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SOME CONSTRUCTIONS OF IMPLICATIVE/COMMUTATIVE *d*-ALGEBRAS

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Abstract. In this paper, we give some constructions of implicative/commutative d-algebras which are not BCK-algebras. This demonstrate that the notion of implicative/commutative d-algebras are indeed generalizations of the same in BCK-algebras.

1. Preliminaries

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([5, 6]). BCK-algebras have some connections with other areas: D. Mundici [10] proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras, and J. Meng [8] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. Z. Riečanová [14] showed that extendable commutative BCK-algebras directed upwards are equivalent to generalized MV-effect algebras. G. Georgescu and A. Iorgulescu [2] introduced the notion of pseudo-BCK algebras as an extension of BCK-algebras. X. H. Zhang and W. H. Li [15] established the connections between BCC-algebras, pseudo-BCK algebras, pseudo-BL algebras and weak pseudo-BL algebras (pseudo-MTL algebras). J. Neggers and H. S. Kim introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and then investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs [12]. After that some further aspects were studied ([7, 11, 13]). J. S. Han et al. [3] defined a variety of special d-algebras, such as strong d-algebras, (weakly) selective d-algebras and others. The main assertion is that the squared algebra $(X;\square,0)$ of a d-algebra is a d-algebra if and only if the root $(X;*,0)$ of the squared algebra $(X;\square,0)$ is a strong d-algebra.

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implicative/commutative d-algebras are indeed generalizations of the same in BCK-algebras.

2. Introduction

An *(ordinary) d-algebra* ([12, 13]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(A) $x * x = 0$,
(B) $0 * x = 0$,
(C) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A BCK-algebra is a d-algebra X satisfying the following additional axioms:

(D) $(x * y) * (x * z)) * (z * y) = 0$,
(E) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

Example 2.1 ([3]). Consider the real numbers $\mathbb{R}$, and suppose that $(\mathbb{R}; *, e)$ has the multiplication

$$x * y = (x - y)(x - e) + e.$$

Then $x*x = e; e*x = e; x*y = y*x = e$ yields $(x-y)(x-e) = 0$, $(y-x)(y-e) = e$ and $x = y$ or $x = e = y$, i.e., $x = y$, i.e., $(\mathbb{R}; *, e)$ is a d-algebra.

Theorem 2.2 ([4, p. 162]). *Let X be a set with $0 \in X$. If we define a binary operation $*$ on X by*

$$x * y := \begin{cases} 0 & \text{if } x = y, \\ x & \text{if } x \neq y, \end{cases}$$

then $(X, *, 0)$ is an implicative BCK-algebra.

3. Commutative d-algebra

Definition 3.1. A field $(X, +, \cdot)$ is called $\sqrt{3}$-*exponential* if there is a function $\varphi : X \to X$ such that

(E1) $\varphi(\varphi(x)) = x^3$,
(E2) $\varphi(xy) = \varphi(x)\varphi(y)$,
(E3) if $x \neq 0$, then $\varphi(x) \neq 0$,
(E4) $\varphi(0) = 0$

for any $x, y \in X$.

Example 3.2. Let $X := \mathbf{R}$ be the set of all real numbers. If we define a map $\varphi : X \to X$ by

$$\varphi(x) := \begin{cases} x^{\sqrt{3}} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -y^{\sqrt{3}} & \text{if } x = -y < 0, \end{cases}$$

then $(\mathbf{R}, +, \cdot)$ is $\sqrt{3}$-exponential.

Proposition 3.3. *Let $(X, +, \cdot)$ be a $\sqrt{3}$-exponential field. If we define a new binary operation $\star$ on X by $x \star y := x^2\varphi(y)y$ for any $x, y \in X$, then $x \star (x \star y) = y \star (y \star x)$ for any $x, y \in X$.*

Proof. Given $x, y \in X$, we have

$$\begin{aligned} x \star (x \star y) &= x^2\varphi(x \star y)(x \star y) \\ &= x^2\varphi(x^2\varphi(y)y)(x^2\varphi(y)y) \\ &= x^4y^4\varphi(x)^2\varphi(y)^2. \end{aligned}$$

Similarly, we obtain $y \star (y \star x) = y^4x^4\varphi(y)^2\varphi(x)^2$, proving the proposition. □

Using the notion of $\sqrt{3}$-exponential field, we construct a commutative d-algebra which is not a BCK-algebra.

Theorem 3.4. *Let $(X, +, \cdot)$ be a $\sqrt{3}$-exponential field and let $x \star y := x^2\varphi(y)y$ for any $x, y \in X$. If we define a binary operation "$*$" on X by*

$$x * y := \begin{cases} 0 & \text{if } x = 0 \text{ or } x = y, \\ x & \text{if } y = 0, \\ x \star y & \text{otherwise,} \end{cases}$$

then $(X, *, 0)$ is a commutative d-algebra.

Proof. Let $x * y = y * x = 0$. If $x = 0$ or $y = 0$, then it is easy to see that $x = y$. If we assume that $xy \neq 0$ and $x \neq y$, then $x^2\varphi(y)y = y^2\varphi(x)x = 0$, which leads to $\varphi(x) = \varphi(y) = 0$. By (E3) we obtain $x = y$, a contradiction. Hence $(X, *, 0)$ is a d-algebra.

We claim that $(X, *)$ is commutative. If $xy \neq 0$ and $x \neq y$, then $x * (x * y) = x \star (x \star y) = y \star (y \star x) = y * (y * x)$ by Proposition 3.3. The other cases are trivial. This proves the theorem. □

Note that the commutative d-algebra $(X, *, 0)$ described in Theorem 3.4 need not be a BCK-algebra, as in Example 3.2, where $(2 * (2 * 1)) * 1 = 2^8(2^{\sqrt{3}})^4 = 2^{8+4\sqrt{3}} \neq 0$. Moreover, it is not implicative, since $x * (y * x) = x^{6+2\sqrt{3}}y^{2\sqrt{3}} \neq x$.

We give another method for finding commutative d-algebras which are not BCK-algebras.

4. (Positive-)Implicative d-algebras

Proposition 4.1. *Let X be a field and let $x, y \in X$. If we define*

$$x * y := x(x - y)\varphi(x, y) \tag{1}$$

where $\varphi : X \times X \to X$ is a function with $\varphi(x, y) \neq 0$ for any $x, y \in X$. Then $(X, *, 0)$ is a d-algebra.

Proof. If we assume that $x * y = y * x = 0$, then $x(x-y)\varphi(x,y) = 0$ and $y(y-x)\varphi(y,x) = 0$ and hence $x(x-y) = 0 = y(y-x)$. This leads to $x = y$, since $x - y \neq 0$ implies $x = 0, y = 0$, i.e., $x = y$, a contradiction. Hence $(X, *, 0)$ is a d-algebra. □

The d-algebra $(X, *, 0)$ described in Proposition 4.1 is called a *φ-function d-algebra.*

A d/BCK-algebra $(X, *, 0)$ is said to be *implicative* ([2, 10]) if $x = x*(y*x)$ for any $x, y \in X$.

Proposition 4.2. *If $(X, *, 0)$ is an implicative d-algebra, then $x * 0 = x$ for any $x \in X$.*

Proof. If X is implicative, then $x = x*(y*x)$ for any $x, y \in X$. If we let $y := x$, then $x = x * (x * x) = x * 0$, proving the proposition. □

Proposition 4.3. *Let $(X, *, 0)$ be a φ-function d-algebra. Then $(X, *, 0)$ is implicative if and only if φ satisfies the condition:*

$$\varphi(x, y * x) := \begin{cases} \frac{1}{x - y*x} & \text{if } x \neq 0, \\ a & \text{otherwise,} \end{cases}$$

where a is an arbitrary element of X.

Proof. Straightforward. □

Note that if $x \neq 0$, then $x \neq y * x$ in Proposition 4.3. A d/BCK-algebra $(X, *, 0)$ is said to be *positive implicative* ([2, 10]) if $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$.

Proposition 4.4. *There are no positive implicative φ-function d-algebras which are not BCK-algebras.*

Proof. Assume that the implicative φ-function d-algebra $(X, *, 0)$ which is not a BCK-algebra is positive implicative. Then $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$. If we let $z := x$, then $(x * y) * x = (x * x) * (y * x) = 0 * (y * x) = 0$, i.e., $(x * y) * x = 0$. Since $(X, *, 0)$ is a φ-function d-algebra, we have $0 = (x * y)[(x * y) - x]\varphi(x * y, x)$. Since $\varphi(x, y) \neq 0, \forall x, y \in X$, we obtain $0 = (x * y)[x * y - x]$. Therefore, either $x * y = 0$ or $x * y = x$, i.e., $x * y \in \{0, x\}, \forall x, y \in X$. Assume that there are $x, y \in X$ such that $x \neq 0, x \neq y$ and $x * y = 0$. Then $0 = x * y = x(x - y)\varphi(x, y) \neq 0$, a contradiction. Hence we have $x * y = 0$ if $x = y$ and $x * y = x$ if $x \neq y$, i.e., $(X, *, 0)$ is an implicative BCK-algebra by Theorem 2.2, a contradiction. □

Theorem 4.5 ([9]). *A BCK-algebra X is positive implicative if and only if $(x * y) * y = x * y$ for any $x, y \in X$.*

Theorem 4.6. *If the φ-function d-algebra $(X, *, 0)$ is implicative, then $(x * y) * y = x * y$ for any $x, y \in X$.*

Proof. Let the φ-function d-algebra $(X, *, 0)$ be implicative. Then we have $x = x(x - y * x)\varphi(x, y * x)$ for any $x, y \in X$. Assume that $x \neq 0$, since $x = 0$ implies $(0 * y) * z = 0 = (0 * z) * (y * z)$. Then we have $1 = (x - y * x)\varphi(x, y * x)$. Hence, $\varphi(x, y * x) = \frac{1}{(x-y*x)}$. Also, $y * x \neq 0$ and $y * x \neq x * (y * x)$. Then $\varphi(y*x, x) = \varphi(y*x, x*(y*x)) = \frac{1}{y*x-x*(y*x)} = \frac{1}{y*x-x} = -\frac{1}{x-y*x} = -\varphi(x, y*x)$, i.e., we obtain

$$\varphi(y * x, x) = -\varphi(x, y * x). \tag{2}$$

Given $x, y \in X$, we have

$$\begin{aligned} (y * x) * x &= (y * x)(y * x - x)\varphi(y * x, x) \\ &= (y * x)(y * x - x)[-\varphi(x, y * x)] \qquad \text{by (2)} \\ &= (y * x)(x - y * x)\varphi(x, y * x). \end{aligned}$$

Since $x = x * (y * x) = x(x - y * x)\varphi(x, y * x)$, we have

$$\begin{aligned} x - (y * x) * x &= (y * x - x)^2\varphi(x, y * x) \\ &= (y * x - x)^2 \frac{1}{x - y * x} \\ &= x - y * x, \end{aligned}$$

proving the theorem. □

Note that in BCK-algebras, the condition $(x * y) * (x * z) = (x * y) * x$ is equivalent to the condition $(x*y)*y = x*y$, but it is not equivalent in d-algebras in general. This can be demonstrated by Theorem 4.6 and Example 4.8.

Example 4.7. If we define a map $\varphi : X \to X$ by

$$\varphi(x, y) := \begin{cases} \frac{1}{x-y} & \text{if } x(x - y) \neq 0, \\ a & \text{if } x = y, \\ b & \text{if } x = 0, \end{cases}$$

then the function φ satisfies the conditions of Proposition 3.2, and so it defines a φ-function d-algebra $(X, *, 0)$ where

$$x * y := \begin{cases} x & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

which is an implicative BCK-algebra as described in Theorem 2.2.

We need to find an implicative d-algebra which is not a BCK-algebra. Consider the following example.

Example 4.8. If we define a map φ on X by

$$\varphi(x, y) := \begin{cases} \frac{-y}{x(y-x)} & \text{if } y(y - x) \neq 0, \\ a & \text{otherwise} \end{cases}$$

for an arbitrary element a in X, then

$$x * y := \begin{cases} -y & \text{if } y(y-x) \neq 0, \\ 0 & \text{if } x = 0 \text{ or } x = y, \\ x & \text{if } y = 0 \end{cases}$$

leads to a d-algebra. If $y(y-x) \neq 0$, then $x * (y * x) = x * (-x) = x$ for any $x, y \in X$, showing that $(X, *, 0)$ is an implicative d-algebra. Indeed, it is not a BCK-algebra, since $((3 * 4) * (3 * 5)) * (5 * 4) = 4 \neq 0$.

Example 4.9. If we apply Example 4.8 to the finite field $\mathbb{Z}_5$, then we obtain the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	3	2	1
2	2	4	0	2	1
3	3	4	3	0	1
4	4	4	3	2	0

Then it is an implicative d-algebra, which is not a BCK-algebra, since $((3 * 4) * (3 * 2)) * (2 * 4) = 4 \neq 0$. Moreover, it is not positive implicative, since $(3 * 4) * 5 = -4 * 5 = -5$ and $(3 * 5) * (4 * 5) = -5 * -5 = 5$.

Remark. In BCK-algebras, X is an implicative BCK-algebra if and only if it is both a positive implicative and a commutative BCK-algebra. But this does not hold in d-algebras. See Example 4.9.

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