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ON WEAK ARMENDARIZ RINGS

Young Cheol Jeon, Hong Kee Kim, Yang Lee, and Jung Sook Yoon

Abstract. In the present note we study the properties of weak Armendariz rings, and the connections among weak Armendariz rings, Armendariz rings, reduced rings and IFP rings. We prove that a right Ore ring R is weak Armendariz if and only if so is Q, where Q is the classical right quotient ring of R. With the help of this result we can show that a semiprime right Goldie ring R is weak Armendariz if and only if R is Armendariz if and only if R is reduced if and only if R is IFP if and only if Q is a finite direct product of division rings, obtaining a simpler proof of Lee and Wong's result. In the process we construct a semiprime ring extension that is infinite dimensional, from given any semiprime ring. We next find more examples of weak Armendariz rings.

1. Introduction

Throughout this note each ring is associative with identity unless otherwise stated. Given a ring R, the polynomial ring with an indeterminate x over R is denoted by $R[x]$. Due to Rege and Chhawchharia [15], a ring R is called *Armendariz* if for given $f(x) = a_0+a_1x+\cdots+a_mx^m$ and $g(x) = b_0+b_1x+\cdots+b_nx^n \in R[x]$, $f(x)g(x) = 0$ implies that $a_ib_j = 0$ for each i, j (the converse is obviously true). Due to Lee and Wong [12], a ring R is called *weak Armendariz* if for given $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x \in R[x]$, $f(x)g(x) = 0$ implies that $a_ib_j = 0$ for each i, j (the converse is obviously true). It is obvious that Armendariz rings are weak Armendariz and that subrings of (weak) Armendariz rings are still (weak) Armendariz. There is a weak Armendariz ring but not Armendariz by [12, Example 3.2]. The structure of (weak) Armendariz rings was also observed by Anderson and Camillo [2], containing the relations between closely related rings.

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A ring is called *reduced* if it has no nonzero nilpotent elements. Reduced rings are Armendariz by [3, Lemma 1]. A ring is called *abelian* if every idempotent is central. Weak Armendariz rings are abelian by [12, Lemma 3.4(3)].

Due to Bell [4], a right (or left) ideal I of a ring R is said to have the *insertion-of-factors-property* (simply, *IFP*) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. So a ring R is called *IFP* if the zero ideal of R has the IFP. Shin [16] used the term *SI* for the IFP, while Narbonne [14] called IFP rings *semicommutative.* Simple computations give that reduced rings are IFP and IFP rings are abelian. Subrings of IFP rings are also IFP obviously. Note that a ring R is IFP if and only if any right annihilator is an ideal if and only if any left annihilator is an ideal if and only if $ab = 0$ implies $aRb = 0$ for $a, b \in R$ [16, Lemma 1.2].

We summarize preliminary facts in the following.

Lemma 1.1. (1) *Armendariz rings are weak Armendariz.*

(2) *The class of (weak) Armendariz rings is closed under subrings and direct products.*

(3) *Reduced rings are Armendariz.*

(4) *Reduced rings are IFP.*

(5) *IFP rings are abelian.*

(6) *Weak Armendariz rings are abelian.*

Proof. The proofs of (1) and (2) are trivial. The proofs of (3) and (6) are done by [3, Lemma 1] and [12, Lemma 3.4(3)], respectively.

(4) Let R be a reduced ring and $ab = 0$ for $a, b \in R$. For any $r \in R$ we have $bar = 0 \Rightarrow arb = 0$ from $baba = 0 \Rightarrow ba = 0$. Thus R is IFP.

(5) Let R be an IFP ring and $0 \neq e = e^2 \in R$. Then $eR(1-e) = 0 = (1-e)Re$. So for each $r \in R$, $er(1-e) = 0 = (1-e)re$ implies that e is central. So R is abelian. □

In the following we note that the converses of (1), (3), (4), (5) and (6) need not hold, and that the classes of (weak) Armendariz rings and IFP rings do not contain each other.

Example 1.2. (1) Let $R = \mathbb{Z}_3[x,y]/(x^3, x^2y^2, y^3)$, where $\mathbb{Z}_3$ is the Galois field of order 3, $\mathbb{Z}_3[x,y]$ is the polynomial ring with two indeterminates x, y over $\mathbb{Z}_3$, and (x^3, x^2y^2, y^3) is the ideal of $\mathbb{Z}_3[x,y]$ generated by x^3, x^2y^2, y^3. Let $R[t]$ be the polynomial ring with an indeterminate t over R. Since $(\bar{x} + \bar{y}t)^3 = (\bar{x} + \bar{y}t)(\bar{x}^2 + 2\bar{x}\bar{y}t + \bar{y}^2t^2) = 0$ with $\bar{x}\bar{y}^2 \neq 0$, R is not Armendariz. But R is weak Armendariz by [12, Example 3.2].

(2) Let R be a reduced ring. Then

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

is IFP and Armendariz by [10, Proposition 1.2] and [9, Proposition 2], respectively.

(3) Let F be a field and $A = F[a, b, c]$ be the free algebra of polynomials with zero constant terms in noncommuting indeterminates a, b, c over F. Note that A is a ring without identity and consider an ideal of $F + A$, say I, generated by cc, ac and crc for all $r \in A$. Let $R = (F + A)/I$. First notice that R is not IFP because $ac \in I$ but $abc \notin I$ (hence $(a+I)(c+I) = 0$ but $(a+I)(b+I)(c+I) \neq 0$ in R). However R is an Armendariz ring by the assertion in [7, Example 14].

(4) Commutative rings (hence IFP) need not be weak Armendariz. According to [15, Example 3.2], let $\mathbb{Z}_8$ be the ring of integers modulo 8 and $R = T(\mathbb{Z}_8, \mathbb{Z}_8)$ be the trivial extension of $\mathbb{Z}_8$. Consider the polynomial $f(x) = (4, 0) + (4, 1)x$ over R. The square of $f(x)$ is zero but the product $(4, 0)(4, 1) = (0, 4)$ is not zero. Thus R is not weak Armendariz.

(5) Let S be an abelian ring and $R = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in S \right\}$. Then R is abelian by [6, Lemma 2]. Due to [10, Example 1.3], consider

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0$$

and

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0.$$

Then R is not IFP.

(6) Let R be the abelian ring as in (5). Due to [9, Example 3], consider

$$f(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x,$$

$$g(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$$

in $R[x]$. Then $f(x)g(x) = 0$, but $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$. So R is not weak Armendariz.

2. Weak Armendariz rings and related rings

In this section we continue the study of (weak) Armendariz rings, concentrating on the conditions under which weak Armendariz rings, Armendariz rings, reduced rings and IFP rings are equivalent.

The prime radical of a ring is the set of all strongly nilpotent elements by [11, Proposition 3.2.1]. A ring is called *semiprime* if the prime radical is zero.

Lemma 2.1. *For a semiprime ring R the following conditions are equivalent:*
(1) *R is weak Armendariz;*
(2) *If $a, b, c \in R$ is such that $ac = 0 = b^n$ with $n \geq 1$, then $abc = 0$;*
(3) *If $a, b, c \in R$ is such that $ac = 0 = b^2$, then $abc = 0$.*

Proof. (1)⇒(2) is proved by [12, Lemma 3.9], and (2)⇒(3) is trivial. (3)⇒(1) is proved by [12, Remark 3.5]. □

If a ring R satisfies the condition (3) in Lemma 2.1, then R is abelian by [12, Lemma 3.4(3)]. So it is natural to conjecture that abelian semiprime rings are weak Armendariz. But the answer is negative as we see in the following arguments.

Denote by U_n the 2^n by 2^n upper triangular matrix ring over a ring S, where n is a positive integer. Define a ring extension of S, that is a subring of U_n,

$$D_n = \{M \in U_n \mid \text{the diagonal entries of } M \text{ are equal}\}.$$

Theorem 2.2. (1) *Let S be a semiprime ring. Define a map $\sigma : U_n \to U_{n+1}$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$, then U_n can be considered as a subring of U_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in U_n$). Set R be the direct limit of the direct system (U_n, σ_{ij}) with $\sigma_{ij} = \sigma^{j-i}$. Then R is a semiprime ring.*

(2) *Let S be a semiprime ring. Define a map $\sigma : D_n \to D_{n+1}$ by $B \mapsto \left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right)$, then D_n can be considered as a subring of D_{n+1} via σ (i.e., $B = \sigma(B)$ for $B \in D_n$). Set R be the direct limit of the direct system (D_n, σ_{ij}), where $\sigma_{ij} = \sigma^{j-i}$. Then R is a semiprime ring.*

Proof. (1) First note $R = \bigcup_{n=1}^{\infty} U_n$, via $\sigma : U_n \hookrightarrow U_{n+1}$. Let $0 \neq A \in R$. Then $A = (a_{st}) \in U_n$ for some n. Set i be smallest such that the i-th row of A contains a nonzero entry, and j be smallest such that $a_{ij} \neq 0$ in the i-th row. Put $a = a_{ij}$. Since S is semiprime, there is a non-stationary sequence $(a_0, a_1, \ldots, a_y, \ldots)$ such that $a_0 = a$ and $a_y = a_{y-1}s_{y-1}a_{y-1}$ for some $s_{y-1} \in S$, where $y = 1, 2, \ldots$. Use e_{uv} to denote the square matrix in which (u, v)-entry is 1 and zero elsewhere.

Suppose that the diagonal of A is nonzero, say $a_{ii} = a$. In this case we compute in U_n. Let $A_0 = A$ and $A_1 = A_0(s_0e_{ii})A_0 \in A_0RA_0$, then the (i, i)-entry of A_1 is $a_0s_0a_0 = a_1 \neq 0$. Next let $A_2 = A_1(s_1e_{ii})A_1 \in A_1RA_1$, then the (i, i)-entry of A_2 is $a_1s_1a_1 = a_2 \neq 0$. Proceeding in this manner, we obtain that the (i, i)-entry of A_k is $a_{k-1}s_{k-1}a_{k-1} = a_k \neq 0$ for any k. Thus we can obtain inductively a non-stationary sequence (A_k) such that $A_0 = A$ and $A_{k+1} \in A_kRA_k$ for $k = 0, 1, \ldots$.

Suppose that the diagonal of A is zero. Then $i < j$ and $(i + 2^k, j + 2^k)$-entry of A in U_{k+1} is also a for $k = n, n + 1, n + 2, \ldots$. Let $A_0 = A$ and $A_1 = A_0(s_0B_0)A_0 \in A_0RA_0$, where A_0 is considered in R_{n+1} and $B_0 = e_{j(i+2^n)} \in U_{n+1}$. Say $A_1 = (b_{st})$. Then i is smallest such that the i-th row of A_1 contains a nonzero entry and $j+2^n$ is smallest such that $b_{i(j+2^n)} = a_0s_0a_0 = a_1 \neq 0$ in the i-th row. Next let $A_2 = A_1(s_1B_1)A_1 \in A_1RA_1$, where $B_1 = e_{(j+2^n)(i+2^{n+1})} \in U_{n+2}$. Say $A_2 = (c_{st})$. Then i is smallest such that the i-th row of A_2 contains a nonzero entry and $j+2^n+2^{n+1}$ is smallest such that $b_{i(j+2^n+2^{n+1})} = a_1s_1a_1 = a_2 \neq 0$ in the i-th row. Proceeding in this manner, we obtain that the $(i, j + 2^n + 2^{n+1} + \cdots + 2^{n+(k-1)})$-entry of A_k is $a_{k-1}s_{k-1}a_{k-1} = a_k \neq 0$ for any k. Thus we can obtain inductively a non-stationary sequence (A_k) such that $A_0 = A$ and $A_{k+1} \in A_kRA_k$ for $k = 0, 1, \ldots$.

Therefore A is not strongly nilpotent, concluding that R is semiprime.

(2) The proof is similar to (1). □

Note that the ring R in Theorem 2.2(2) is infinite dimensional and non-reduced. With the help of Theorem 2.2 there is a semiprime abelian ring that is not weak Armendariz.

Example 2.3. Let S be a reduced ring and consider the direct limit R over S as in Theorem 2.2(2). Then R is semiprime by Theorem 2.2 since reduced rings are semiprime, but R is not weak Armendariz by the same computation as in Example 1.2(6).

Reduced rings are abelian by Lemma 1.1, and so every D_n is abelian by [6, Lemma 2] such that every idempotent in D_n is of the form

$$\begin{pmatrix} f & 0 & 0 & \cdots & 0 \\ 0 & f & 0 & \cdots & 0 \\ 0 & 0 & f & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f \end{pmatrix}$$

with $f^2 = f \in S$. Thus R is abelian.

Armendariz rings are weak Armendariz but the converse need not be true by [12, Example 3.2]. In the remainder of this section we study when weak Armendarizness and related concepts are equivalent.

A ring R is called *von Neumann regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$. von Neumann regular rings are semiprime by [5, Corollary 1.2]. A ring R is called *abelian regular* if R is von Neumann regular and abelian. A ring is called *right (left) duo* if each right (left) ideal is two-sided. A prime ideal P of a ring R is called *completely prime* if R/P is a domain. Our conditions in this note coincide when given rings are von Neumann regular as follows. The following extends [2, Theorem 6].

Lemma 2.4. *Let R be a von Neumann regular ring. Then the following conditions are equivalent:*

(1) *R is right (left) duo;*

(2) *R is reduced;*

(3) *R is IFP;*

(4) *R is Armendariz;*
(5) *R is weak Armendariz;*
(6) *If $a, b, c \in R$ is such that $ac = 0 = b^n$ with $n \geq 1$, then $abc = 0$;*
(7) *If $a, b, c \in R$ is such that $ac = 0 = b^2$, then $abc = 0$;*
(8) *R is an abelian ring;*
(9) *R is a subdirect product of division rings.*

Proof. (3)⇒(9): Let R be IFP. Then the prime radical of R contains all nilpotent elements by [16, Theorem 1.5]. But von Neumann regular rings are semiprime by [5, Corollary 1.2], and so each minimal prime ideal of R is completely prime by [16, Proposition 1.11]. Thus $R \cong R/0$ is a subdirect product of domains, since the prime radical of R is zero. But each factor ring of R is also von Neumann regular; hence if P is a minimal prime ideal of R, then R/P must be a division ring, since regular elements of von Neumann regular rings are invertible.

(9)⇒(2) is obvious. (2)⇒(3), (2)⇒(4), (4)⇒(5) and (5)⇒(8) are obtained from Lemma 1.1.

The conditions (1), (2) and (8) are equivalent by [5, Theorem 3.2]. The conditions (5), (6) and (7) are equivalent by Lemma 2.1. □

A ring R is called *π-regular* if for each $a \in R$ there exist a positive integer n, depending on a, and $b \in R$ such that $a^n = a^n b a^n$. It is easy to show that the Jacobson radical of a π-regular ring is nil. Since von Neumann regular rings are π-regular, one may ask if abelian π-regular rings are (weak) Armendariz, based on Lemma 2.4. However the answer is negative by the ring R in Example 1.2(5) over a division ring S. In fact R is abelian by [6, Lemma 2], and π-regular because each element in R is either invertible or nilpotent; but R is not weak Armendariz by Example 1.2(6).

Next we observe the classical right quotient rings of weak Armendariz rings, and as a corollary obtain a situation for which weak Armendariz rings, Armendariz rings, reduced rings and IFP rings are equal.

The Armendarizness can go up to classical right quotient rings by [7, Theorem 12]. In the following we show that the weak Armendarizness also can go up to classical right quotient rings.

Theorem 2.5. *Let R be a right Ore ring with the classical right quotient ring Q. Then R is weak Armendariz if and only if so is Q.*

Proof. It suffices to show by Lemma 1.1(2) that if R is weak Armendariz then so is Q. We apply the proof of [7, Theorem 12]. Consider $f(x) = \sum_{i=0}^{1} \alpha_i x^i, g(x) = \sum_{j=0}^{1} \beta_j x^j \in Q[x]$ such that $f(x)g(x) = 0$. By [13, Proposition 2.1.16], we can assume that $\alpha_i = a_i u^{-1}, \beta_j = b_j v^{-1}$ with $a_i, b_j \in R$ for all i, j and regular $u, v \in R$. Also, by [13, Proposition 2.1.16], for each j there exist $c_j \in R$ and regular $w \in R$ such that $u^{-1} b_j = c_j w^{-1}$. Put

$m(x) = \sum_{i=0}^{1} a_i x^i, \ell(x) = \sum_{j=0}^{1} c_j x^j \in R[x]$. Then we have

$$0 = f(x)g(x) = \sum_{i=0}^{1}\sum_{j=0}^{1} \alpha_i\beta_j x^{i+j} = \sum_{i=0}^{1}\sum_{j=0}^{1} a_i(u^{-1}b_j)v^{-1}x^{i+j}$$
$$= \sum_{i=0}^{1}\sum_{j=0}^{1} a_i c_j (vw)^{-1} x^{i+j} = m(x)\ell(x)(vw)^{-1};$$

hence $m(x)\ell(x) = \sum_{i=0}^{1}\sum_{j=0}^{1} a_i c_j\ x^{i+j} = 0$ in $R[x]$. Since R is weak Armendariz, $a_i c_j = 0$ for all i, j and so $\alpha_i\beta_j = a_i u^{-1} b_j v^{-1} = a_i c_j w^{-1} v^{-1} = 0$ for all i, j. Therefore Q is also weak Armendariz. □

As a well-known Goldie's theorem, R is a semiprime right Goldie ring if and only if there exists the classical right quotient ring of R which is semisimple Artinian [13, Theorem 2.3.6]. For a semiprime ring R, notice that R is reduced if and only if R is IFP. Through the following we can extend [7, Corollary 13].

Theorem 2.6. *Suppose that R is a semiprime right Goldie ring with Q its classical right quotient ring. Then the following conditions are equivalent:*

(1) *R is a weak Armendariz ring;*
(2) *R is an Armendariz ring;*
(3) *R is a reduced ring;*
(4) *R is an IFP ring;*
(5) *Q is a weak Armendariz ring;*
(6) *Q is an Armendariz ring;*
(7) *Q is a reduced ring;*
(8) *Q is an IFP ring;*
(9) *Q is an abelian ring;*
(10) *Q is a finite direct product of division rings.*

Proof. Since Q is semisimple Artinian, Q is von Neumann regular by [5, Theorem 1.7]. So the conditions (5), (6), (7), (8), (9) and (10) are equivalent by Lemma 2.4.

(1)⇒(5) is proved by Theorem 2.5. (2)⇒(1), (3)⇒(2) and (3)⇒(4) are obtained from Lemma 1.1. Since R is semiprime, we obtain (4)⇒(3) and [16, Theorem 1.5]. (7)⇒(3) is obvious. □

By this theorem we can obtain the following Lee and Wong's result independently.

Corollary 2.7 ([12, Theorem 3.3]). *A semiprime right Goldie ring is weak Armendariz if and only if it is reduced.*

From Theorem 2.6, one may conjecture that right Goldie weak Armendariz rings are Armendariz. But Example 1.2(1) erases the possibility; actually the ring R in Example 1.2(1) is Noetherian (hence Goldie) and weak Armendariz, but it is not Armendariz. Thus the semiprimeness in Theorem 2.6 is not superfluous.

Let S be a ring and denote the ring extension

$$\left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in S \right\}$$

by R_n. Then we have another equivalence between weak Armendarizness and related concepts through R_3.

Proposition 2.8. *For a ring S and R_3 over S the following conditions are equivalent:*

(1) *S is a reduced ring;*
(2) *R_3 is Armendariz;*
(3) *R_3 is weak Armendariz;*
(4) *R_3 is IFP.*

Proof. (1)⇒(2), (2)⇒(3) and (1)⇒(4) are proved by Lemma 1.1 and Example 1.2(2).

(3)⇒(1): Let R_3 be weak Armendariz, and assume on the contrary that there is a nonzero $a \in S$ with $a^2 = 0$. Put $u = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ and $v = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in R_3. Then $u^2 = 0 = v^2$ and $uv = vu \neq 0$; hence R_3 is not weak Armendariz from $(u + vx)(u - vx) = 0$, where x is an indeterminate over R_3. We get a contradiction.

(4)⇒(1): Let R_3 be IFP, and assume on the contrary that there is a nonzero $a \in S$ with $a^2 = 0$. Take $A = \begin{pmatrix} a & a & -1 \\ 0 & a & -1 \\ 0 & 0 & a \end{pmatrix}$, $B = \begin{pmatrix} a & 0 & a \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$ in R_3. Then $AB = 0$ but

$$\begin{pmatrix} a & a & -1 \\ 0 & a & -1 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & a \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0,$$

a contradiction to the IFPness of R_3. Thus S is reduced. □

Based on Proposition 2.8, one may ask whether R_n is also (weak) Armendariz and IFP for $n \geq 4$ when S is a reduced ring. However the answer is negative by Example 1.2 (5, 6).

A ring is called *semiprimitive* if the Jacobson radical is zero. Given a ring R, $R[X]$ denotes the polynomial ring with X a set of commuting indeterminates over R (possibly infinite).

Proposition 2.9. *If a ring R is semiprime weak Armendariz, then $R[X]$ is semiprimitive.*

Proof. By [12, Theorem 3.10] R has no nonzero nil one-sided ideals since R is semiprime weak Armendariz. So by Amitsur [1], $R[X]$ is semiprimitive. $\square$

The condition "semiprime" is not superfluous as can be seen by R_n $(2 \leq n \geq 3)$ over a reduced ring S, e.g., R_3 is Armendariz by [9, Proposition 2] but the Jacobson radical of $R[X]$ contains $N[X]$ with $N = \begin{pmatrix} 0 & S & S \\ 0 & 0 & S \\ 0 & 0 & 0 \end{pmatrix}$.

A proper ideal I of a ring is called *(weak) Armendariz* if I is (weak) Armendariz as a ring without identity. It is natural to ask whether given a ring R is (weak) Armendariz when R/I and I are (weak) Armendariz for any nonzero proper ideal I of R. However the answer is negative by [9, Example 14] and Lemma 1.1(6), letting R be the 2 by 2 upper triangular matrix ring over a field. But when I is reduced as a ring then it is proved by [7, Theorem 11] that R is Armendariz when R/I is Armendariz. We show that this result also holds for weak Armendariz rings in the following.

Proposition 2.10. *Let R be a ring such that R/I is weak Armendariz for some proper ideal I of R. If I is reduced, then R is weak Armendariz.*

Proof. Let $a, b \in R$. If $ab = 0$, then $bIa = 0$ by the proof of [7, Theorem 11]. We use this fact freely. Put $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{1} a_i x^i, g(x) = \sum_{j=0}^{1} b_j x^j \in R[x]$. Then we have $b_0 I a_0 = 0$. Since R/I is weak Armendariz, $a_i b_j \in I$ for all i, j. It suffices to show $a_0 b_1 = 0 = a_1 b_0$. Assume on the contrary $a_0 b_1 \neq 0$. Then $(a_1 b_0)(a_0 b_1)^2 = a_1(b_0 a_0 b_1 a_0) b_1 = 0$ from $b_0 I a_0 = 0$ and $a_0 b_1 \in I$; hence we have $0 = f(x)g(x)(a_0 b_1)^2 = (a_0 b_1 + a_1 b_0)x(a_0 b_1)^2 = (a_0 b_1)(a_0 b_1)^2 x$ and $(a_0 b_1)^3 = 0$. But I is reduced and so $a_0 b_1 = 0$, a contradiction. Thus R is weak Armendariz. $\square$

Applying the method in the proof of [7, Proposition 10], we obtain the following.

Proposition 2.11. *For an abelian ring R, the following conditions are equivalent:*

(1) *R is weak Armendariz;*
(2) *eR and $(1-e)R$ are weak Armendariz for every $e = e^2 \in R$;*
(3) *eR and $(1-e)R$ are weak Armendariz for some $e = e^2 \in R$.*

Note that the preceding result also holds for Armendariz rings by [7, Proposition 10].

3. More examples of weak Armendariz rings

In this section we extend the class of weak Armendariz rings.

Proposition 3.1. *Let R be a ring and Δ be a multiplicative monoid in R consisting of central regular elements. Then R is (weak) Armendariz if and only if so is $\Delta^{-1}R$.*

Proof. Let R be Armendariz and $S = \Delta^{-1}R$. Put $f(x)g(x) = 0$ where $f(x) = \sum_{i=0}^{m} \alpha_i x^i, g(x) = \sum_{j=0}^{n} \beta_j x^j \in S[x]$. We can assume that $\alpha_i = a_i u^{-1}, \beta_j = b_j v^{-1}$ with $a_i, b_j \in R$ for all i, j and $u, v \in \Delta$. Then we have

$$0 = f(x)g(x) = \sum_{i=0}^{m}\sum_{j=0}^{n} \alpha_i \beta_j x^{i+j} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_i b_j u^{-1} v^{-1} x^{i+j}$$
$$= (\sum_{i=0}^{m}\sum_{j=0}^{n} a_i b_j x^{i+j})(uv)^{-1};$$

hence $\sum_{i=0}^{m}\sum_{j=0}^{n} a_i b_j x^{i+j} = 0$ in $R[x]$. Since R is Armendariz, $a_i b_j = 0$ for all i, j and so $\alpha_i \beta_j = a_i u^{-1} b_j v^{-1} = a_i b_j u^{-1} v^{-1} = 0$ for all i, j. Thus S is Armendariz. The converse is obtained by Lemma 1.1(2). The proof for weak Armendariz rings is similar. □

The ring of *Laurent* polynomials in x, coefficients in a ring R, consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers; denotes it by $R[x; x^{-1}]$.

Corollary 3.2. (1) *A commutative ring R is (weak) Armendariz if and only if so is the total quotient ring of R.*

(2) *Let R be a ring. $R[x]$ is (weak) Armendariz if and only if so is $R[x; x^{-1}]$.*

Proof. It suffices to show the necessity by Lemma 1.1(2). (1) Let Δ be the multiplicative monoid of all regular elements in R. Then $\Delta^{-1}R$ is the total quotient ring of R and hence the result holds by Proposition 3.1.

(2) Let $\Delta = \{1, x, x^2, \ldots\}$. Then Δ is a multiplicative monoid in $R[x]$ consisting of central regular elements. Note that $R[x; x^{-1}] = \Delta^{-1}R[x]$. If $R[x]$ is (weak) Armendariz, so is $\Delta^{-1}R[x]$ by Proposition 3.1. □

Due to Kaplansky [8], a ring is called *Baer* if the right annihilator of every nonempty subset is generated by an idempotent. The concept of Baer rings is left-right symmetric by [8, Theorem 3]. The class of Baer rings contain domains, the ring of all linear transformations on a vector space over a division ring, and the ring of all bounded operators on a Hilbert space.

From Example 1.2(1), homomorphic images of Armendariz rings (moreover domains) need not be Armendariz. For weak Armendariz rings we also get a negative situation as in the following.

Example 3.3. Let $R = \mathbb{Z}_2[x, y]/(x^2, y^2)$, where $\mathbb{Z}_2$ is the Galois field of order 2, $\mathbb{Z}_2[x, y]$ is the polynomial ring with two indeterminates x, y over $\mathbb{Z}_2$, and (x^2, y^2) is the ideal of $\mathbb{Z}_2[x, y]$ generated by x^2, y^2. Let $R[t]$ be the polynomial ring with an indeterminate t over R. Since $(\bar{x} + \bar{y}t)^2 = 0$ and $\bar{x}\bar{y} \neq 0$, R is not weak Armendariz.

But [12, Lemma 3.6] showed that a factor ring of R by a left (or right) annihilator of an ideal is (weak) Armendariz, where R is a (weak) Armendariz ring. Thereby we have the following.

Proposition 3.4. (1) *If a ring R is IFP and (weak) Armendariz, then R/A is (weak) Armendariz for the one-sided annihilator A of every nonempty subset in R.*

(2) *If a ring R is Baer and (weak) Armendariz, then R/A is (weak) Armendariz for the one-sided annihilator A of every nonempty subset in R.*

Proof. (1) Any one-sided annihilator in an IFP ring is two-sided by [16, Lemma 1.2], and so the result holds by [12, Lemma 3.6].

(2) Let R be a Baer and (weak) Armendariz ring. Then R is abelian by Lemma 1.1(6), and so the one-sided annihilator A of every nonempty subset in R is two-sided. Thus we have the result by [12, Lemma 3.6]. □

We end this note with raising following questions:

(1) If R is a weak Armendariz ring then is $R[x]$ weak Armendariz?

(2) Are semiprime weak Armendariz rings Armendariz?

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Young Cheol Jeon
Department of Mathematics
Korea Science Academy
Busan 614-103, Korea
E-mail address: jachun@chol.com

Hong Kee Kim
Department of Mathematics and RINS
Gyeongsang National University
Jinju 660-701, Korea
E-mail address: hkkim@gnu.ac.kr

Yang Lee
Department of Mathematics Education
Busan National University
Busan 609-735, Korea
E-mail address: ylee@pusan.ac.kr

Jung Sook Yoon
Department of Mathematics Education
Busan National University
Busan 609-735, Korea
E-mail address: gamsas@hanmail.net