Bull. Korean Math. Soc. **46** (2009), No. 1, pp. 87–97

THE STABILITY OF THE SINE AND COSINE FUNCTIONAL EQUATIONS IN SCHWARTZ DISTRIBUTIONS

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Abstract. We prove the Hyers-Ulam stability of the sine and cosine functional equations in the spaces of generalized functions such as Schwartz distributions, Fourier hyperfunctions, and Gelfand generalized functions.

1. Introduction

Let $f, g : \mathbb{R}^n \to \mathbb{C}$ and denote by $T_j(f,g)$, $j = 1, 2$, the sine and cosine differences, respectively,

$$T_1(f,g) := f(x+y) - f(x)g(y) - g(x)f(y), \tag{1.1}$$

$$T_2(f,g) := g(x+y) - g(x)g(y) + f(x)f(y). \tag{1.2}$$

In [17] L. Székelyhidi proved the Hyers-Ulam stability for the sine and cosine functional equations. As a special case of his result it is obtained that if $T_j(f,g)$, $j = 1,2$, is a bounded function on $\mathbb{R}^{2n}$, then either there exist λ, $\mu \in \mathbb{C}$, not both zero, such that $\lambda f - \mu g$ is a bounded function on $\mathbb{R}^n$, or else $T_j(f,g) = 0$, $j = 1,2$, respectively. In this paper we consider the Hyers-Ulam stability problems of the sine and cosine functional equations in the spaces of generalized functions such as the Schwartz tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, Fourier hyperfunctions $\mathcal{F}'(\mathbb{R}^n)$ and Gelfand generalized functions ${\mathcal{S}'}_{1/2}^{1/2}(\mathbb{R}^n)$. Following the formulation as in [3, 5, 6, 7] we generalize the differences (1.1) and (1.2) to the spaces of generalized functions u, v as:

$$T_1(u,v) := u \circ A - u \otimes v - v \otimes u, \tag{1.3}$$

$$T_2(u,v) := v \circ A - v \otimes v + u \otimes u, \tag{1.4}$$

Received May 1, 2008; Revised August 6, 2008.

2000 *Mathematics Subject Classification.* 39B82, 46F12.

Key words and phrases. Hyers-Ulam stability, trigonometric functional equation, distributions, Fourier hyperfunctions, Gelfand generalized functions, heat kernel.

The first author was supported by the Korean Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2007-331-C00025).

The second author was supported by the Korean Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2007-521-C00016).

where $A(x,y) = x+y$, $x, y \in \mathbb{R}^n$ and $\circ$ denotes the pullback, $\otimes$ denotes the tensor product of generalized functions [9]. For some related Hyers-Ulam stability problems we refer the reader to [10, 12, 13, 14, 15, 17, 18, 19, 20].

We denote by $L^\infty(\mathbb{R}^n)$ the space of all bounded measurable functions on $\mathbb{R}^n$. As results we prove the followings.

Theorem 1.1. *Let $u, v \in \mathcal{S}'^{1/2}_{1/2}$ satisfy $T_1(u,v) \in L^\infty(\mathbb{R}^{2n})$. Then u and v satisfy one of the followings:*

(i) $u = 0$, v : *arbitrary,*
(ii) u *and* v *are bounded measurable functions,*
(iii) $u = c \cdot x\, e^{ia\cdot x} + B(x)$, $v = e^{ia\cdot x}$,
(iv) $u = \lambda(e^{c\cdot x} - B(x))$, $v = \frac{1}{2}(e^{c\cdot x} + B(x))$,
(v) $u = \lambda(e^{b\cdot x} - e^{c\cdot x})$, $v = \frac{1}{2}(e^{b\cdot x} + e^{c\cdot x})$,
(vi) $u = b \cdot x\, e^{c\cdot x}$, $v = e^{c\cdot x}$,

where $a \in \mathbb{R}^n$, b, $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, and B is a bounded measurable function.

Theorem 1.2. *Let $u, v \in \mathcal{S}'^{1/2}_{1/2}$ satisfy $T_2(u,v) \in L^\infty(\mathbb{R}^{2n})$. Then u and v satisfy one of the followings:*

(i) u *and* v *are bounded measurable functions,*
(ii) $v = e^{c\cdot x}$ *and* u *is a bounded measurable function,*
(iii) $v = c \cdot x\, e^{ia\cdot x} + B(x)$, $u = \pm[(1 - c\cdot x)e^{ia\cdot x} - B(x)]$,
(iv) $v = \frac{e^{c\cdot x}+\lambda B(x)}{1-\lambda^2}$, $u = \frac{\lambda e^{c\cdot x}+B(x)}{1-\lambda^2}$,
(v) $v = (1 - b\cdot x)e^{c\cdot x}$, $u = \pm b \cdot x\, e^{c\cdot x}$,
(vi) $v = e^{b\cdot x}[\cos(c\cdot x) + \lambda \sin(c \cdot x)]$, $u = \sqrt{\lambda^2 + 1}\, e^{b\cdot x}\sin(c\cdot x)$,

where $a \in \mathbb{R}^n$, b, $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, and B is a bounded measurable function.

2. Generalized functions

For the spaces of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ we refer the reader to [8, 9, 16]. Here we briefly introduce the spaces of Gelfand generalized functions and Fourier hyperfunctions. Here we use the following notations: $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial x_j}$.

Definition 2.1. For given r, $s \geq 0$ we denote by $\mathcal{S}_r^s$ or $\mathcal{S}_r^s(\mathbb{R}^n)$ the space of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ such that there exist positive constants A and B satisfying

$$\|\varphi\|_{h,k} := \sup_{x\in\mathbb{R}^n,\, \alpha,\beta\in\mathbb{N}_0^n} \frac{|x^\alpha \partial^\beta \varphi(x)|}{h^{|\alpha|} k^{|\beta|} \alpha!^r \beta!^s} < \infty. \tag{2.1}$$

The topology on the space $\mathcal{S}_r^s$ is defined by the seminorms $\|\cdot\|_{h,k}$ in the left-hand side of (2.1) and the elements of the dual space $\mathcal{S}'^s_r$ of $\mathcal{S}_r^s$ are called *Gelfand–Shilov generalized functions.* In particular, we denote $\mathcal{S}'^1_1$ by $\mathcal{F}'$ and call its elements *Fourier hyperfunctions.*

It is known that if $r > 0$ and $0 \le s < 1$, the space $\mathcal{S}_r^s(\mathbb{R}^n)$ consists of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ that can be continued to an entire function on $\mathbb{C}^n$ satisfying

$$|\varphi(x+iy)| \le C\exp(-a|x|^{1/r} + b|y|^{1/(1-s)}) \tag{2.2}$$

for some a, $b > 0$.

It is well known that the following topological inclusions hold:

$$\mathcal{S}_{1/2}^{1/2} \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{S}, \qquad \mathcal{S}' \hookrightarrow \mathcal{F}' \hookrightarrow \mathcal{S'}_{1/2}^{1/2}.$$

We refer the reader to [9], chapter V–VI, for tensor products and pullbacks of generalized functions.

3. Proofs

For the proof of the theorems we employ the following n-dimensional heat kernel

$$E_t(x) = (4\pi t)^{-n/2}\exp(-|x|^2/4t), \quad t > 0.$$

In view of (2.2) it is easy to see that for each $t > 0$, E_t belongs to the Gelfand–Shilov space $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^n)$. Thus the convolution $(u * E_t)(x) := \langle u_y, E_t(x-y)\rangle$ is well defined for all $u \in \mathcal{S'}_{1/2}^{1/2}$. It is well known that $U(x,t) = (u * E_t)(x)$ is a smooth solution of the heat equation $(\partial/\partial_t - \Delta)U = 0$ in $\{(x,t) : x \in \mathbb{R}^n,\ t > 0\}$ and $(u * E_t)(x) \to u$ as $t \to 0^+$ in the sense of generalized functions, that is, for every $\varphi \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^n)$,

$$\langle u, \varphi\rangle = \lim_{t\to 0^+} \int (u * E_t)(x)\varphi(x)\,dx.$$

We call $(u * E_t)(x)$ *the Gauss transform* of u. Let $\langle G, +\rangle$ be a semigroup, and $\mathbb{C}$ be the field of complex numbers. A function $l : G \to \mathbb{C}$ is said to be *additive* provided $l(x+y) = l(x)+l(y)$ and $m : G \to \mathbb{C}$ is said to be *exponential* provided $m(x+y) = m(x)m(y)$.

We first discuss the solutions of the trigonometric functional equations (1.3) and (1.4) in the space $\mathcal{S'}_{1/2}^{1/2}$ of Gelfand generalized functions. As a consequence of the result [3] we have the following.

Lemma 3.1. *The solutions u, $v \in \mathcal{S'}_{1/2}^{1/2}$ of the equation $T_j(u,v) = 0,\ j = 1,2$ are equal, respectively, to the smooth solution f, g of corresponding classical functional equations $T_j(f,g) = 0,\ j = 1,2$.*

Remark. We refer the reader to Aczél ([1], p. 180) and Aczél–Dombres ([2], pp. 209–217) for the general solutions and measurable solutions of the equations $T_j(f,g) = 0,\ j = 1,2$.

For the proof of Theorem 1.1 we need the following lemmas.

Lemma 3.2 ([11]). *Let G be a semigroup and $\mathbb{C}$ be the field of complex numbers. If $f, g : G \to \mathbb{C}$ satisfy the inequality; for each $y \in G$ there exists a positive constant M_y such that*

$$|f(x+y) - f(x)g(y)| \le M_y. \tag{3.1}$$

Then either f is bounded function or g is an exponential function.

Proof. Suppose that g is not exponential. Then there are y, $z \in G$ such that $g(y+z) \neq g(y)g(z)$. Now we have

$$\begin{aligned} &f(x+y+z) - f(x+y)g(z) \\ &= \big(f(x+y+z) - f(x)g(y+z)\big) - g(z)\big(f(x+y) - f(x)g(y)\big) \\ &\quad + f(x)\big(g(y+z) - g(y)g(z)\big), \end{aligned}$$

and hence

$$\begin{aligned} f(x) = {} & \big(g(y+z) - g(y)g(z)\big)^{-1}\Big(\big(f(x+y+z) - f(x+y)g(z)\big) \\ & - \big(f(x+y+z) - f(x)g(y+z)\big) + g(z)\big(f(x+y) - f(x)g(y)\big)\Big). \end{aligned} \tag{3.2}$$

In view of (3.1) the right hand side of (3.2) is bounded as a function of x. Consequently, f is bounded. □

Lemma 3.3. *Let $U, V : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ satisfy the inequality; there exists a positive constant M such that*

$$|U(x+y, t+s) - U(x,t)V(y,s) - V(x,t)U(y,s)| \le M \tag{3.3}$$

for all $x, y \in \mathbb{R}^n, t, s > 0$. Then either there exist ρ. $\nu \in \mathbb{C}$, not both zero, and $L > 0$ such that

$$|\rho U(x,t) - \nu V(y,s)| \le L, \tag{3.4}$$

or else

$$U(x+y, t+s) - U(x,t)V(y,s) - V(x,t)U(y,s) = 0 \tag{3.5}$$

for all x, $y \in \mathbb{R}^n$, $t, s > 0$.

Also the inequality (3.4) *together with* (3.3) *implies one of the followings:*

(i) $U = 0$, V; *arbitrary,*

(ii) U *and* V *are bounded functions,*

(iii) V *is a bounded exponential function and* U *is an unbounded function,*

(iv) $U(x,t) = \lambda(m(x,t) - B(x,t))$, $V = \frac{1}{2}(m(x,t) + B(x,t))$, *where $\lambda \in \mathbb{C}$ and B is a bounded function, m is an exponential function.*

Proof. First we prove that the equation (3.5) is satisfied if the inequality (3.4) fails. Assume that the inequality (3.4) holds only when $\rho = \nu = 0$ and let

$$F(x,y,t,s) = U(x+y, t+s) - U(x,t)V(y,s) - V(x,t)U(y,s). \tag{3.6}$$

Choose y_1 and s_1 satisfying $U(y_1, s_1) \neq 0$. Then we have the equalities

$$V(x,t) = \lambda_0 U(x,t) + \lambda_1 U(x+y_1, t+s_1) - \lambda_1 F(x, y_1, t, s_1), \tag{3.7}$$

where $\lambda_0 = -\frac{V(y_1,s_1)}{U(y_1,s_1)}$ and $\lambda_1 = \frac{1}{U(y_1,s_1)}$. From (3.6) and (3.7) we have

$$\begin{aligned} &U\big((x+y)+z, (t+s)+r\big) \\ &= U(x+y, t+s)V(z,r) + V(x+y,t+s)U(z,r) + F(x+y, z, t+s, r) \\ &= U(x+y,t+s)V(z,r) \\ &\quad + \Big(\lambda_0 U(x+y,t+s) + \lambda_1 U(x+y+y_1, t+s+s_1) \\ &\qquad - \lambda_1 F(x+y, y_1, t+s, s_1)\Big)U(z,r) \\ &\quad + F(x+y, z, t+s, r) \\ &= \Big(U(x,t)V(y,s) + V(x,t)U(y,s) + F(x,y,t,s)\Big)V(z,r) \\ &\quad + \lambda_0\Big(U(x,t)V(y,s) + V(x,t)U(y,s) + F(x,y,t,s)\Big)U(z,r) \\ &\quad + \lambda_1\Big(U(x,t)V(y+y_1, s+s_1) + V(x,t)U(y+y_1,s+s_1) \\ &\quad + F(x, y+y_1, t, s+s_1) - F(x+y, y_1, t+s, s_1)\Big)U(z,r) \\ &\quad + F(x+y, z, t+s, r), \end{aligned} \tag{3.8}$$

and also we have

$$\begin{aligned} &U\big(x+(y+z), t+(s+r)\big) \\ &= U(x,t)V(y+z, s+r) + V(x,t)U(y+z,s+r) + F(x, y+z, t, s+r). \end{aligned} \tag{3.9}$$

Thus we have

$$\begin{aligned} &U(x,t)\Big(V(y,s)V(z,r) + \lambda_0 V(y,s)U(z,r) + \lambda_1 V(y+y_1, s+s_1)U(z,r) \\ &\quad - V(y+z, s+r)\Big) + V(x,t)\Big(U(y,s)V(z,r) + \lambda_0 U(y,s)U(z,r) \\ &\quad + \lambda_1 U(y+y_1, s+s_1)U(z,r) - U(y+z, s+r)\Big) \\ &= F(x, y+z, t, s+r) - F(x+y, z, t+s, r) - F(x,y,t,s)V(z,r) \\ &\quad - \lambda_0 F(x,y,t,s)U(z,r) \\ &\quad - \lambda_1\Big(F(x, y+y_1, t, s+s_1) - F(x+y, y_1, t+s, s_1)\Big)U(z,r). \end{aligned} \tag{3.10}$$

If we fix y, z, s, r, the right hand side of (3.10) is bounded. Thus by the assumption, we have

(3.11)

$$F(x, y+z, t, s+r) - F(x+y, z, t+s, r)$$

$$= \Big(\lambda_0 F(x,y,t,s) + \lambda_1 F(x, y+y_1, t, s+s_1) - \lambda_1 F(x+y, y_1, t+s, s_1)\Big) U(z,r) + F(x,y,t,s)V(z,r).$$

Since the left hand side of (3.11) is bounded, our assumption implies $F \equiv 0$. Now we assume that the inequality (3.4) holds. Obviously, we have the case (i) as one of the possible cases. If U is a nontrivial bounded function, it follows from (3.3) that V is bounded, which gives (ii). If U is unbounded and V is bounded, then in view of (3.3), $U(x+y, t+s) - U(x,t)V(y,s)$ is bounded function of (x,t) for each (y,s). By Lemma 3.2, V is an exponential function, which gives the case (iii). Finally, we consider the case that both U and V are unbounded functions. For this case, the inequality (3.4) implies

$$V(x,t) = \mu U(x,t) + B(x,t) \tag{3.12}$$

for some $\mu \in \mathbb{C}$, $\mu \neq 0$ and a bounded function B. Putting (3.12) in (3.3) and using the triangle inequality we have

$$|U(x+y, t+s) - U(x,t)\,(B(y,s) + 2\mu U(y,s))\,| \le |U(y,s)B(x,t)| + M \tag{3.13}$$

for all x, $y \in \mathbb{R}^n$, $t, s > 0$. Thus, the left hand side of (3.13) is bounded for each $y \in \mathbb{R}^n$ and $s > 0$. Applying Lemma 3.2, we have

$$B(y,s) + 2\mu U(y,s) = m(y,s) \tag{3.14}$$

for all $y \in \mathbb{R}^n$, $s > 0$, where m is an exponential function on $\mathbb{R}^n \times (0,\infty)$. Thus the case (iv) follows immediately from (3.12) and (3.14) with $\lambda = 1/2\mu$. □

The case (iii) of the above lemma can be interpreted more precisely if some regularities of U, V are given. In fact, we can obtain the following.

Lemma 3.4. *Let $U, V : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be continuous functions satisfying the inequality* (3.3). *If V is a bounded exponential function and U is an unbounded function, then there exist $a \in \mathbb{R}^n$, $c \in \mathbb{C}$, $b \in \mathbb{C}$ and $C > 0$ such that*

$$|U(x,t) - c \cdot x\; e^{ia\cdot x + bt}| \le C \tag{3.15}$$

for all $x \in \mathbb{R}^n$, $0 < t < 1$.

Proof. Since $V(x,t)$ is a continuous exponential function we have $V(x,t) = e^{c\cdot x + bt}$ for some $c \in \mathbb{C}^n$, $b \in \mathbb{C}$. Furthermore, since V is bounded, it follows that $c = ia$, $a \in \mathbb{R}^n$, $\Re b < 0$. Put $y = 0$ in (3.3) and divide the result by $|V(0,s)|$ to get

$$|U(x,t)| \le \frac{|U(x, t+s) - U(0,s)V(x,t)| + M}{|V(0,s)|}. \tag{3.16}$$

Thus it follows that

$$\limsup_{t\to 0^+} U(x,t) := f(x)$$

exists. Put $y = 0$ and $t \to 0^+$ so that $U(x,t) \to f(x)$ in (3.3) and use the triangle inequality to get

$$|U(x,s) - f(x)V(0,s)| \le M + |U(0,s)|. \tag{3.17}$$

It follows from (3.3), (3.17) and the triangle inequality

$$(3.18)\quad |f(x+y)V(0,t+s)-f(x)V(y,s)V(0,t)-f(y)V(x,t)V(0,s)| \le M^*(t,s),$$

where $M^*(t,s) = 4M + |U(0,t)| + |U(0,s)| + U(0,t+s)|$. Letting $t, s \to 0^+$ in (3.18) we have

$$(3.19)\qquad |f(x+y) - f(x)V(y,0) - f(y)V(x,0)| \le M_1$$

for some $M_1 > 0$, which implies

$$(3.20)\qquad |g(x+y) - g(x) - g(y)| \le M_1,$$

where $g(x) = e^{-ia\cdot x}f(x)$. Thus it follows from the Hyers-Ulam stability theorem [10], there exists an additive function $A(x)$ such that

$$|g(x) - A(x)| \le M_1, \text{ or } |f(x) - A(x)V(x,0)| \le M_1.$$

Since g is continuous we must have $A(x) = c \cdot x$ for some $c \in \mathbb{C}^n$ and that

$$(3.21)\qquad |f(x) - c\cdot x\, e^{a\cdot x}| \le M_1$$

for all $x \in \mathbb{R}^n$. It follows from (3.17) and (3.21) that

$$(3.22)\qquad |U(x,t) - c\cdot x\, e^{a\cdot x+bt}| \le M_2 + |U(0,t)|,$$

where $M_2 = M + M_1$. It follows from (3.16) that $U(0,t)$ is bounded in $(0,1)$. This completes the proof. □

Proof of Theorem 1.1. Convolving in (1.3) the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels we have in view of the semigroup property $(E_t * E_s)(x) = E_{t+s}(x)$ of the heat kernel

$$(3.23)\qquad \begin{aligned}[(u\circ A) * (E_t(\xi)E_s(\eta))](x,\, y) &= \langle u_\xi,\, \int E_t(x-\xi+\eta)E_s(y-\eta)\,d\eta\rangle \\ &= \langle u_\xi,\, (E_t * E_s)(x+y-\xi)\rangle \\ &= U(x+y, t+s).\end{aligned}$$

Similarly we have

$$(3.24)\qquad \begin{aligned}[(u\otimes v) * (E_t(\xi)E_s(\eta))](x,\, y) &= U(x,t)V(y,s), \\ [(v\otimes u) * (E_t(\xi)E_s(\eta))](x,\, y) &= V(x,t)U(y,s),\end{aligned}$$

where $U(x,t), V(x,t)$ are the Gauss transforms of u, v, respectively. Thus by our assumption we have the inequality (3.3). We first consider the cases when U, V satisfy the inequality (3.4). The case (i) follows immediately from Lemma 3.3 (i). If U, V are bounded functions, then by the result [21, p. 123, Theorem 1] the initial values u, v of U, V are bounded measurable functions, which gives (ii). For the case (iii), letting $t \to 0^+$ in (3.15) it follows that $u - c\cdot x\, e^{a\cdot x} := B(x)$ is a bounded measurable function, which gives (iii). Letting $t \to 0^+$ in (iv) of Lemma 3.3 the case (iv) follows, since $m(x,t) = e^{a\cdot x+bt}$ for some $a \in \mathbb{C}^n$, $b \in \mathbb{C}$ from the continuity of U, V. Now we consider the case that U, V satisfy the equation (3.5). Letting t, $s \to 0^+$ in (3.5) we have

$$u \circ A - u \otimes v - v \otimes u = 0. \tag{3.25}$$

By Lemma 3.1 and the result in [2, p. 180], the solutions of the equation (3.25) are given by (i), (v), (vi) or

$$u = \lambda e^{a \cdot x}, \ v = \frac{1}{2} e^{a \cdot x} \tag{3.26}$$

for some $a \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$. Now, the solution (3.26) is contained in the case (iv). This completes the proof. □

Secondly we prove Theorem 1.2. We first prove the following.

Lemma 3.5. *Let $U, V : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ satisfy the inequality; there exists a positive constant M such that*

$$|V(x+y, t+s) - V(x,t)V(y,s) + U(x,t)U(y,s)| \le M \tag{3.27}$$

for all $x, y \in \mathbb{R}^n, t, s > 0$. Then U, V satisfy one of the followings:

(i) *U and V are bounded functions,*
(ii) *V is exponential and U is a bounded function,*
(iii) *$V + U$ or $V - U$ is a bounded exponential function,*
(iv) *$V = \frac{m + \lambda B}{1 - \lambda^2}$, $U = \frac{\lambda m + B}{1 - \lambda^2}$,*
(v) *$V(x+y, t+s) - V(x,t)V(y,s) + U(x,t)U(y,s) = 0$ for all $x, y \in \mathbb{R}^n$, $t, s > 0$,*

where a, $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ and l, m, B are, respectively, additive, unbounded exponential, and bounded functions on $\mathbb{R}^n \times (0, \infty)$.

Proof. As in the proof of Lemma 3.3 we first prove that either there exist ρ, $\nu \in \mathbb{C}$, not both zero, and $L > 0$ such that

$$|\rho U(x,t) - \nu V(y,s)| \le L, \tag{3.28}$$

or else

$$V(x+y, t+s) - V(x,t)V(y,s) - U(x,t)U(y,s) = 0 \tag{3.29}$$

for all x, $y \in \mathbb{R}^n$, $t, s > 0$. Assume that $|\rho U(x,t) - \nu V(y,s)| \le L$ for some ρ, $\nu \in \mathbb{C}$ implies ρ, $\nu = 0$. For this case, we can choose y_1 and s_1 satisfying $U(y_1, s_1) \ne 0$. Let

$$F(x,y,t,s) = V(x+y, t+s) - V(x,t)V(y,s) + U(x,t)U(y,s). \tag{3.30}$$

Then we have

$$U(x,t) = \lambda_0 V(x,t) + \lambda_1 V(x + y_1, t + s_1)) - \lambda_1 F(x, y_1, t, s_1), \tag{3.31}$$

where $\lambda_0 = \frac{V(y_1, s_1)}{U(y_1, s_1)}$ and $\lambda_1 = -\frac{1}{U(y_1, s_1)}$. Using (3.30) and (3.31) we have

$$\begin{aligned} &V\big((x+y)+z, (t+s)+r\big) \\ &= V(x+y, t+s)V(z,r) - U(x+y, t+s)U(z,r) + F(x+y, z, t+s, r) \end{aligned} \tag{3.32}$$

$$
\begin{aligned}
&= \Big(V(x,t)V(y,s) - U(x,t)U(y,s) + F(x,y,t,s)\Big)V(z,r) \\
&\quad - \Big(\lambda_0 V(x+y,t+s) + \lambda_1 V(x+y+y_1,t+s+s_1) \\
&\qquad - \lambda_1 F(x+y,y_1,t+s,s_1)\Big)U(z,r) \\
&\quad + F(x+y,z,t+s,r) \\
&= \Big(V(x,t)V(y,s) - U(x,t)U(y,s) + F(x,y,t,s)\Big)V(z,r) \\
&\quad - \lambda_0\Big(V(x,t)V(y,s) - U(x,t)U(y,s) + F(x,y,t,s)\Big)U(z,r) \\
&\quad - \lambda_1\Big(V(x,t)V(y+y_1,s+s_1) - U(x,t)U(y+y_1,s+s_1) \\
&\qquad + F(x,y+y_1,t,s+s_1)\Big)U(z,r) \\
&\quad + \lambda_1 F(x+y,y_1,t+s,s_1)U(z,r) + F(x+y,z,t+s,r).
\end{aligned}
$$

On the other hand, we can write

(3.33)

$$
\begin{aligned}
&V\big(x+(y+z),t+(s+r)\big) \\
&= V(x,t)V(y+z,s+r) - U(x,t)U(y+z,s+r) + F(x,y+z,t,s+r).
\end{aligned}
$$

By equating the right hand sides of (3.32) and (3.33) we have

(3.34)

$$
\begin{aligned}
&V(x,t)\Big(V(y,s)V(z,r) - \lambda_0 V(y,s)U(z,r) - \lambda_1 V(y+y_1,s+s_1)U(z,r) \\
&- V(y+z,s+r)\Big) + U(x,t)\Big(-U(y,s)V(z,r) + \lambda_0 U(y,s)U(z,r) \\
&+ \lambda_1 U(y+y_1,s+s_1)U(z,r) + U(y+z,s+r)\Big) \\
&= -F(x,y,t,s)V(z,r) + \lambda_0 F(x,y,t,s)U(z,r) + \lambda_1 F(x,y+y_1,t,s+s_1)U(z,r) \\
&\quad -\lambda_1 F(x+y,y_1,t+s,s_1)U(z,r) - F(x+y,z,t+s,r) + F(x,y+z,t,s+r).
\end{aligned}
$$

When y, s, z, r are fixed, the right side of (3.34) is bounded. Thus by our assumption that $|\rho U(x,t) - \nu V(y,s)| \le L$ for some ρ, $\nu \in \mathbb{C}$ implies ρ, $\nu = 0$, we have

$$
\begin{aligned}
(3.35)\qquad & F(x,y+z,t,s+r) - F(x+y,z,t+s,r) \\
&= F(x,y,t,s)V(z,r) + \Big(-\lambda_0 F(x,y,t,s) - \lambda_1 F(x,y+y_1,t,s+s_1) \\
&\qquad + \lambda_1 F(x+y,y_1,t+s,s_1)\Big)U(z,r).
\end{aligned}
$$

Since F is bounded, our assumption implies $F \equiv 0$. This gives the case (v). Now we assume that the inequality (3.28) holds for some ρ, ν, not both zero. If U is bounded, the cases (i) and (ii) follow immediately from Lemma 3.2. It remains to consider the case when both U and V are unbounded functions. For this case, as in the proof of Lemma 3.3 we have

$$U(x,t) = \lambda V(x,t) + B(x,t) \tag{3.36}$$

for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and a bounded function B. Putting (3.36) in (3.27) and using the triangle inequality we have

$$\begin{aligned} &|V(x+y,t+s) - V(x,t)\left((1-\lambda^2)V(y,s) - \lambda B(y,s)\right)| \\ &\leq |B(x,t)\left(B(y,s) + \lambda V(y,s)\right)| + M \end{aligned} \tag{3.37}$$

for all $x,\ y \in \mathbb{R}^n,\ t,s > 0$. Thus, the left hand side of (3.37) is bounded for each $y \in \mathbb{R}^n$ and $s > 0$. Applying Lemma 3.2, we have

$$(1-\lambda^2)V(y,s) - \lambda B(y,s) = m(y,s) \tag{3.38}$$

for all $y \in \mathbb{R}^n$, $s > 0$, where m is an exponential function on $\mathbb{R}^n \times (0,\infty)$. Thus the case (iii) follows if $\lambda = \pm 1$, and the case (iv) follows if $\lambda \neq \pm 1$. □

Proof of Theorem 1.2. Following the same approach as in Theorem 1.1 the Gauss transform U, V of u, v satisfy one of the conditions (i)∼(v) of Lemma 3.5. By the same reason as in the proof of Theorem 1.1 the conditions (i), (ii), (iv) follow from (i), (ii), (iv) of Lemma 3.5. For the case (iii), assume that U, V satisfy (iii) of Lemma 3.5. Then we may write

$$\pm U = m - V, \tag{3.39}$$

where m is a bounded exponential function. Putting (3.39) in (3.27) and using the triangle inequality we have, for some $M_1 > 0$,

(3.40)

$$|V(x+y,t+s) - V(x,t)m(y,s) - V(y,s)m(x,t)| \leq |m(x,t)m(y,s)| + M \leq M_1$$

for all $x,\ y \in \mathbb{R}^n,\ t,s > 0$. Applying Lemma 3.3 we have

$$V(x,t) = c \cdot x\ e^{ia\cdot x + bt} + B(x,t) \tag{3.41}$$

for some $c \in \mathbb{C}^n$, $a \in \mathbb{R}^n$, $b \in \mathbb{C}$. Letting $t \to 0^+$ in (3.41) we have the case (iii) of Theorem 1.2. Finally, if U, V satisfy the equation (v) in Lemma 3.5, we have

$$v \circ A - v \otimes v - u \otimes u = 0. \tag{3.42}$$

The nontrivial solutions of the equation (3.42) are given by (v), (vi), or contained in the case (iii) or (iv) of Theorem 1.2. This completes the proof. □

References

[1] J. Aczél, *Lectures on Functional Equations and Their Applications*, Mathematics in Science and Engineering, Vol. 19 Academic Press, New York-London, 1966.

[2] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Encyclopedia of Mathematics and its Applications, 31. Cambridge University Press, Cambridge, 1989.

[3] J. A. Baker, *On a functional equation of Aczél and Chung*, Aequationes Math. **46** (1993), no. 1-2, 99–111.

[4] ———, *The stability of the cosine equation*, Proc. Amer. Math. Soc. **80** (1980), no. 3, 411–416.

[5] ———, *Functional equations, tempered distributions and Fourier transforms*, Trans. Amer. Math. Soc. **315** (1989), no. 1, 57–68.
[6] ———, *Distributional methods for functional equations*, Aequationes Math. **62** (2001), no. 1-2, 136–142.
[7] J.-Y. Chung, *A distributional version of functional equations and their stabilities*, Nonlinear Anal. **62** (2005), no. 6, 1037–1051.
[8] I. M. Gelfand and G. E. Shilov, *Generalized Functions. Vol. 2*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1968.
[9] L. Hörmander, *The Analysis of Linear Partial Differential Operators. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 256. Springer-Verlag, Berlin, 1983.
[10] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. **27** (1941), 222–224.
[11] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhauser Boston, Inc., Boston, MA, 1998.
[12] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), no. 2-3, 125–153.
[13] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Inc., Palm Harbor, FL, 2001.
[14] Th. M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), no. 1, 264–284.
[15] ———, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 297–300.
[16] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.
[17] L. Székelyhidi, *The stability of the sine and cosine functional equations*, Proc. Amer. Math. Soc. **110** (1990), no. 1, 109–115.
[18] ———, *The stability of d'Alembert type functional equations*, Acta Sci. Math. (Szeged) **44** (1982), no. 3-4, 313–320.
[19] I. Tyrala, *The stability of d'Alembert's functional equation*, Aequationes Math. **69** (2005), no. 3, 250–256.
[20] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York-London, 1960.
[21] D. V. Widder, *The Heat Equation*, Academic Press, New York, 1975.

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