{s} M_{j-s} \mathbf{m}_s.$$

The matrix on the left is nonsingular, again by condition E. We can now compute  $\mathbf{m}_1, \mathbf{m}_2, \dots$  successively (and uniquely) from (2.12).

*Remark 2.1.* From (2.10), we have a relationship between the continuous moments and the sum of the point values of the multiscaling function  $\phi$  for any orthonormal multiwavelets as

$$(2.13) \quad \mathbf{m}_0^* \sum_{k \in \mathbb{Z}} \phi(k) = 1.$$

### 3. Characterization of orthonormal balanced multiwavelets in terms of moments

In this section, we derive a characterization of orthonormal balanced multiwavelets of order  $p$  in terms of the continuous moments of the multiscaling function  $\phi$ .

The following lemma will be used in the proof of Theorem 3.2. One can easily prove the following lemma or can find the proof in [4].

**Lemma 3.1.** *The following combinatorial identity holds:*

$$\sum_{s=\ell}^{n-1} (-1)^{n-s} \binom{n}{s} \binom{s}{\ell} = -\binom{n}{\ell}$$

for  $\ell = 0, 1, \dots, n - 1$ .

Let  $(\mathbf{m}_n)_j$  be the  $j$ th component of the continuous moments  $\mathbf{m}_n$  of the multiscaling function  $\phi$  and  $\alpha_n$  the first component of the  $\mathbf{m}_n$ ; that is,

$$\begin{aligned} (\mathbf{m}_n)_j &:= \int_{-\infty}^{\infty} x^n \phi_j(x) dx, \\ \alpha_n &:= \int_{-\infty}^{\infty} x^n \phi_1(x) dx = (\mathbf{m}_n)_1. \end{aligned}$$

We are in a position to prove the main result of this paper. Necessity part was proved in [4]. For the completeness of the paper we provide it here, again.

**Theorem 3.2.** *An orthonormal multiwavelet is balanced of order  $p$  if and only if the continuous moments  $\mathbf{m}_n$  of the multiscaling functions  $\phi$  satisfy*

$$(3.1) \quad \mathbf{m}_n = \sum_{i=0}^n \binom{n}{i} \alpha_i \mathbf{u}_{n-i,0}$$

for  $n = 0, 1, \dots, p - 1$ , where  $\mathbf{u}_{n,k}$  is the vector of the discrete-time monomial defined in (2.5); that is,

$$(3.2) \quad (\mathbf{m}_0)_{j+1} = \alpha_0,$$

$$(3.3) \quad (\mathbf{m}_n)_{j+1} = \sum_{i=0}^n \binom{n}{i} \left(\frac{j}{r}\right)^{n-i} \alpha_i$$

for  $j = 0, 1, \dots, r - 1$  and  $n = 1, 2, \dots, p - 1$ .

*Proof.* Necessity: We prove by induction on  $n$ . For  $n = 0$ , by (2.4),

$$(\mathbf{m}_0)_{j+1} = \int_{-\infty}^{\infty} \phi_{j+1}(x) dx = \int_{-\infty}^{\infty} \phi_1(x) dx = \alpha_0.$$

Assume that (3.3) is true up to  $n - 1$ , that is,

$$(3.4) \quad (\mathbf{m}_k)_{j+1} = \sum_{s=0}^k \binom{n}{s} \left(\frac{j}{r}\right)^{k-s} \alpha_s$$

for  $j = 0, 1, \dots, r - 1$  and  $k = 1, 2, \dots, n - 1$ . Note that, by (2.4) and binomial expansion,

$$\begin{aligned} \alpha_n &= \int_{-\infty}^{\infty} x^n \phi_1(x) dx \\ &= \int_{-\infty}^{\infty} \left(x - \frac{j}{r}\right)^n \phi_{j+1}(x) dx \\ &= \sum_{s=0}^n (-1)^{n-s} \binom{n}{s} \left(\frac{j}{r}\right)^{n-s} (\mathbf{m}_s)_{j+1} \\ &= \sum_{s=0}^{n-1} (-1)^{n-s} \binom{n}{s} \left(\frac{j}{r}\right)^{n-s} (\mathbf{m}_s)_{j+1} + (\mathbf{m}_n)_{j+1}. \end{aligned}$$

So, by the assumption (3.4),

$$\begin{aligned} (\mathbf{m}_n)_{j+1} &= \alpha_n - \sum_{s=0}^{n-1} (-1)^{n-s} \binom{n}{s} \left(\frac{j}{r}\right)^{n-s} (\mathbf{m}_s)_{j+1} \\ &= \alpha_n - \sum_{s=0}^{n-1} (-1)^{n-s} \binom{n}{s} \left(\frac{j}{r}\right)^{n-s} \left[ \sum_{\ell=0}^s \binom{s}{\ell} \left(\frac{j}{r}\right)^{s-\ell} \alpha_\ell \right] \\ &= \alpha_n - \sum_{s=0}^{n-1} \sum_{\ell=0}^s (-1)^{n-s} \binom{n}{s} \binom{s}{\ell} \left(\frac{j}{r}\right)^{n-\ell} \alpha_\ell. \end{aligned}$$

By exchanging the order of the double summation and using Lemma 3.1, we have

$$\begin{aligned} (\mathbf{m}_n)_{j+1} &= \alpha_n - \sum_{\ell=0}^{n-1} \left[ \sum_{s=\ell}^{n-1} (-1)^{n-s} \binom{n}{s} \binom{s}{\ell} \right] \left(\frac{j}{r}\right)^{n-\ell} \alpha_\ell \\ &= \alpha_n + \sum_{\ell=0}^{n-1} \binom{n}{\ell} \left(\frac{j}{r}\right)^{n-\ell} \alpha_\ell \\ &= \sum_{\ell=0}^n \binom{n}{\ell} \left(\frac{j}{r}\right)^{n-\ell} \alpha_\ell. \end{aligned}$$

Sufficiency: For  $n = 0, 1, \dots, p-1$  and  $j = 0, 1, \dots, r-1$ , by the binomial expansion,

$$\begin{aligned} \int_{-\infty}^{\infty} \left(x - \frac{j}{r}\right)^n \phi_{j+1}(x) dx &= \sum_{s=0}^n (-1)^{n-s} \binom{n}{s} \left(\frac{j}{r}\right)^{n-s} (\mathbf{m}_s)_{j+1} \\ &= \sum_{s=0}^{n-1} (-1)^{n-s} \binom{n}{s} \left(\frac{j}{r}\right)^{n-s} (\mathbf{m}_s)_{j+1} + (\mathbf{m}_n)_{j+1}. \end{aligned}$$

So, by the assumption,

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(x - \frac{j}{r}\right)^n \phi_{j+1}(x) dx \\ &= (\mathbf{m}_n)_{j+1} + \sum_{s=0}^{n-1} (-1)^{n-s} \binom{n}{s} \left(\frac{j}{r}\right)^{n-s} \left[ \sum_{\ell=0}^s \binom{s}{\ell} \left(\frac{j}{r}\right)^{s-\ell} \alpha_{\ell} \right] \\ &= (\mathbf{m}_n)_{j+1} + \sum_{s=0}^{n-1} \sum_{\ell=0}^s (-1)^{n-s} \binom{n}{s} \binom{s}{\ell} \left(\frac{j}{r}\right)^{n-\ell} \alpha_{\ell}. \end{aligned}$$

By exchanging the order of the double summation and using Lemma 3.1, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(x - \frac{j}{r}\right)^n \phi_{j+1}(x) dx \\ &= (\mathbf{m}_n)_{j+1} + \sum_{\ell=0}^{n-1} \left[ \sum_{s=\ell}^{n-1} (-1)^{n-s} \binom{n}{s} \binom{s}{\ell} \right] \left(\frac{j}{r}\right)^{n-\ell} \alpha_{\ell} \\ &= (\mathbf{m}_n)_{j+1} - \sum_{\ell=0}^{n-1} \binom{n}{\ell} \left(\frac{j}{r}\right)^{n-\ell} \alpha_{\ell} \\ &= \sum_{i=0}^n \binom{n}{i} \left(\frac{j}{r}\right)^{n-i} \alpha_i - \sum_{\ell=0}^{n-1} \binom{n}{\ell} \left(\frac{j}{r}\right)^{n-\ell} \alpha_{\ell} \\ &= \alpha_n \\ &= \int_{-\infty}^{\infty} x^n \phi_1(x) dx. \end{aligned}$$

Hence, it is balanced of order  $p$  by (2.4).  $\square$

**Corollary 3.3.** *An orthonormal multiwavelet is balanced (of order 1) if and only if the zeroth continuous moment  $\mathbf{m}_0$  of the orthonormal multiscaling function  $\phi$  is*

$$(3.5) \quad \mathbf{m}_0 = \frac{1}{\sqrt{r}} [1, 1, \dots, 1]^T,$$

which implies the first component  $\alpha_0$  of  $\mathbf{m}_0$  is

$$(3.6) \quad \alpha_0 = \frac{1}{\sqrt{r}}.$$

*Proof.* It is obvious from (2.9) and (3.2).  $\square$

*Remark 3.4.* 1. An orthonormal multiwavelet is balanced of order  $p$  if and only if the vector notation of the continuous moment  $\mathbf{m}_n$  of  $\phi$  is

$$\begin{aligned} \mathbf{m}_n = & \alpha_n \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \binom{n}{n-1} \frac{\alpha_{n-1}}{r} \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ r-1 \end{pmatrix} + \dots \\ & + \binom{n}{1} \frac{\alpha_1}{r^{n-1}} \begin{pmatrix} 0 \\ 1 \\ 2^{n-1} \\ \vdots \\ (r-1)^{n-1} \end{pmatrix} + \frac{\alpha_0}{r^n} \begin{pmatrix} 0 \\ 1 \\ 2^n \\ \vdots \\ (r-1)^n \end{pmatrix}. \end{aligned}$$

2. If  $\alpha_0, \dots, \alpha_n$ , the first components of the  $j$ th,  $j = 0, \dots, n$ , continuous moments of the balanced orthonormal multiscaling function  $\phi$ , are known, then all the component of the  $n$ th continuous moment  $\mathbf{m}_n$  of the balanced orthonormal multiscaling function  $\phi$  can be computed as a sum of discrete-time polynomials  $\mathbf{u}_{i,0}$  of degree  $i \leq n$ .

3. The main Theorem 3.2 can be used as a criterion for checking balancing of order  $p$  with computed continuous moments  $\mathbf{m}_0, \dots, \mathbf{m}_{p-1}$ .

4. If an orthonormal multiwavelet is balanced, then the relationship between the continuous moments and the sum of point values of the multiscaling function  $\phi$  becomes

$$(3.7) \quad \mathbf{u}_{0,0}^* \sum_{k \in \mathbb{Z}} \phi(k) = [1, 1, \dots, 1] \sum_{k \in \mathbb{Z}} \phi(k) = \frac{1}{\alpha_0} = \sqrt{r}.$$

#### 4. Polynomial reproduction formula

Assume that the multiscaling function  $\phi$  has approximation order  $p$ . One can exactly decompose polynomials of degree  $n < p$  using  $\phi$  and its translates, that is, for  $n = 0, 1, \dots, p-1$

$$x^n = \sum_{k \in \mathbb{Z}} \langle x^n, \phi(x-k) \rangle \phi(x-k).$$

In this section, we show how to express polynomials  $1, x, \dots, x^{p-1}$  in terms of  $\sum_{k \in \mathbb{Z}} \mathbf{u}_{n,k}^* \phi(x-k)$ , when the orthonormal multiwavelets are balanced of order  $p$ . In  $\sum_{k \in \mathbb{Z}} \mathbf{u}_{n,k}^* \phi(x-k)$ , the discrete-time monomials  $\mathbf{u}_{n,k}^*$  are used as expansion coefficients with  $\phi(x-k)$ .



For notational simplicity, we define

$$(4.1) \quad \Pi_n(x) := \sum_{k \in \mathbb{Z}} \mathbf{u}_{n,k}^* \phi(x - k).$$

**Lemma 4.1.** *The following equality holds:*

$$(4.2) \quad \mathbf{u}_{n,k}^* = \sum_{i=0}^n \binom{n}{i} k^{n-i} \mathbf{u}_{i,0}^*.$$

*Proof.* Using binomial expansion, we have

$$\begin{aligned} \mathbf{u}_{n,k}^* &= \left[ k^n, \left(k + \frac{1}{r}\right)^n, \left(k + \frac{2}{r}\right)^n, \dots, \left(k + \frac{r-1}{r}\right)^n \right] \\ &= \sum_{i=0}^n \binom{n}{i} k^{n-i} \left[ 0^i, \left(\frac{1}{r}\right)^i, \left(\frac{2}{r}\right)^i, \dots, \left(\frac{r-1}{r}\right)^i \right] \\ &= \sum_{i=0}^n \binom{n}{i} k^{n-i} \mathbf{u}_{i,0}^*. \end{aligned} \quad \square$$

**Theorem 4.2.** *An orthonormal multiwavelet is balanced of order  $p$ . Then for  $n = 0, 1, \dots, p-1$ ,*

$$(4.3) \quad x^n = \sum_{j=0}^n \binom{n}{j} \alpha_j \Pi_{n-j}(x).$$

*Proof.* For  $n = 0, 1, \dots, p-1$ ,

$$\begin{aligned} x^n &= \sum_{k \in \mathbb{Z}} \langle x^n, \phi(x - k) \rangle \phi(x - k) \\ &= \sum_{k \in \mathbb{Z}} \langle (x + k)^n, \phi(x) \rangle \phi(x - k) \\ &= \sum_{k \in \mathbb{Z}} \left[ \sum_{j=0}^n \binom{n}{j} k^{n-j} \langle x^j, \phi(x) \rangle \right] \phi(x - k) \\ &= \sum_{k \in \mathbb{Z}} \left[ \sum_{j=0}^n \binom{n}{j} k^{n-j} \mathbf{m}_j^* \right] \phi(x - k). \end{aligned}$$

By the main Theorem 3.2,

$$\begin{aligned} x^n &= \sum_{k \in \mathbb{Z}} \left[ \sum_{j=0}^n \binom{n}{j} k^{n-j} \sum_{i=0}^j \binom{j}{i} \alpha_{j-i} \mathbf{u}_{i,0}^* \right] \phi(x - k) \\ &= \sum_{k \in \mathbb{Z}} \left[ \sum_{j=0}^n \sum_{i=0}^j \binom{n}{j} \binom{j}{i} \alpha_{j-i} k^{n-j} \mathbf{u}_{i,0}^* \right] \phi(x - k). \end{aligned}$$

By expanding the double summation and adding up diagonally, we have

$$\begin{aligned}
 x^n &= \sum_{k \in \mathbb{Z}} \left[ \sum_{s=0}^n \sum_{t=0}^{n-s} \binom{n}{s+t} \binom{s+t}{t} \alpha_s k^{n-s-t} \mathbf{u}_{t,0}^* \right] \phi(x-k) \\
 &= \sum_{k \in \mathbb{Z}} \left[ \sum_{j=0}^n \sum_{i=0}^{n-j} \binom{n}{j+i} \binom{j+i}{i} \alpha_j k^{n-j-i} \mathbf{u}_{i,0}^* \right] \phi(x-k) \\
 &= \sum_{k \in \mathbb{Z}} \left[ \sum_{j=0}^n \sum_{i=0}^{n-j} \binom{n}{j} \binom{n-j}{i} \alpha_j k^{n-j-i} \mathbf{u}_{i,0}^* \right] \phi(x-k) \\
 &= \sum_{k \in \mathbb{Z}} \left[ \sum_{j=0}^n \binom{n}{j} \alpha_j \sum_{i=0}^{n-j} \binom{n-j}{i} k^{n-j-i} \mathbf{u}_{i,0}^* \right] \phi(x-k).
 \end{aligned}$$

By Lemma 4.1,

$$\begin{aligned}
 x^n &= \sum_{k \in \mathbb{Z}} \left[ \sum_{j=0}^n \binom{n}{j} \alpha_j \mathbf{u}_{n-j,k}^* \right] \phi(x-k) \\
 &= \sum_{j=0}^n \binom{n}{j} \alpha_j \Pi_{n-j}(x).
 \end{aligned}$$

□

For future reference, we list some in detail.

$$\begin{aligned}
 1 &= \frac{1}{\sqrt{r}} \sum_{k \in \mathbb{Z}} \mathbf{u}_{0,k}^* \phi(x-k) = \frac{1}{\sqrt{r}} \Pi_0(x), \\
 x &= \sum_{k \in \mathbb{Z}} \left[ \frac{1}{\sqrt{r}} \mathbf{u}_{1,k}^* + \alpha_1 \mathbf{u}_{0,k}^* \right] \phi(x-k) \\
 &= \frac{1}{\sqrt{r}} \Pi_1(x) + \alpha_1 \Pi_0(x) = \frac{1}{\sqrt{r}} \Pi_1(x) + \alpha_1 \sqrt{r}, \\
 x^2 &= \sum_{k \in \mathbb{Z}} \left[ \frac{1}{\sqrt{r}} \mathbf{u}_{2,k}^* + 2\alpha_1 \mathbf{u}_{1,k}^* + \alpha_2 \mathbf{u}_{0,k}^* \right] \phi(x-k) \\
 &= \frac{1}{\sqrt{r}} \Pi_2(x) + 2\alpha_1 \Pi_1(x) + \alpha_2 \sqrt{r}.
 \end{aligned}$$

## 5. Preservation of discrete polynomials

Assume that the orthonormal multiwavelets are balanced of order  $p$ . In [9], balancing of order  $p$  implies that the discrete-time monomials  $\mathbf{u}_{n,k}$  ( $n < p$ ) as expansion coefficients with  $\phi(x-k)$  gives a polynomial of degree  $n$ , that is,

$$\Pi_n(x) = \sum_{k \in \mathbb{Z}} \mathbf{u}_{n,k}^* \phi(x-k) \in P_n(\mathbb{R}),$$

where  $P_n(\mathbb{R})$  is the set of polynomials of degree  $n$  on  $\mathbb{R}$ . In this section, we derive an algorithm for finding the polynomial  $\Pi_n(x)$  of degree  $n$  for  $n = 0, 1, \dots, p-1$  in terms of the first components  $\alpha_n$  of the continuous moments  $\mathbf{m}_n$  of the multiscaling function  $\phi$ .

**Algorithm 5.1.** Let an orthonormal multiscaling function  $\phi$  be balanced of order  $p$ . Then an algorithm for finding the polynomial

$$\Pi_n(x) = \sum_{k \in \mathbb{Z}} \mathbf{u}_{n,k}^* \phi(x - k)$$

of degree  $n$  in terms of the first component  $\alpha_n$  of the  $n$ th continuous moment of  $\phi$  for  $n = 0, 1, \dots, p-1$  is:

- step 1: for  $n = 0$ , we compute

$$(5.1) \quad \Pi_0(x) = \frac{1}{\alpha_0} = \sqrt{r}.$$

- step 2: for  $n = 1, 2, \dots, p-1$ , we recursively compute

$$(5.2) \quad \Pi_n(x) = \frac{1}{\alpha_0} \left[ x^n - \sum_{j=1}^n \binom{n}{j} \alpha_j \Pi_{n-j}(x) \right].$$

*Proof.* From Theorem 4.2, it is obvious that for  $n = 0$ ,

$$\Pi_0(x) = \sum_{k \in \mathbb{Z}} \mathbf{u}_{0,k}^* \phi(x - k) = \frac{1}{\alpha_0} = \sqrt{r}.$$

For  $n = 1, 2, \dots, p-1$ ,

$$x^n = \sum_{j=0}^n \binom{n}{j} \alpha_j \Pi_{n-j}(x) = \alpha_0 \Pi_n(x) + \sum_{j=1}^n \binom{n}{j} \alpha_j \Pi_{n-j}(x).$$

Hence,

$$\Pi_n(x) = \frac{1}{\alpha_0} \left[ x^n - \sum_{j=1}^n \binom{n}{j} \alpha_j \Pi_{n-j}(x) \right]$$

for  $n = 1, 2, \dots, p-1$ . □

For future reference, we list some in detail.

$$\begin{aligned} \Pi_0(x) &= \frac{1}{\alpha_0} = \sqrt{r}, \\ \Pi_1(x) &= \frac{1}{\alpha_0} [x - \alpha_1 \Pi_0(x)] = \sqrt{r} x - r \alpha_1, \\ \Pi_2(x) &= \frac{1}{\alpha_0} [x^2 - 2 \alpha_1 \Pi_1(x) - \alpha_2 \Pi_0(x)] \\ &= \sqrt{r} [x^2 - 2 \alpha_1 (\sqrt{r} x - r \alpha_1) - \alpha_2 \sqrt{r}] \\ &= \sqrt{r} x^2 - 2 r \alpha_1 x + (2 r \sqrt{r} \alpha_1^2 - r \alpha_2). \end{aligned}$$

## 6. Examples

In this section, we give three examples to illustrate the general theory.

We base our examples on the orthonormal high-order balanced multiscaling functions  $\phi$  by Selesnick given in [9] and by Lebrun and Vetterli given in [6, 7] with a stronger condition. These functions have multiplicity  $r = 2$  and dilation factor  $d = 2$ .

**Example 6.1.** In this example we take the multiscaling function  $\phi$  supported on  $[0, 3]$  given in [9]. In [9], the constant  $A$  is given by  $A = \pm \frac{1}{2} \sqrt{-8 + 6\sqrt{3}}$ . In this example, we consider only for  $A = -\frac{1}{2} \sqrt{-8 + 6\sqrt{3}}$ , because the multiscaling function  $\phi$  for this case is smoother than the other case. The nonzero recursion coefficients are

$$\begin{aligned} h_0 &= \begin{pmatrix} 0 & \frac{7}{12} - \frac{1}{4}\sqrt{3} + \frac{A}{6} \\ \frac{1}{16} - \frac{1}{16}\sqrt{3} & -\frac{5}{48} + \frac{1}{16}\sqrt{3} - \frac{A}{12} \end{pmatrix}, \\ h_1 &= \begin{pmatrix} -\frac{A}{3} + \frac{1}{3} & \frac{1}{12} + \frac{1}{4}\sqrt{3} - \frac{1}{6} \\ \frac{7}{48} - \frac{1}{16}\sqrt{3} + \frac{A}{6} & \frac{19}{48} - \frac{3}{16}\sqrt{3} - \frac{A}{12} \end{pmatrix}, \\ h_2 &= \begin{pmatrix} 0 & 0 \\ \frac{7}{16} + \frac{1}{16}\sqrt{3} & \frac{1}{48} + \frac{3}{16}\sqrt{3} - \frac{A}{12} \end{pmatrix}, \\ h_3 &= \begin{pmatrix} 0 & 0 \\ \frac{1}{48} + \frac{1}{16}\sqrt{3} + \frac{A}{6} & \frac{1}{48} - \frac{1}{16}\sqrt{3} + \frac{1}{12} \end{pmatrix}. \end{aligned}$$

Discrete moments  $M_n$  of the multiscaling function  $\phi$  for  $n = 0, 1, 2$  are

$$\begin{aligned} M_0 &= \frac{1}{3} \begin{pmatrix} 1 - A & 2 + A \\ 2 + A & 1 - A \end{pmatrix}, \\ M_1 &= \frac{1}{12} \begin{pmatrix} 4 - 4A & 1 + 3\sqrt{3} + 4A \\ 13 + 3\sqrt{3} + 8A & 6 - 6A \end{pmatrix}, \\ M_2 &= \frac{1}{12} \begin{pmatrix} 4 - 4A & 1 + 3\sqrt{3} + 2A \\ 25 + 9\sqrt{3} + 20A & 8 - 14A \end{pmatrix}. \end{aligned}$$

Continuous moments  $m_n$  of the multiscaling function  $\phi$  for  $n = 0, 1, 2$  are

$$\begin{aligned} m_0 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad m_1 = \frac{1}{8} \begin{pmatrix} 3\sqrt{2} + \sqrt{6} \\ 5\sqrt{2} + \sqrt{6} \end{pmatrix}, \\ m_2 &= \frac{1}{55504} \begin{pmatrix} 20526\sqrt{2} + 11044\sqrt{6} + 578A\sqrt{2} - 98A\sqrt{6} \\ 48854\sqrt{2} + 16708\sqrt{6} - 578A\sqrt{2} + 98A\sqrt{6} \end{pmatrix}. \end{aligned}$$

Hence, the first components  $\alpha_n$  of the  $n$ th continuous moment  $\mathbf{m}_n$  of the multiscaling function  $\phi$  for  $n = 0, 1, 2$  are  $\alpha_0 = \frac{1}{\sqrt{2}}$ ,  $\alpha_1 = \frac{1}{8} (3\sqrt{2} + \sqrt{6})$  and  $\alpha_2 = \frac{1}{55504} (20526\sqrt{2} + 11044\sqrt{6} + 578A\sqrt{2} - 98A\sqrt{6})$ .

One can easily check that equation (3.1), the main result of this paper, is satisfied for  $n = 0, 1$ , that is,

$$\begin{aligned} \alpha_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{m}_0, \\ \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\alpha_0}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} \alpha_1 \\ \alpha_1 + \frac{\alpha_0}{2} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 3\sqrt{2} + \sqrt{6} \\ 5\sqrt{2} + \sqrt{6} \end{pmatrix} = \mathbf{m}_1, \\ \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\frac{\alpha_1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{\alpha_0}{2^2} \begin{pmatrix} 0 \\ 1^2 \end{pmatrix} &= \begin{pmatrix} \alpha_2 \\ \alpha_2 + \alpha_1 + \frac{\alpha_0}{4} \end{pmatrix} \neq \mathbf{m}_2. \end{aligned}$$

Hence, it is balanced of order 2.

We have the polynomial reproduction formula as

$$\begin{aligned} 1 &= \sum_{k \in \mathbb{Z}} \left[ \frac{1}{\sqrt{2}} \phi_1(x - k) + \frac{1}{\sqrt{2}} \phi_2(x - k) \right], \\ x &= \sum_{k \in \mathbb{Z}} \frac{\sqrt{2}}{8} \left[ (4k + 3\sqrt{2} + \sqrt{6}) \phi_1(x - k) + (4k + 3\sqrt{2} + \sqrt{6} + 2) \phi_2(x - k) \right]. \end{aligned}$$

We have the preservation of discrete polynomials as

$$\begin{aligned} \Pi_0(x) &= \sum_{k \in \mathbb{Z}} [\phi_1(x - k) + \phi_2(x - k)] = \sqrt{2}, \\ \Pi_1(x) &= \sum_{k \in \mathbb{Z}} \left[ k \phi_1(x - k) + \left( k + \frac{1}{2} \right) \phi_2(x - k) \right] = \sqrt{2}x - \frac{3\sqrt{2} + \sqrt{6}}{4}. \end{aligned}$$

Hence,

$$\sum_{k \in \mathbb{Z}} [2k \phi_1(x - k) + (2k + 1) \phi_2(x - k)] = 2\sqrt{2}x - \frac{3\sqrt{2} + \sqrt{6}}{2}.$$

**Example 6.2.** In this example we take the multiscaling function  $\phi$  supported on  $[0, 2]$  given in [6, 7]. This is called the *Bat* of order 1, which is a stronger condition than the balanced of order 1, in [6, 7]. For the *Bat* of order 1, the nonzero recursion coefficients are

$$h_0 = \frac{1}{4\sqrt{2}} \begin{pmatrix} 0 & 2 + \sqrt{7} \\ 0 & 2 - \sqrt{7} \end{pmatrix}, \quad h_1 = \frac{1}{4\sqrt{2}} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad h_2 = \frac{1}{4\sqrt{2}} \begin{pmatrix} 2 - \sqrt{7} & 0 \\ 2 + \sqrt{7} & 0 \end{pmatrix}.$$

These differ from Lebrun and Vetterli by a factor of  $\sqrt{2}$ , due to differences in notation.

Discrete moments  $M_n$  of the multiscaling function  $\phi$  for  $n = 0, 1, 2$  are

$$M_0 = \frac{1}{8} \begin{pmatrix} 5 - \sqrt{7} & 3 + \sqrt{7} \\ 3 + \sqrt{7} & 5 - \sqrt{7} \end{pmatrix}, \quad M_1 = \frac{1}{8} \begin{pmatrix} 7 - 2\sqrt{7} & 1 \\ 5 + 2\sqrt{7} & 3 \end{pmatrix}, \quad M_2 = \frac{1}{8} \begin{pmatrix} 11 - 4\sqrt{7} & 1 \\ 9 + 4\sqrt{7} & 3 \end{pmatrix}.$$

Continuous moments  $\mathbf{m}_n$  of the multiscaling function  $\phi$  for  $n = 0, 1, 2$  are

$$\mathbf{m}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{m}_1 = \frac{1}{6\sqrt{2}} \begin{pmatrix} 7 - \sqrt{7} \\ 5 + \sqrt{7} \end{pmatrix}, \quad \mathbf{m}_2 = \frac{1}{18\sqrt{2}} \begin{pmatrix} 28 - 7\sqrt{7} \\ 16 + 5\sqrt{7} \end{pmatrix}.$$

Hence, the first components  $\alpha_n$  of the  $n$ th continuous moment  $\mathbf{m}_n$  of the multiscaling function  $\phi$  for  $n = 0, 1, 2$  are  $\alpha_0 = \frac{1}{\sqrt{2}}$ ,  $\alpha_1 = \frac{7-\sqrt{7}}{6\sqrt{2}}$  and  $\alpha_2 = \frac{28-7\sqrt{7}}{18\sqrt{2}}$ . One can easily check that equation (3.1), the main result of this paper, is satisfied for  $n = 0$  only, that is,

$$\alpha_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{m}_0,$$

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\alpha_0}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_1 + \frac{\alpha_0}{2} \end{pmatrix} = \frac{1}{6\sqrt{2}} \begin{pmatrix} 7 - \sqrt{7} \\ 10 - \sqrt{7} \end{pmatrix} \neq \frac{1}{6\sqrt{2}} \begin{pmatrix} 7 - \sqrt{7} \\ 5 + \sqrt{7} \end{pmatrix} = \mathbf{m}_1.$$

Hence, the *Bat* of order 1 is balanced of order 1.

We have the polynomial reproduction formula as

$$1 = \sum_{k \in \mathbb{Z}} \left[ \frac{1}{\sqrt{2}} \phi_1(x - k) + \frac{1}{\sqrt{2}} \phi_2(x - k) \right].$$

We have the preservation of discrete polynomials as

$$\Pi_0(x) = \sum_{k \in \mathbb{Z}} [\phi_1(x - k) + \phi_2(x - k)] = \sqrt{2}.$$

**Example 6.3.** In this example we take the multiscaling function  $\phi$  supported on  $[0, 3]$  given in [6, 7]. This is called the *Bat* of order 2, which is a stronger condition than the balanced of order 2, in [6, 7]. For the *Bat* of order 2, the nonzero recursion coefficients are

$$h_0 = \sqrt{2} \begin{pmatrix} 0 & b_0 \\ 0 & a_4 \end{pmatrix}, \quad h_1 = \sqrt{2} \begin{pmatrix} a_1 & b_1 \\ b_3 & a_3 \end{pmatrix}, \quad h_2 = \sqrt{2} \begin{pmatrix} a_2 & b_2 \\ b_2 & a_2 \end{pmatrix},$$

$$h_3 = \sqrt{2} \begin{pmatrix} a_3 & b_3 \\ b_1 & a_1 \end{pmatrix}, \quad h_4 = \sqrt{2} \begin{pmatrix} a_4 & 0 \\ b_0 & 0 \end{pmatrix},$$

where

$$a_1 = \frac{93-13\sqrt{31}}{640}, \quad b_0 = \frac{-31+\sqrt{31}}{640},$$

$$a_2 = \frac{341-11\sqrt{31}}{640}, \quad b_1 = \frac{217+23\sqrt{31}}{640},$$

$$a_3 = \frac{11-11\sqrt{31}}{640}, \quad b_2 = \frac{23+7\sqrt{31}}{640},$$

$$a_4 = \frac{-13+3\sqrt{31}}{640}, \quad b_3 = \frac{-1+\sqrt{31}}{640}.$$

These differ from Lebrun and Vetterli by a factor of  $\sqrt{2}$ , due to differences in notation.

Discrete moments  $M_n$  of the multiscaling function  $\phi$  for  $n = 0, 1, 2$  are

$$\begin{aligned} M_0 &= \frac{1}{40} \begin{pmatrix} 27 - 2\sqrt{31} & 13 + 2\sqrt{31} \\ 13 + 2\sqrt{31} & 27 - 2\sqrt{31} \end{pmatrix}, \\ M_1 &= \frac{1}{160} \begin{pmatrix} 189 - 14\sqrt{31} & 65 + 10\sqrt{31} \\ 143 + 22\sqrt{31} & 243 - 18\sqrt{31} \end{pmatrix}, \\ M_2 &= \frac{1}{160} \begin{pmatrix} 337 - 27\sqrt{31} & 75 + 15\sqrt{31} \\ 387 + 63\sqrt{31} & 553 - 43\sqrt{31} \end{pmatrix}. \end{aligned}$$

Continuous moments  $\mathbf{m}_n$  of the multiscaling function  $\phi$  for  $n = 0, 1, 2, 3$  are

$$\begin{aligned} \mathbf{m}_0 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{m}_1 = \frac{1}{4\sqrt{2}} \begin{pmatrix} 7 \\ 9 \end{pmatrix}, \quad \mathbf{m}_2 = \frac{1}{16\sqrt{2}} \begin{pmatrix} 49 \\ 81 \end{pmatrix}, \\ \mathbf{m}_3 &= \frac{1}{293391\sqrt{2}} \begin{pmatrix} 1590697 + 1548\sqrt{31} \\ 3401607 - 1548\sqrt{31} \end{pmatrix}. \end{aligned}$$

Hence, the first components  $\alpha_n$  of the  $n$ th continuous moment  $\mathbf{m}_n$  of the multiscaling function  $\phi$  for  $n = 0, 1, 2, 3$  are  $\alpha_0 = \frac{1}{\sqrt{2}}$ ,  $\alpha_1 = \frac{7}{4\sqrt{2}}$ ,  $\alpha_2 = \frac{49}{16\sqrt{2}}$  and  $\alpha_3 = \frac{1590697+1548\sqrt{31}}{293391\sqrt{2}}$ . One can easily check that equation (3.1), the main result of this paper, is satisfied for  $n = 0, 1, 2$ , that is,

$$\begin{aligned} \alpha_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{m}_0, \\ \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\alpha_0}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \frac{1}{4\sqrt{2}} \begin{pmatrix} 7 \\ 9 \end{pmatrix} = \mathbf{m}_1. \end{aligned}$$

Hence, the *Bat* of order 2 is balanced of order 2.

We have the polynomial reproduction formula as

$$\begin{aligned} 1 &= \sum_{k \in \mathbb{Z}} \left[ \frac{1}{\sqrt{2}} \phi_1(x - k) + \frac{1}{\sqrt{2}} \phi_2(x - k) \right], \\ x &= \sum_{k \in \mathbb{Z}} \left[ \frac{4k + 7}{4\sqrt{2}} \phi_1(x - k) + \frac{4k + 9}{4\sqrt{2}} \phi_2(x - k) \right]. \end{aligned}$$

We have the preservation of discrete polynomials as

$$\begin{aligned} \Pi_0(x) &= \sum_{k \in \mathbb{Z}} [\phi_1(x - k) + \phi_2(x - k)] = \sqrt{2}, \\ \Pi_1(x) &= \sum_{k \in \mathbb{Z}} \left[ k \phi_1(x - k) + \left( k + \frac{1}{2} \right) \phi_2(x - k) \right] = \sqrt{2}x - \frac{7\sqrt{2}}{4}. \end{aligned}$$

Hence,

$$\sum_{k \in \mathbb{Z}} [2k \phi_1(x - k) + (2k + 1) \phi_2(x - k)] = 2\sqrt{2}x - \frac{7\sqrt{2}}{2}.$$

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