Bull. Korean Math. Soc. **46** (2009), No. 1, pp. 67–69

FOOTNOTE TO A MANUSCRIPT BY GWENA AND TEIXIDOR I BIGAS

EDOARDO BALLICO AND CLAUDIO FONTANARI

ABSTRACT. Recent work by Gwena and Teixidor i Bigas provides a characteristic-free proof of a part of a previous theorem by one of us, under a stronger numerical assumption. By using an intermediate result from the mentioned manuscript, here we present a simpler, characteristic-free proof of the whole original statement.

This short note is a small fruit of the nice workshop *Moduli Spaces of Vector Bundles: Algebro-Geometric Aspects*, held in Barcelona on March, 12–14, 2008, where Montserrat Teixidor i Bigas held a stimulating lecture about her recent joint work with Tawanda Gwena [2].

In particular, Proposition 2.5 in [2] provides a characteristic-free proof of the first statement in [1, Theorem 0.1], under the stronger numerical assumption $d \geq (2r-1)g+1$.

Here we present a simpler, characteristic-free proof of both statements in [1, Theorem 0.1], using [2, Proposition 2.2].

Theorem 1. *Let E be a generic stable vector bundle of rank r and degree d on a smooth curve C of genus g. Then the natural multiplication map*

$$\mu_E : H^0(C,E) \otimes H^0(C,K) \to H^0(C, K \otimes E)$$

is injective for $d \leq rg + r$ and surjective for $d \geq rg + r$.

As a consequence, the same proof of [2, Theorem 2.4] gives us the following result (cf. [2, Theorem 2.6]).

Theorem 2. *Let E be a generic stable vector bundle of rank r and degree $d \geq rg + r$ on any smooth curve C of genus g. If the vector bundle M_E is defined by the exact sequence*

$$0 \to M_E \to H^0(C,E) \otimes \mathcal{O}_C \to E \to 0,$$

then we have $h^0(C,(M_E)^*) = h^0(C,E)$.

Finally, since trivially $E = M^*_{(M_E)^*}$, we improve the main result of [2] as follows.

Received March 28, 2008.
2000 *Mathematics Subject Classification.* 14H60.
Key words and phrases. algebraic curve, vector bundle, multiplication map.

Theorem 3. *Notation as above. If $d \geq rg + r$, then the natural map $E \mapsto M_E$ is generically injective.*

Proof of Theorem 1. Since E is general, we have $h^0(C, E) = 0$ for $d \leq r(g-1)$ and $h^1(C, E) = 0$ for $d \geq r(g-1)$. In particular, if $d \leq r(g-1)$, then μ_E is trivially injective.

Hence in order to prove the first statement we may assume $r(g-1) \leq d \leq r(g+1)$ and fix r general line bundles L_i, $i = 1, \ldots, r$, on C of degree d_i with $g - 1 \leq d_i \leq g + 1$ such that $d = \sum_{i=1}^{r} d_i$. Since L_i is general, we have $h^1(C, L_i) = 0$.

We are going to show that every μ_{L_i} is injective. Indeed, if $d_i = g - 1$, then $h^0(C, L_i) = 0$ and there is nothing to prove. Next, if $d_i = g$, then $h^0(C, L_i) = 1$ and μ_{L_i} is obviously injective. Finally, if $d_i = g + 1$, then μ_{L_i} is surjective by the base point free pencil trick (cf. [2, Proposition 2.2]), hence injective by dimension reasons.

Define the rank r vector bundle $F := \bigoplus_{i=1}^{r} L_i$ and notice that the corresponding multiplication map μ_F is injective since every μ_{L_i} is.

Obviously, F itself is *not* stable, but as every vector bundle on a smooth curve it is a flat limit of stable vector bundles by [4, Proposition 2.6] (in characteristic zero) or [3, Corollary 2.2] (in positive characteristic).

On the other hand, the injectivity of μ_{G_λ} is an open condition in flat families $\{G_\lambda\}$ of vector bundles with $h^1(G_\lambda) = 0$ and also the vanishing of h^1 is an open condition, hence our first claim is checked.

Assume now $d \geq r(g+1)$ and fix r general line bundles L_i, $i = 1, \ldots, r$, on C of degree $d_i \geq g + 1$ (in particular, L_i is non-special and spanned) such that $d = \sum_{i=1}^{r} d_i$. If $F := \bigoplus_{i=1}^{r} L_i$, then μ_F is surjective since every μ_{L_i} is by [2, Proposition 2.2]. Now we can apply the same degeneration argument as above and we are done. □

By the way, such a result raises a number of natural questions, which we list here for the interested reader. Let C be a smooth projective curve of genus $g \geq 2$ and fix an integer $r \geq 1$.

(1) What is the minimal integer $\delta_{C,r}$ such that the multiplication map μ_E is surjective for every spanned rank r vector bundle E of degree $d \geq \delta_{C,r}$ on C such that $h^1(C, E) = 0$?
(2) Is it possible to classify all spanned rank r vector bundles E of degree $d \geq \delta_{C,r} - 1$ on C such that $h^1(C, E) = 0$ and μ_E is *not* surjective?
(3) Is the answer to both questions the same for every smooth curve of genus g, with the possible exception of a few well classified (low gonality, low Clifford index) curves?

References

[1] E. Ballico, *The rank of the multiplication map for sections of bundles on curves*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. **(8) 4** (2001), no. 3, 677–683.

[2] T. Gwena and M. Teixidor i Bigas, *Maps between moduli spaces of vector bundles and the base locus of the theta divisor*, arXiv:0706.3953 (2007).
[3] A. Hirschowitz, *Probleme de Brill-Noether in rang superieur*, Prepublication Mathematiques 91, Nice, 1986.
[4] M. S. Narasimhan and S. Ramanan, *Deformations of the moduli space of vector bundles over an algebraic curve*, Ann. Math. (2) **101** (1975), 391–417.

Edoardo Ballico
Dipartimento di Matematica
Università degli Studi di Trento
Via Sommarive 14
38100 Povo, Italy
E-mail address: ballico@science.unitn.it

Claudio Fontanari
Dipartimento di Matematica
Universita' degli Studi di Trento
Via Sommarive 14
38100 Povo, Italy
E-mail address: fontanar@science.unitn.it