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EMBEDDINGS OF LINE IN THE PLANE AND ABHYANKAR-MOH EPIMORPHISM THEOREM

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Abstract. In this paper, we consider the parameter space of the rational plane curves with uni-branched singularity. We show that such a parameter space is decomposable into irreducible components which are *rational* varieties. Rational parametrizations of the irreducible components are given in a constructive way, by a repeated use of Abhyankar-Moh Epimorphism Theorem. We compute an enumerative invariant of this parameter space, and include explicit computational examples to recover some classically-known invariants.

1. Introduction

Let C be a curve in the complex affine plane $\mathbb{A}^2_{\mathbb{C}}$. We call C an *unparametrized affine embedding of line* of degree d if it is isomorphic to $\mathbb{A}^1_{\mathbb{C}}$ and its projective closure $\overline{C}$ is a curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$. We call such a curve C a smooth rational affine plane curve of degree d. Denote by $\mathcal{A}(d)$ the space of all unparametrized affine embeddings of line of degree d. Note that the notion of the degree of C is well-defined with a fixed affine structure on $\mathbb{A}^2_{\mathbb{C}}$ since any affine automorphism can be extended to an automorphism of $\mathbb{P}^2_{\mathbb{C}}$.

Many results are available on the local properties of the parameter space $\mathcal{A}(d)$. For example, one knows that it is a union of smooth irreducible varieties [9]. Our main interest in this paper is in studying some of its global properties. As a result, we show that $\mathcal{A}(d)$ is a disjoint union of smooth irreducible *rational* varieties. In fact, we give explicit rational parametrizations of all of its irreducible components, and compute the dimensions of the components. In particular, we show that the (maximal) dimension of $\mathcal{A}(d)$ is $d+2$ for $d \geq 2$, and that there are exactly $(\frac{d(d+1)}{2} - 1)$ distinct unparametrized embeddings of line (i.e., embeddings of line up to reparametrization) passing through general $d+2$ points in $\mathbb{A}^2_{\mathbb{C}}$ for $d \geq 2$.

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The main ingredient of the proof is the Abhyankar-Moh Epimorphism Theorem [1]. To explain this connection more explicitly, let us start by recalling a basic definition.

Definition 1. A polynomial map $\varphi = (f, g) : \mathbb{C}^1 \to \mathbb{C}^2$ is called a *parametrized affine embedding of line*, or simply an embedding of line, if $\mathbb{C}[t] = \mathbb{C}[f(t), g(t)]$.

Note that the above condition amounts to the condition that the image curve in $\mathbb{C}^2$ is isomorphic to $\mathbb{C}^1$ under the polynomial map. The Abyhankar-Moh Epimorphism Theorem provides a necessary condition on the degrees of f and g for (f, g) to be an embedding of line.

Theorem 1 ([1]). *Suppose f, g are in $\mathbb{C}[t]$ with $m := \deg g \leq n := \deg f$. If $\mathbb{C}[f(t), g(t)] = \mathbb{C}[t]$, then m divides n.*

Among the curves parametrically defined by $\varphi = (f, g) : \mathbb{C}^1 \to \mathbb{C}^2$ with m dividing n, there are smooth curves and non-smooth curves. An obvious question arises now:

How many such curves are smooth?

Since $\mathcal{A}(d)$ can be viewed as the space of all *embeddings of line* of degree d up to reparametrization, this question naturally prompts one to study the parameter space $\mathcal{A}(d)$. In fact, this was the initial motivation for the authors to study the parameter space $\mathcal{A}(d)$.

We also study the parameter space $\mathcal{M}(d)$ of unicuspidal rational curves with maximal tangent [4]. $\mathcal{A}(d)$ can be considered as a sub-variety of $\mathcal{M}(d)$, consisting of the curves whose singular points lie on the line at infinity which is the maximal tangent line. Here, the maximal tangent line means the line which intersect the curve only at the singular point.

The variety $\mathcal{A}(d)$ has the main component $\mathcal{A}_d$ which has exactly the largest dimension among the irreducible components. The explicit parametrization of a Zariski open subset $\mathcal{A}_d^\circ$ of the main component $\mathcal{A}_d$ is given by monomial terms with one exception in the homogeneous coordinates of $\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(d))$. So, in a sense the main component is almost a toric variety. Using this parametrization, we compute the degree of the variety $\overline{\mathcal{A}}(d)$. Two proofs are given: one by using a projection and Kouchinirenko theorem on the number of solutions of a system of equations [16, 7], the other one by a direct method.

The same parametrization can be used for degree counting for the parameter space of rational projective plane curves with unique irreducible singular points. As an illustration of our technique, we attempt to compute the degree of the parameter space $\mathcal{M}(3)$ of cuspidal cubic curves in $\mathbb{P}^2_{\mathbb{C}}$. An explicit computation with the computer algebra package *Singular* [10] shows that this degree is 24, coinciding with a classical result [18, 15, 2].

After writing up this paper, we learned that the degree of $\mathcal{A}(d)$ had been independently obtained in a recent paper [13], as an example. The approaches, however, appear to be substantially different. It is to be noted that our method produces not only the degree of the parameter space $\mathcal{A}(d)$ but also an explicit parametrization of $\mathcal{A}(d)$ itself.

2. Decomposition of embeddings of line

Let us start by recalling basic terminologies.

Definition 2 (c.f. [17]). A polynomial map $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ is called an *elementary polynomial map* of $\mathbb{C}^2$ if $\Phi(x,y)$ is $(x, y+f(x))$ or $(x+g(y), y)$ for some polynomial $f(x) \in \mathbb{C}[x]$ or $g(y) \in \mathbb{C}[y]$.

The Abyhyankar-Moh can be used to reduce the degrees of φ by applying an elementary polynomial map.

Lemma 1. *Let $\varphi = (f,g) : \mathbb{C}^1 \to \mathbb{C}^2$ be an embedding of line in $\mathbb{C}^2$ where*

$$\begin{aligned} f(t) &= a_0t^n + a_1t^{n-1} + \cdots + a_n \\ g(t) &= b_0t^m + b_1t^{m-1} + \cdots + b_m. \end{aligned}$$

Suppose $\deg f = n = k \cdot \deg g = km$ *for a positive integer k. Then there is a unique elementary polynomial map $\Phi(x,y) = (x - c(y), y)$ such that $\Phi \circ \varphi(t) = (h(t), g(t))$, $c(0) = 0$, and* $\deg h$ *is strictly less than m. Furthermore, if f and g are viewed as polynomials in $\mathbb{Z}[a_0, \ldots, a_n, b_0, \ldots, b_m][t]$, then c is a polynomial in $\mathbb{Z}[a_0, \ldots, a_n, b_0, \ldots, b_n][\frac{1}{b_0}][t]$.*

Proof. There are unique $c_0, c_1, \ldots, c_{k-1} \in \mathbb{C}$ such that the degree of

$$h(t) := f(t) - (c_0g(t)^k + c_1g(t)^{k-1} + \cdots + c_{k-1}g(t))$$

is less than the degree of g. This is possible due to a successive application of Theorem 1. Let $c_0 = a_0/b_0^k$, and suppose $c_0, c_1, \ldots, c_{l-1}$ are defined then c_l is determined as

$$c_l = (a_{lm} - c_0b_{k,n-lm} - c_1b_{k-1,n-lm} - \cdots - c_{l-1}b_{k-l+1,n-lm})/b_0^{k-l},$$

where $g(t)^p = (\sum_{i=0}^m b_it^{m-i})^p = \sum_{j=0}^n b_{p,j}t^j$. □

Definition 3. (1) Denote by $\mathcal{R} \subset \mathrm{Mor}(\mathbb{C}^1, \mathbb{C}^2)$ the locus of all embeddings of parametrized affine line and define

$$\mathcal{R}(n, \cdot) = \{(f,g) \in \mathcal{R} \mid \deg f = n,\ \deg g < n\}.$$

Similarly, define

$$\mathcal{R}(n,m) = \{(f,g) \in \mathcal{R} \mid \deg f = n,\ \deg g = m\}$$

and

$$\mathcal{R}^\circ(n) = \{(f,g) \in \mathcal{R} \mid \deg f = n,\ \deg g \le n\}.$$

(2) Let $C_k = \mathbb{C}^* \times \mathbb{C}^{k-1}$ for $k > 1$ and $C_{k_1,\ldots,k_q} = \prod_{i=1}^q C_{k_i}$ for $k_i > 1$.

Note that $\mathcal{R}(n,\cdot) = \coprod_{m|n,\, m<n} R(n,m)$. Lemma 1 will be used to decompose $\mathcal{R}(n,\cdot)$ further.

Define, for $k > 1$,

$$\begin{aligned} \tau_{k,m} : C_k \times \mathcal{R}(m,\cdot) &\rightarrow \mathcal{R}(mk,\cdot) \\ (c,g,h) &\mapsto (h + \sum_{i=0}^{k-1} c_i g^{k-i}, g). \end{aligned}$$

Given ordered positive integers $k_1, \ldots, k_q$ with $\prod_{i=1}^q k_i = n$ and $k_i > 1$, by iteratively composing τ's we obtain a map

$$\tau_{(k_1,\ldots,k_q)} : C_{k_1} \times (C_{k_2} \times \cdots \times (C_{k_q} \times \mathcal{R}(1,\cdot)) \cdots) \rightarrow \mathcal{R}(n,\cdot),$$

where $\tau_{(k_1,\ldots,k_q)} := \tau_{k_1,k_2\cdots k_q} \circ \cdots \circ \tau_{k_{q-1},k_q} \circ \tau_{k_q,1}$.

Definition 4. Denote the image of this composition map by $\mathcal{R}^\circ_{\mathbf{k}}$ where $\mathbf{k} = (k_1, \ldots, k_q)$.

Lemma 2. *$\mathcal{R}^\circ_{\mathbf{k}}$ is a locally closed subset of $\mathbb{C}^{n+1} \times \mathbb{C}^{m+1}$ under $\tau_{\mathbf{k}}$, where $n = \prod_{i=1}^q k_i$, $m = \prod_{i=2}^q k_i$. Furthermore it is isomorphic to $C_{\mathbf{k}} \times \mathcal{R}(1,\cdot)$ as a variety.*

Proof. Consider

$$\tau_{\mathbf{k}} : C_{\mathbf{k}} \times \mathcal{R}(1,\cdot) \rightarrow \mathbb{C}^{n+1} \times \mathbb{C}^{m+1}$$

and its *converse*

$$\eta_{\mathbf{k}} : U \rightarrow C_{\mathbf{k}} \times \mathcal{R}(1,\cdot)$$

in some open neighborhood U of $\mathcal{R}^\circ_{\mathbf{k}}$ in $\mathbb{C}^{n+1} \times \mathbb{C}^{m+1}$. Then $\eta_{\mathbf{k}} \circ \tau_{\mathbf{k}}$ is the identity. However $\tau_{\mathbf{k}} \circ \eta_{\mathbf{k}}(\phi) = \phi$ if and only if $\phi \in \mathcal{R}^\circ_{\mathbf{k}}$. This shows that $\mathcal{R}^\circ_{\mathbf{k}}$ is a closed subset of U. □

Lemma 3. *Let $k > 1$. Endow $\mathcal{R}(l,\cdot)$ with the induced Zariski topology as a subspace of $\mathbb{C}^{l+1} \times \mathbb{C}^l$. Then the polynomial map $\tau : C_k \times \mathcal{R}(m,\cdot) \rightarrow \mathcal{R}(mk,\cdot)$ sending $\tau(c,g,h) \rightarrow (h + \sum_{i=0}^{k-1} c_i g^{k-i}, g)$ is a closed map.*

Proof. Let $n = mk$. Consider the sequences $(c^{(j)}, g^{(j)}, h^{(j)})$ such that the limit $(f^{(\infty)}, g^{(\infty)})$ of $\tau(c^{(j)}, g^{(j)}, h^{(j)})$ exists in $\mathcal{R}(m,\cdot)$. Then the sequence $g^{(j)}$ has the limit $g^{(\infty)} = b_0 t^m + \cdots + b_m$ in $\mathbb{C}^{m+1}$. Since the first component of $\tau(c^{(j)}, h^{(j)}, g^{(j)}))$ has degree $mk > 1$, $g^{(\infty)}$ must not be zero.

Case I: Suppose that $b_0 = 0$. We will show that this cannot happen. Since the coefficient of t^{mk} in $c_0^{(j)}(g^{(j)})^k$ converges to a nonzero number, $|c_0^{(j)}|$ goes to ∞. By a change of coordinate t, we may assume that $g^{(\infty)}(t) = b_{m-l}t^l + O(t^{l+1})$ where $n > l \geq 1$ and $b_{m-l} \neq 0$. Now consider the coefficients of t^{mk}, $t^{mk-1}, \ldots,$

$t^{m(k-1)+l}$ in $h^{(j)} + \sum c_i^{(j)}(g^{(j)})^{k-i}$, which are

$c_0^{(j)}(b_0^{(j)})^k$,

$c_0^{(j)}(b_0^{(j)})^{k-1}b_1^{(j)}$,

$c_0^{(j)}(k(b_0^{(j)})^{k-1}b_2^{(j)} + \frac{k(k-1)}{2}(b_0^{(j)})^{k-2}(b_1^{(j)})^2)$,

$c_0^{(j)}(k(b_0^{(j)})^{k-1}b_3^{(j)} + k(k-1)(b_0^{(j)})^{k-2}b_1^{(j)}b_2^{(j)} + \frac{k(k-1)(k-2)}{3!}(b_0^{(j)})^{k-3}(b_1^{(j)})^3)$,

$\vdots$

$c_0^{(j)}(k(b_0^{(j)})^{k-1}b_{m-l}^{(j)} + \alpha(b_0^{(j)}, \ldots, b_{m-l+1}^{(j)}))$,

where α is a homogeneous polynomial of degree k. It shows that $|b_0^{(j)}|$ decreases as fast as $1/|c_0^{(j)}|^{\frac{1}{k}}$, $|b_1^{(j)}|$ decreases as fast as (or faster than) $1/|c_0^{(j)}|^{\frac{1}{k}}$, ..., $|b_{m-l}^{(j)}|$ decreases as fast as (or faster than) $1/|c_0^{(j)}|^{\frac{1}{k}}$. This is a contradiction to $b_{m-l}^{(j)} \to b_{m-l} \neq 0$.

Case II: Suppose that $b_0 \neq 0$. Let $f^{(\infty)} = a_0t^n + \cdots + a_n$ with $n = mk$. Then $c_0^{(j)}$ goes to $a_0/b_0^k \neq 0$, and c_l goes to

$$(a_{lm} - c_0b_{k,n-lm} - c_1b_{k-1,n-lm} - \cdots - c_{l-1}b_{k-l+1,n-lm})/b_0^{k-l},$$

where $(\sum_{i=0}^m b_it^{m-i})^p = \sum_{j=0}^n b_{p,j}t^j$. Thus, $c_i^{(j)}$ has the (bounded) limit for all i and also the sequence $h^{(j)}$ has the limit in $\mathbb{C}^m$.

Since the Zariski closure and the closure by strong topology of a constructible subset in a variety coincide, the above arguments prove the lemma. □

Remark 1. (1) $\mathcal{R}(n,n) \cong \mathbb{C}^* \times \mathcal{R}(n,\cdot)$ by sending (f,g) to $(\frac{b_0}{a_0}, (f, g - \frac{b_0}{a_0}f))$, where $f = \sum_{i=0}^n a_it^{n-i}$ and $g = \sum_{i=0}^n b_it^{n-i}$. Here the equivalence symbol $\cong$ means that there are regular maps between open subsets containing $\mathcal{R}_{n,n}$ and $\mathbb{C}^* \times \mathcal{R}(n,\cdot)$ in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ such that those maps are inverses to each other when restricted to $\mathcal{R}(n,n)$ and $\mathbb{C}^* \times \mathcal{R}(n,\cdot)$. Later we will show that $\mathcal{R}(n,n)$ and $\mathcal{R}(n,\cdot)$ are subvarieties (but not closed ones).

(2) Similarly $\mathcal{R}^\circ(n) \cong \mathbb{C} \times \mathcal{R}(n,\cdot)$ by sending (f,g) to $(\frac{b_0}{a_0}, (f, g - \frac{b_0}{a_0}f))$ where $g = \sum_{i=0}^n b_it^{n-i}$. Note that b_0 here could possibly be zero.

In what follows, we will call the maximal dimension (or simply, dimension) of a variety X to be the maximum among the dimensions of the irreducible components of X.

Theorem 2. (1) *The parameter space* $\mathcal{R}^\circ_{(k_1,\ldots,k_q)}$ *in* $\mathbb{C}^{n+1} \times \mathbb{C}^{m+1}$ *is isomorphic to*

$$\mathcal{R}^\circ_{(k_1,\ldots,k_q)} \cong \prod_{i=1}^q C_{k_i} \times C_3$$

of dimension $(\sum k_i) + 3$.

(2) $\mathcal{R}(n,\cdot) \cong \coprod_{\mathbf{k},\prod k_i = n} \mathcal{R}^\circ_{\mathbf{k}}$.

(3) $\mathcal{R}^\circ(n) \cong \mathbb{C} \times \left(\coprod_{\mathbf{k},\prod k_i = n,} \mathcal{R}^\circ_{\mathbf{k}} \right)$ *of the maximal dimension* $n+4$ *if* $n \geq 2$ *or* 4 *if* $n = 1$.

Proof. The proof of statement (1) is proven in Lemma 2. The second statement follows from Lemmas 2 and 3. For the proof of statement (3). Consider

$$\alpha : (\mathbb{C}^* \times \mathbb{C}^n) \times \mathbb{C}^{n+1} \to \mathbb{C} \times (\mathbb{C}^* \times \mathbb{C}^n) \times \mathbb{C}^n$$

sending $(\mathbf{a}, \mathbf{b})$ to $(b_0/a_0, \mathbf{a}, \mathbf{b} - \frac{b_0}{a_0}(a_1, \ldots, a_n))$. It has the inverse $\beta(c, \mathbf{a}, \mathbf{d}) = (\mathbf{a}, \mathbf{d} + c\mathbf{a})$. □

Note that the maximal dimension of $\mathcal{R}(n,m)$ is $(n/m) + m + 3$, and the maximal dimension of $\mathcal{R}(n,n)$ is $n+4$, and the maximal dimension of $\mathcal{R}(n,\cdot)$ is $n+3$.

Example 1.

$$\begin{aligned}
\mathcal{R}(8,8) &= \mathbb{C}^* \times R(8,\cdot) \\
&= \mathbb{C}^* \times (C_4 \times C_2 \coprod C_2 \times C_2 \times C_2 \coprod C_2 \times C_4 \coprod C_8) \times \mathcal{R}(1,\cdot), \\
\mathcal{R}(8,4) &= (C_2 \times C_2 \times C_2 \coprod C_2 \times C_4) \times \mathcal{R}(1,\cdot), \\
\mathcal{R}^\circ(8) &= \mathbb{C} \times (C_4 \times C_2 \coprod C_2 \times C_2 \times C_2 \coprod C_2 \times C_4 \coprod C_8) \times \mathcal{R}(1,\cdot).
\end{aligned}$$

Note that $\mathcal{R} \subset \mathrm{Mor}(\mathbb{C}^1, \mathbb{C}^2)$ has an action by $\mathrm{Aff}(\mathbb{C}^2) = GL_2(\mathbb{C}) \ltimes \mathbb{C}^2$ induced from its action of affine automorphisms of the target $\mathbb{C}^2$. Since $\mathrm{Aff}(\mathbb{C}^2)$ is a connected algebraic group and $\mathcal{R}^\circ$ is a disjoint union of components of types, we obtain the following definition.

Definition 5. An element of $\mathcal{R}^\circ_{\mathbf{k}}$ is called a parametrized affine embedding of line of type $\mathbf{k}$. In general we define the type of a closed parametrized affine embedding of the line $\mathbb{A}^1_{\mathbb{C}}$ to the affine plane $\mathbb{A}^2_{\mathbb{C}}$ after choosing appropriate affine coordinates of $\mathbb{A}^1_{\mathbb{C}}$ and $\mathbb{A}^2_{\mathbb{C}}$.

3. Applications

3.1. Parameter spaces of (unparametrized affine) embeddings of line

Given $\varphi(t) : \mathbb{C}^1 \to \mathbb{C}^2$, consider its unique projectivization $\bar{\varphi}$:

$$\begin{array}{ccc}
\varphi(t) : \mathbb{C}^1 & \to & \mathbb{C}^2 \\
\cap & & \cap \\
\bar{\varphi} : \mathbb{P}^1 & \to & \mathbb{P}^2.
\end{array}$$

In terms of homogeneous coordinates of $\mathbb{P}^1$, the projectivization $\bar{\varphi}$ can be written as

$$\bar{\varphi}([t:s]) = [a_0t^n + a_1t^{n-1}s + \cdots + a_ns^n : b_0t^ms^{n-m} + b_1t^{m-1}s^{n-m+1} + \cdots + b_ms^n : s^n],$$

where $\varphi(t) = (a_0t^n + a_1t^{n-1} + \cdots + a_n, b_0t^m + b_1t^{m-1} + \cdots + b_m)$.

This rational projective curve of degree n meets the line at infinity exactly at one point, which is $[a_0 : b_0 : 0]$ if $n = m$, $[1 : 0 : 0]$ if $n > m$. It is obvious that this point is the unique singular point of the given curve, if there is any. Moreover the line l_∞ at infinity is a unique tangent line at this singular point of the curve. The intersection multiplicity $C \cdot l_\infty$ is n. Let us denote that cuspidal singularity of plane curves defined to be irreducible singular point. Also unicuspidal curve meant to be a curve having only one singular point.

Definition 6. Denote by $\mathcal{M}(n) \subset \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(n))$ the space of unicuspidal rational curves of degree n with maximal tangent line.

Example 2 ([8])**.** There is a rational plane curve with one cusp and the unique tangent line at the cuspidal point.

In $\mathbb{P}^2$, consider the plane curve C defined by the equation

$$(x^2 - yz)^2 - zy^3 = 0.$$

By a parametrization, we see that it is a rational curve having only one singular point $(0,0,1)$ with a unique tangent line:

$$x = t(t^2-1), \ y = (t^2-1)^2, \ z = 1.$$

However the curve $C \setminus (0,0,1)$ is not isomorphic to $\mathbb{C}$: Otherwise, there is a regular map $\phi : \mathbb{P}^1 \setminus \{\pm 1\} \to \mathbb{C} \subset \mathbb{P}^1$. Now considering the degree of the extension $\overline{\phi} : \mathbb{P}^1 \to \mathbb{P}^1$, we obtain a contradiction.

Example 3. There is a rational unicuspidal quintic not having maximal tangent line [6, 20]. The equation is the following:

$$(y - x^2)(y - x^2 - 2xy) + y^5 = 0.$$

The affine plane curve given by the parametrization $\varphi(t) : \mathbb{C}^1 \to \mathbb{C}^2$ has the defining equation

$$h(x,y) := \mathrm{Res}_t(f(t) - x, g(t) - y) \in \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)),$$

where $\mathrm{Res}_t(f(t) - x, g(t) - y)$ is the resultant of $f(t) - x$ and $g(t) - y$ with respect to t (c.f. Theorem 0.4 of [19]). Define $\mathcal{A}(n,m)$ by

$$\mathcal{A}(n,m) = \{h(x,y) = \mathrm{Res}_t(f(t) - x, g(t) - y) | (f,g) \in \mathcal{R}(n,m)\},$$

which is the locus of the curves corresponding to $\mathcal{R}(n,m)$. We also define $\mathcal{A}^\circ_{\mathbf{k}}$ to be the locus corresponding to $\mathcal{R}^\circ_{\mathbf{k}}$. The locus $\mathcal{A}(n,m)$ is viewed as a subset of $\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n))$. Let $\mathcal{M}_{\mathbf{k}}$ be the subset of $\mathcal{M}(n)$ consisting of those that can be transformed to elements of $\mathcal{A}^\circ_{\mathbf{k}}$ by an automorphism of $\mathbb{P}^2$.

Theorem 3. *Each component $\mathcal{M}_{\mathbf{k}}$ is a rational variety of dimension $4+\sum k_i$ for $n \geq 3$.*

Proof. Note that there is a fibration $\pi : \mathcal{M}_{\mathbf{k}} \to F(3)$ where $F(3)$ is a flag manifold and π assigns to a curve the flag of its singular point and the tangent line at the singular point. The fibration has a rational section by the following construction. Near the flag

$$(\langle(0,1,0)\rangle, \langle(0,1,0),(1,0,0)\rangle)$$

of a point in $F(3)$, there is an affine chart of $(\alpha,\beta,\gamma) \in \mathbb{C}^3$ parametrizing flags $(\langle\alpha,1,\beta\rangle, \langle(\alpha,1,\beta),(1,0,\gamma)\rangle)$. Pick any $f(x,y,z) \in \mathcal{E}_{\mathbf{k}}$, then $f(x-\alpha, y, z-\gamma x+(\alpha\gamma-\beta)y)$ defines a rational section. Thus $\mathcal{M}_{\mathbf{k}}$ is a rational variety. □

Corollary 1. *$\mathcal{A}(n) = \coprod_{\mathbf{k}} \mathcal{A}_{\mathbf{k}}$ is a quasi projective subvariety, and each component $\mathcal{A}_{\mathbf{k}}$ is a rational variety of dimension $2+\sum k_i$ for $n \geq 2$.*

Example 4. Note that $\mathcal{A}(1)$ is $\mathbb{P}^2$ minus one point. In general $\mathcal{A}_n$ is an affine bundle over $\mathbb{P}^1$ with fiber $\mathbb{C}^* \times \mathbb{C}^n$ for $n \geq 2$.

3.2. The degree of $\overline{\mathcal{A}}(n)$

Lemma 4. *Let n be an nonnegative integer. The lattice volume (c.f. [7]) of the convex hull of the exponents of*

$$1, y^{k_0}, x_1, x_1y^{k_1}, x_2, x_2y^{k_2}, \ldots, x_n, x_ny^{k_n}$$

in $\mathbb{Z}^{n+1}$ is $\sum_{i=0}^n k_i$ if k_i are nonnegative integers not vanishing simultaneously.

Proof. When $n=0$, it is clear. Assume that $n>0$. By the Kouchnirenko theorem [16], it is equal to the number N of common zeros of generic $n+2$ equations $\sum a_\omega \mathbf{x}^\omega = 0$, where ω runs over all exponents in the set of monomials above.

Note that the equations can be rewritten as $Ax=0$, where

$$x = (1, x_1, \ldots, x_n)^{\text{transpose}}$$

and the entries a_{ij} of the matrix A satisfies $a_{ij} \in \mathbb{C} + \mathbb{C}y^{k_j}$ if $n \geq j \geq 0$. By taking the determinant of A, we conclude that the number N is less than or equal to the degree of the determinant of A in y. Thus $N \leq \sum_{i=1}^n k_i$. However we can achieve the upper bound by simply taking A as the maximal cyclic form

$$\begin{pmatrix} y^{k_0} & 0 & 0 & \cdots & 0 & -1 \\ -1 & y^{k_1} & 0 & & 0 & 0 \\ 0 & -1 & y^{k_2} & \cdots & & 0 \\ \vdots & & & \vdots & & \\ 0 & & & & -1 & y^{k_n} \end{pmatrix}$$

□

Theorem 4. *The degree of (the maximal dimension component of) $\overline{\mathcal{A}}(n)$ is $\frac{n(n+1)}{2} - 1$.*

Proof. Since we have

$$\mathbb{C} \times \mathcal{A}_{n,1}^{\circ} = \left\{ \sum_{i=0}^{n} a_i(y - \lambda x)^{n-i} - x \in \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(n)) \,|\, \mathbf{a} \in C_n, \lambda \in \mathbb{C} \right\},$$

it is enough to find the degree of the closure of the image

$$\phi : (\mathbb{C}^*)^{n+2} \to \mathbb{P}^{\binom{n+2}{2}-1}$$

defined by sending $(a_0, \ldots, a_n, \lambda)$ to $(a_0, a_0\lambda, \ldots, a_0\lambda^n, a_1, a_1\lambda, \ldots, a_1\lambda^{n-1}, \ldots, a_{n-2}, a_{n-2}\lambda, a_{n-2}\lambda^2, a_{n-1}, a_{n-1}\lambda - 1, a_n)$.

Let π be the composition $\pi' \circ A$ of the affine transformation

$$A(z_0, \ldots, z_N, z_{N+1}) \mapsto (z_0, \ldots, z_N - z_{N+1}, z_{N+1})$$

of $\mathbb{P}^{N+1}$ and the projection of $\mathbb{P}^{N+1}$ to $\mathbb{P}^N$ centered at $(0, \ldots, 0, 1)$ along the subspace $\mathbb{P}(\mathbb{C}^N \times \{0\})$. Consider

$$\psi : (\mathbb{C}^*)^{n+2} \to \mathbb{P}^N$$

sending $(a_0, \ldots, a_n, \lambda)$ to $(a_0, a_0\lambda, \ldots, a_0\lambda^n, a_1, a_1\lambda, \ldots, a_1\lambda^{n-1}, \ldots, a_{n-2}, a_{n-2}\lambda, a_{n-2}\lambda^2, a_{n-1}, a_{n-1}\lambda, a_n, 1)$.

Note that ψ is defined by monomials and $\phi = \pi \circ \psi$. Now applying Lemma 4, one notes that the degree of the closure of $\mathrm{Im}\psi$ is $\frac{n(n+1)}{2}$. Since this variety is defined by the obvious quadratic equations, it is easy to check that the center $(0, \ldots, 0, 1, 0, 1)$ is a nonsingular point, which, together with the injectivity of ϕ, implies that the degree of $\pi \circ \mathrm{Im}\psi$ is one less than the degree of $\mathrm{Im}\psi$. □

Another Proof of Theorem 4. Denote the general hyperplane $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n))$ by H_i. To compute the intersection number

$$H_1 \cdots H_{n+2} \cdot \overline{\mathcal{A}}(n),$$

we may assume that all the intersection points occur in

$$\mathbb{C} \times \mathcal{A}_{n,1}^{\circ} = \left\{ \sum_{i=0}^{n} a_i(y - \lambda x)^{n-i} - x \in \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(n)) | \mathbf{a} \in C_n, \lambda \in \mathbb{C} \right\}.$$

The equations can be expressed by a matrix equation $Ab = 0$ where A has entries a_{ij} that are polynomials in λ with degree k_j and $j = 0, 1, \ldots, n+1$. Note that the first column entries a_{i0} are constants, and b is the transpose of

$$(1, b_1, \ldots, b_{n+1}) := (1, a_n, a_{n-1}, \ldots, a_0).$$

There is one more constraint that the constant term a_{i0} equals the negative of the leading coefficient of a_{i1}. By considering the determinant of A, it is clear that there are at most $(\sum_{i=0}^{n+1} k_i) - 1$ solutions. However one can achieve the upper bound by taking the specialization of A to

$$\begin{pmatrix} 1 & -\lambda^{k_1} & 0 & \cdots & 0 & -1 \\ 0 & \lambda^{k_1-1} & 0 & & 0 & -1 \\ 0 & -1 & \lambda^{k_2} & \cdots & & 0 \\ \vdots & & & \vdots & & \\ 0 & & & & -1 & \lambda^{k_{n+1}} \end{pmatrix}.$$

□

The above theorem combined with Kleiman's transversality theorem [14, 5] applied to $(\mathbb{P}^2)^{n+2}$ with the group action $(\mathrm{GL}_2(\mathbb{C}))^{n+2}$ implies the following.

Corollary 2. *Let $n \geq 2$. There are exactly $(\frac{n(n+1)}{2} - 1)$ distinct plane curves in $\mathcal{A}(n)$, passing through general* $\dim\mathcal{A}(n)$ *points in* $\mathbb{P}^2$.

3.3. The degree of $\overline{\mathcal{M}}(3)$

Corollary 3 ([15, 2]). *Let $\mathcal{M}(3)$ be the space of cuspidal cubic curves. Then the* $\deg \overline{\mathcal{M}}(3) = 24$.

Let $(f(t), g(t))$ be a smooth polynomial parametrization of a cubic curve. Then its general form must be $(a_0t^3 + a_1t^2 + a_2t + a_3, t)$. And their defining equations are parametrized as follows:

$$a_0y^3 + a_1y^2 + a_2y + a_3 = x \Rightarrow a_0y^3 + a_1y^2z + a_2yz^2 + a_3z^3 = xz^2.$$

The defining equations near singular point $[1 : 0 : 0]$ are

$$a_0y^3 + a_1y^2z + a_2yz^2 + a_3z^3 - z^2 = f(y, z).$$

Now we consider $\mathrm{PGL}(3, \mathbb{C})$ action on them. The local action changes the singular point and the unique singular tangent:

$$f(y, z) \Rightarrow f(y + b_1, \lambda y + z + b_2).$$

Hence we have local parametrization of cuspidal cubics as follows:

$$\begin{aligned} &a_0(y + b_1)^3 + a_1(y + b_1)^2(\lambda y + z + b_2) + a_2(y + b_1)(\lambda y + z + b_2)^2 \\ &+a_3(\lambda y + z + b_2)^3 - (\lambda y + z + b_2)^2 = 0. \end{aligned}$$

We can use "deg lex order" $y > z$ with descending order.

$$\begin{gathered} (a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3)y^3 \\ +(a_1 + 2a_2\lambda + 3a_3\lambda^2)y^2z \\ +(a_2 + 3a_3\lambda)yz^2 \\ +a_3z^3 \\ +(3a_0b_1 + a_1b_2 + 2a_1b_1\lambda + 2a_2b_2\lambda + a_2b_1\lambda^2 + 3a_3b_2\lambda^2 - \lambda^2)y^2 \\ +(2a_1b_1 + 2a_2b_2 + 2a_2b_1\lambda + 6a_3b_2\lambda - 2\lambda)yz \\ +(a_2b_1 + 3a_3b_2 - 1)z^2 \\ +(3a_0b_1^2 + a_1b_1^2\lambda + 2a_1b_1b_2 + a_2b_2^2 + 2a_2b_1b_2\lambda + 3a_3b_2^2\lambda - 2b_2\lambda)y \\ +(a_1b_1^2 + 2a_2b_1b_2 + 3a_3b_2^2 - 2b_2)z \end{gathered}$$

$$+(a_0b_1^3 + a_1b_1^2b_2 + a_2b_1b_2^2 + a_3b_2^3 - b_2^2).$$

Define polynomials $P_{i,j}(a_0, a_1, a_2, a_3, b_1, b_2, \lambda)$ by

$$f(y + b_1, \lambda y + z + b_2) = \sum_{i+j\leq 3} P_{i,j}(a_0, a_1, a_2, a_3, b_1, b_2, \lambda)y^iz^j.$$

Hence the projective closure, $\overline{\mathrm{Im}(\Phi)}$, of the image of the map

$$\Phi : \mathbb{C}^* \times \mathbb{C}^6 \to \mathbb{C}^9 \subset \mathbb{C}P^9$$

is the parameter space of cuspidal cubic curves in $\mathbb{C}P^2$. It is an explicit rational parametrization of the parameter space of cuspidal cubic curves.

An explicit computation with the computer algebra package *Singular* [10] shows that the degree of $\overline{\mathrm{Im}(\Phi)}$ is 24. Moreover another numerical invariant which counts the number of cuspidal cubics with a fixed singular point can be computed by computing the degree of the subvariety $\overline{\mathrm{Im}(\Phi(b_1 = b_2 = 0))}$. A computation shows that this number is 2. We also have checked the degree of the subvariety $\overline{\mathrm{Im}(\Phi(b_1 = 0)}$. This number counts the cuspidal cubics whose singular points lie on a fixed line. Our computation shows it is 12.

Remark 2. (1) These numerical invariants have been known for years [18, 15, 2, 11, 12], going all the way back to a very old Enriques' formula.

(2) After writing up this paper, we learned that the degree of $\mathcal{A}(d)$ had been independently obtained in a recent paper [13] as an example. The approaches, however, appear to be substantially different.

(3) Note that our method produces not only the degree of the parameter space $\mathcal{A}(d)$ but also an explicit parametrization of $\mathcal{A}(d)$ itself.

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