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ON (DISK, ANNULUS) PAIRS OF HEEGAARD SPLITTINGS THAT INTERSECT IN ONE POINT

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ABSTRACT. Let $M = H_1 \cup_S H_2$ be a Heegaard splitting of a 3-manifold M, D be an essential disk in H_1 and A be an essential annulus in H_2. Suppose D and A intersect in one point.

First, we show that a Heegaard splitting admitting such a (D, A) pair satisfies the disjoint curve property, yet there are infinitely many examples of strongly irreducible Heegaard splittings with such (D, A) pairs.

In the second half, we obtain another Heegaard splitting $M = H_1' \cup_{S'} H_2'$ by removing the neighborhood of A from H_2 and attaching it to H_1, and show that $M = H_1' \cup_{S'} H_2'$ also has a (D, A) pair with $|D \cap A| = 1$.

1. Introduction

A *Heegaard splitting* $M = H_1 \cup_S H_2$ of a 3-manifold M is a decomposition of M into two handlebodies H_1 and H_2. In [3], Hempel defined *distance* $d(S)$ of a Heegaard splitting $H_1 \cup_S H_2$ as a complexity of Heegaard splitting, which is a nonnegative integer.

There are several notions on Heegaard splittings corresponding to $d = 0, 1, 2$. A Heegaard splitting $H_1 \cup_S H_2$ is *reducible* if there are essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ with $\partial D_1 = \partial D_2$. A reducible Heegaard splitting corresponds to $d = 0$. If a Heegaard splitting is not reducible, it is called *irreducible*. A Heegaard splitting $H_1 \cup_S H_2$ is *weakly reducible* if there are essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ with $\partial D_1 \cap \partial D_2 = \emptyset$. A weakly reducible Heegaard splitting corresponds to $d \leq 1$. Casson and Gordon [1] showed that a weakly reducible Heegaard splitting is either reducible or the manifold contains an incompressible surface. If a Heegaard splitting is not weakly reducible, it is called *strongly irreducible*. A Heegaard splitting $H_1 \cup_S H_2$ is said to have the *disjoint curve property*, which was introduced by Thompson [8], if there are essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ and an essential loop $\gamma \subset S$ with $\gamma \cap (\partial D_1 \cup \partial D_2) = \emptyset$. A Heegaard splitting having the disjoint curve property corresponds to $d \leq 2$. Note that a strongly irreducible Heegaard splitting

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corresponds to $d \geq 2$. Schleimer [6] showed that for a given 3-manifold, the number of Heegaard splittings with $d \geq 3$ is finite.

In this paper, we consider a Heegaard splitting $H_1 \cup_S H_2$ having an essential disk $D \subset H_1$ and an essential annulus $A \subset H_2$ with $|D \cap A| = 1$. We denote it as a Heegaard splitting having a (D, A) pair with $|D \cap A| = 1$ for short. First we consider relations of a Heegaard splitting having a (D, A) pair with $|D \cap A| = 1$ with previously known reducibility notions.

A Heegaard splitting $H_1 \cup_S H_2$ is *stabilized* if there are essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ with $|\partial D_1 \cap \partial D_2| = 1$. We show that for a genus $g \geq 2$ splitting, a stabilized Heegaard splitting has a (D, A) pair with $|D \cap A| = 1$, and a splitting having a (D, A) pair with $|D \cap A| = 1$ has the disjoint curve property.

In Section 3, we give infinitely many examples of strongly irreducible Heegaard splittings with genus $g \geq 2$ having a (D, A) pair with $|D \cap A| = 1$ by $\frac{1}{q}$-surgery ($|q| \geq 6$) on certain pretzel knots in S^3 and using a theorem due to Casson and Gordon [2].

Theorem 1.1. *There are infinitely many examples of strongly irreducible Heegaard splittings with genus $g \geq 2$ having a (D, A) pair with $|D \cap A| = 1$.*

For a Heegaard splitting $M = H_1 \cup_S H_2$ having a (D, A) pair with $|D \cap A| = 1$, we can make another Heegaard splitting $M = H_1' \cup_{S'} H_2'$ by removing the neighborhood of A from H_2 and attaching it to H_1. We show that $H_1' \cup_{S'} H_2$ also has a (D, A) pair with $|D \cap A| = 1$, and hence has the disjoint curve property. This makes sense in the way that the distance of Heegaard splitting is bounded by the alternate Heegaard genus [5].

Theorem 1.2. *Suppose $M = H_1' \cup_{S'} H_2'$ is a Heegaard splitting obtained from a splitting $M = H_1 \cup_S H_2$ having a (D, A) pair with $|D \cap A| = 1$, by removing the neighborhood of A from H_2 and attaching it to H_1. Then $M = H_1' \cup_{S'} H_2'$ also has a (D, A) pair with $|D \cap A| = 1$.*

2. Heegaard splitting having a (D, A) pair with $|D \cap A| = 1$

We begin with a sufficient condition for a (D, A) pair with $|D \cap A| = 1$.

Lemma 2.1. *A genus $g \geq 2$ stabilized Heegaard splitting $H_1 \cup_S H_2$ has a (D, A) pair with $|D \cap A| = 1$.*

Proof. Let $D_1 \subset H_1$ and $D_2 \subset H_2$ be essential disks with $|\partial D_1 \cap \partial D_2| = 1$. ∂D_1 is non-separating in S. If we cut S by ∂D_1, we get a twice punctured genus $g - 1$ surface S', and the arc ∂D_2 cut at the point $\partial D_1 \cap \partial D_2$ connects the two punctures of S'. Let H_1' be a genus $g - 1$ handlebody obtained by compressing H_1 by D_1. Take an arc γ properly embedded in S' with both of its endpoints in the same puncture of S' such that

- γ is disjoint from the arc $\partial D_2 - (\partial D_1 \cap \partial D_2)$.

- the loop which is the union of γ and a subarc of ∂D_1 cut by two endpoints of γ does not bound a disk in H_1'.

Note that if a handlebody is compressed along an essential disk and a loop on its boundary does not bound a disk in the compressed handlebody, the loop does not bound a disk in the original handlebody. Attach a band to D_1 along γ and push the band to the interior of H_1 to get an annulus A. By above arguments we can see that each of the loops of ∂A does not bound a disk in H_1. Hence A is incompressible in H_1. By the construction, D_2 intersects only one component of ∂A, hence A is not ∂-parallel. So (D_2, A) is the desired pair with $|D_2 \cap A| = 1$. □

It is well known that for genus $g \geq 2$ Heegaard splittings of irreducible manifolds, stabilized Heegaard splittings are equivalent to reducible Heegaard splittings. So Lemma 2.1 can also be stated as follows: A genus $g \geq 2$ reducible Heegaard splitting $H_1 \cup_S H_2$ of an irreducible manifold has a (D, A) pair with $|D \cap A| = 1$.

Next we consider a necessary condition for a (D, A) pair with $|D \cap A| = 1$.

Lemma 2.2. *A Heegaard splitting $H_1 \cup_S H_2$ having a (D, A) pair with $|D \cap A| = 1$ has the disjoint curve property.*

Proof. Since A is an essential annulus in a handlebody, it is boundary compressible. Boundary compress A and let D' be the resulting disk. Note that D' is essential since A is not boundary parallel. By isotopy we make D' to be disjoint from ∂A. Let γ be the component of ∂A that is disjoint from D. Then the triple (D, D', γ) satisfies the disjoint curve property. □

3. Strongly irreducible Heegaard splittings having (D, A) pairs with $|D \cap A| = 1$

Let $K = K(a_1, a_2, \ldots, a_{2n+1})$ be a pretzel knot where each a_i is odd, and for some j a_j is 1. (Here each a_j represents the number of half twists.) Suppose the canonical Seifert surface F by Seifert algorithm on standard diagram of K is a minimal genus surface. Then F is incompressible in $S^3 - N(F)$. Also F is incompressible in the product neighborhood $N(F) = F \times I$. We can see that $N(F)$ and $cl(S^3 - N(F))$ are handlebodies. Hence this gives a Heegaard splitting $N(F) \cup_\Sigma N(F)^c$ of S^3.

Now we are going to construct a strongly irreducible Heegaard splitting from Σ by Dehn surgery on K. Remove a neighborhood $N(K)$ from S^3. Let $K(1/q)$ denote the manifold obtained by $1/q$-filling on $S^3 - N(K)$. We can assume that the filling solid torus T is attached to $N(F) = F \times I$ along an annulus. Note that if we perform $1/q$-surgery, a meridian curve $(1, 0)$ of the filling solid torus is mapped to $(1, q)$ curve and longitude $(0, 1)$ of filling solid torus is mapped to longitude $(0, 1)$. So $N(F) \cup T$ is a handlebody. Then we get the Heegaard splitting $(N(F) \cup T) \cup_{\Sigma'} N(F)^c$ for $K(1/q)$. (Alternatively, we can regard Σ' is obtained from Σ by Dehn twists on K $|q|$ times.) By a theorem due to Casson and Gordon [2], Σ' is a strongly irreducible Heegaard splitting if $|q| \geq 6$. Here we refer the statement in ([4], Appendix).

Theorem 3.1 (Casson-Gordon). *Suppose $M = H_1 \cup_\Sigma H_2$ is a weakly reducible Heegaard splitting for the closed manifold M. Let K be a simple closed curve in Σ such that $\Sigma - N(K)$ is incompressible in both H_1 and H_2. Then Σ', for all $|q| \geq 6$, is a strongly irreducible Heegaard splitting for the Dehn filled manifold $M(1/q)$.*

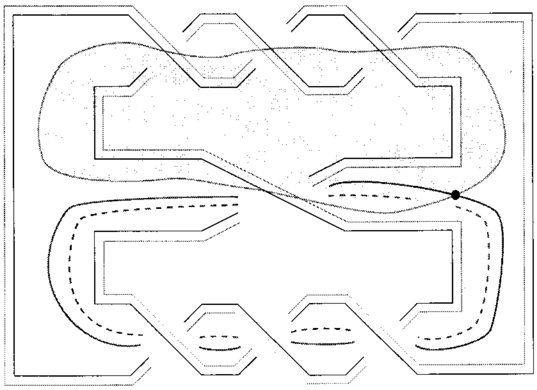

FIGURE 1. (D, A) pair with $|D \cap A| = 1$ for $K(3, 1, 3)$

We show the Heegaard splitting $(N(F) \cup T) \cup_{\Sigma'} N(F)^c$ has a (D, A) pair with $|D \cap A| = 1$. The knot $K = K(a_1, a_2, \ldots, a_{2n+1})$ has the property that there exists an essential disk $D \subset N(F)^c$ and an essential annulus $A \subset F \times I$. Fig. 1 illustrates an example where ∂A consists of two bottom curves. We can see that A is essential in $N(F) \cup T$ also. Hence we obtain the result of Theorem 1.1.

Fig. 2 shows the relations of Heegaard splittings having (D, A) pairs with $|D \cap A| = 1$ with other notions.

4. Alternate Heegaard splittings

If we attach a 1-handle $D^2 \times I$ to a handlebody, the result is obviously a handlebody. The following lemma considers attaching (annulus $\times$ I) to a handlebody.

Lemma 4.1. *Let γ_1, γ_2 be two disjoint essential loops on the boundary of a handlebody H and D be an essential disk of H such that $|\partial D \cap \gamma_1| = 1$ and $\partial D \cap \gamma_2 = \emptyset$. Let A be an annulus.*

If we attach $A \times I$ to H along $A \times \partial I$ so that $A \times \{0\}$ is attached to $N(\gamma_1; \partial H)$ and $A \times \{1\}$ to $N(\gamma_2; \partial H)$, then the resulting manifold is a handlebody of same genus with H.

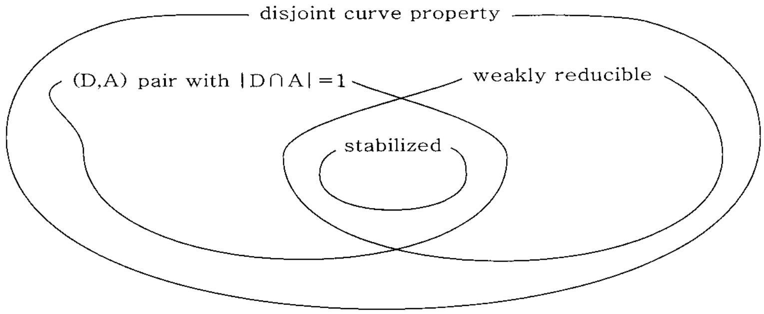

FIGURE 2. genus $g \geq 2$ Heegaard splittings

Proof. Consider the neighborhood $D \times I$ in H and $\gamma_1 \times I$ in ∂H. We can assume that $\partial(D \times I) \cap (\gamma_1 \times I)$ is a rectangle R since $|\partial D \cap \gamma_1| = 1$. Let R' be the rectangle in $A \times \{0\}$ that is attached to R. Note that $cl((A \times \{0\}) - R')$ is a disk.

Observe that $(D \times I) \cup (R' \times I)$, where $R' \times I$ is taken in $A \times I$, is a 3-ball. Let H' be $cl(H - (D \times I))$. Since $|\partial D \cap \gamma_1| = 1$, D is a non-separating disk. Hence H' is a handlebody with genus one less than genus of H. We can also observe that $cl((A \times \{0\}) - R') \times I$ taken in $A \times I$ is a (disk $\times I$) attached along (disk $\times \partial I$) to $\partial H'$. Hence $H' \cup (cl((A \times \{0\}) - R') \times I)$ is a handlebody of same genus with H.

The intersection of $(D \times I) \cup (R' \times I)$ and $H' \cup (cl((A \times \{0\}) - R') \times I)$ is $(D \times \partial I) \cup ((R' \cap cl((A \times \{0\}) - R')) \times I) \cup (R' \times \{1\})$, which is a disk in the boundary of a 3-ball. So we conclude that the resulting manifold after attaching $A \times I$ along $A \times \partial I$ is a handlebody of same genus with H. □

Now we consider removing a properly embedded essential annulus from a handlebody.

Proposition 4.2. *Let A be an essential annulus properly embedded in a handlebody H. Then A cuts H into handlebodies (or a handlebody).*

Proof. By standard innermost disk and outermost arc argument, we can find an essential arc γ of $A \cap D$, for a meridian disk D of a meridian disk system of H such that γ is outermost in D. Let Δ be the outermost disk for γ. If we boundary compress A along Δ, we get a disk E. H cut along E is a handlebody or two handlebodies according to the separability of E. H cut along A can be recovered from H cut along E by attaching the neighborhood of Δ, $\Delta \times I$ along $\Delta \times \partial I$ and some isotopies. Hence the result is still a handlebody or two handlebodies. □

By Lemma 4.1 and Proposition 4.2 we can produce another Heegaard splitting from a given one under the conditions as in Lemma 4.1.

Lemma 4.3. *Let $H_1 \cup_S H_2$ be a Heegaard splitting of a 3-manifold M having a (D, A) pair with $|D \cap A| = 1$. Let H_1' be obtained from H_1 by attaching $A \times I \subset H_2$ along $\partial A \times I$ and H_2' be obtained from H_2 by cutting along A. Then $H_1' \cup_{S'} H_2'$ is a Heegaard splitting of same genus with $H_1 \cup_S H_2$.*

Proof. By Lemma 4.1, H_1' is a handlebody of same genus with H_1. By Proposition 4.2 and $\partial H_1'$ being equal to $\partial H_2'$, H_2' is a handlebody. Hence $H_1' \cup_{S'} H_2'$ is a Heegaard splitting of M. □

Remark 4.4. Lemma 4.1 and Lemma 4.3 are generalizations of Definition 14 of [7].

The alternate Heegaard splitting $H_1' \cup_{S'} H_2'$ can possibly be isotopic to $H_1 \cup_S H_2$, otherwise there would be some bounds on the distance of the Heegaard splitting by [5]. In this regard, we show that $H_1' \cup_{S'} H_2'$ also has the disjoint curve property, $d \leq 2$.

Proof of Theorem 1.2. Since A is an essential annulus in H_2, it is boundary compressible. Let Δ be the boundary compressing disk and γ_1 be the corresponding essential arc of A cut by Δ. Then we can see that Δ is an essential disk in H_2' since γ_1 is an essential arc.

Let γ_2 be an essential loop of A. Consider $\gamma_2 \times I$ in the product neighborhood of $A \times I$. $\gamma_2 \times I$ is a properly embedded annulus in H_1' such that $|\Delta \cap (\gamma_2 \times I)| = 1$.

There are two cases to consider.

Case 1. $\gamma_2 \times I$ is compressible in H_1'.

Compress $\gamma_2 \times I$. Then one of the disks E after the compression satisfies $|E \cap \Delta| = 1$. Hence $H_1' \cup_{S'} H_2'$ is stabilized and has a (D, A) pair with $|D \cap A| = 1$ by Lemma 2.1.

Case 2. $\gamma_2 \times I$ is incompressible in H_1'.

$\gamma_2 \times I$ is essential in H_1' and $(\Delta, \gamma_2 \times I)$ is the desired pair. This completes the proof of Theorem 1.2. □

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