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STABILITY OF HOMOMORPHISMS AND DERIVATIONS IN PROPER JCQ^*-TRIPLES ASSOCIATED TO THE PEXIDERIZED CAUCHY TYPE MAPPING

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Abstract. In this paper, we investigate homomorphisms in proper JCQ^*-triples and derivations on proper JCQ^*-triples associated to the following Pexiderized functional equation

$$f(x+y+z) = f_0(x) + f_1(y) + f_2(z).$$

This is applied to investigate homomorphisms and derivations in proper JCQ^*-triples.

1. Introduction and preliminaries

Ulam [14] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

Hyers [7] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

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exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

Th. M. Rassias [13] provided a generalization of Hyers' theorem which allows the *Cauchy difference to be unbounded.*

We recall some basic facts concerning quasi $*$-algebras.

Definition 1.1. Let A be a linear space and A_0 be a $*$-algebra contained in A as a subspace. We say that A is a *quasi $*$-algebra* over A_0 if

(i) the right and left multiplications of an element of A and an element of A_0 are always defined and linear;
(ii) $x_1(x_2a) = (x_1x_2)a$, $(ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$ for all $x_1, x_2 \in A_0$ and all $a \in A$;
(iii) an involution $*$, which extends the involution of A_0, is defined in A with the property $(ab)^* = b^*a^*$, whenever the multiplication is defined.

Quasi $*$-algebras [8, 9] arise in natural way as completions of locally convex $*$-algebras whose multiplication is not jointly continuous; in this case one has to deal with topological quasi $*$-algebras.

A quasi $*$-algebra (A, A_0) is called *topological* if a locally convex topology τ on A is given such that:

(i) the involution $a \mapsto a^*$ is continuous for each $a \in A$,
(ii) the mappings $a \mapsto ab$ and $a \mapsto ba$ are continuous for each $a \in A$ and $b \in A_0$,
(iii) A_0 is dense in $A[\tau]$.

Throughout this paper, we suppose that a locally convex quasi $*$-algebra (A, A_0) is complete. For an overview on partial $*$-algebra and related topics we refer to [1].

In a series of papers [2], [3], [4], [5] many authors have considered a special class of quasi $*$-algebras, called proper CQ^*-algebras, which arise as completions of C^*-algebras. They can be introduced in the following way:

Definition 1.2. Let A be a Banach module over the C^*-algebra A_0 with involution $*$ and C^*-norm $\|\cdot\|_0$ such that $A_0 \subset A$. We say that (A, A_0) is a *proper CQ^*-algebra* if

(i) A_0 is dense in A with respect to its norm $\|\cdot\|$;
(ii) $(ab)^* = b^*a^*$ whenever the multiplication is defined;
(iii) $\|y\|_0 = \max\{\sup_{a\in A, \|a\|\leq 1} \|ay\|, \sup_{a\in A, \|a\|\leq 1} \|ya\|\}$ for all $y \in A_0$.

A proper CQ^*-algebra (A, A_0) is said to have a unit e if there exists an element $e \in A_0$ such that $ae = ea = a$ for all $a \in A$. In this paper we will always assume that the proper CQ^*-algebra under consideration have an identity.

Definition 1.3. A proper CQ^*-algebra (A, A_0), endowed with the Jordan triple product

$$\{z, x, w\} = \frac{1}{2}\{zx^*w + wx^*z\}$$

for all $x \in A$ and all $z, w \in A_0$, is *called a proper JCQ^*-triple*, and denoted by $(A, A_0, \{.,.,.\})$.

Definition 1.4. Let $(A, A_0, \{.,.,.\})$ and $(B, B_0, \{.,.,.\})$ be proper JCQ^*-triples.

(i) A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *proper JCQ^*-triple homomorphism* if $H(z) \in B_0$ and $H(\{z, x, w\}) = \{H(z), H(x), H(w)\}$ for all $z, w \in A_0$ and all $x \in A$.

(ii) A $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ is called a *proper JCQ^*-triple derivation* if

$$\delta(\{w_0, w_1, w_2\}) = \{\delta(w_0), w_1, w_2\} + \{w_0, \delta(w_1), w_2\} + \{w_0, w_1, \delta(w_2)\}$$

for all $w_0, w_1, w_2 \in A_0$.

A. Najati and C. Park [10] investigated homomorphisms in quasi-Banach algebras associated to the Pexiderrized Cauchy function equation. C. Park and Th. M. Rassias [12] investigated homomorphisms in proper JCQ^*-triples and derivations on proper JCQ^*-triples.

In this paper, we investigate homomorphisms and derivations in proper JCQ^*-triples associated to the following Pexiderized Cauchy type functional equation

$$f(x + y + z) = f_0(x) + f_1(y) + f_2(z).$$

Throughout this paper, assume that k is a fixed positive integer.

2. Homomorphisms in proper JCQ^*-triples

Throughout this section, assume that $(A, A_0, \{.,.,.\})$ is a proper JCQ^*-triple with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$, and that $(B, B_0, \{.,.,.\})$ is a proper JCQ^*-triple with C^*-norm $\|\cdot\|_{B_0}$ and norm $\|\cdot\|_B$.

Theorem 2.1. *Let $\varphi : A \times A \times A \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n w_0, x, w_2) = 0 \tag{2.1}$$

for all $w_0, w_2 \in A_0$ and all $x \in A$. Assume that $f, f_i : A \to B$ $(0 \le i \le 2)$ are mappings with $f(0) = 0$ and $f(w), f_0(0), f_2(0) \in B_0$ for all $w \in A_0$ and

$$\|\mu f(x) - f_0(y) - f_1(z) - f_2(t)\|_B \le \left\|kf\left(\frac{\mu x + y + z + t}{k}\right)\right\|_B, \tag{2.2}$$

$$\|f(\{w_0, x, w_2\}) + \{f_0(w_0), f_1(x), f_2(w_2)\}\|_B \le \varphi(w_0, x, w_2) \tag{2.3}$$

for all $\mu \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$, all $w_0, w_2 \in A_0$ and all $x, y, z, t \in A$. Then the mapping $f : A \to B$ is a proper JCQ^*-triple homomorphism. Moreover,

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x) = f_2(0) - f_2(x)$$

for all $x \in A$.

Proof. Letting $\mu = 1$, $x = y = z = t = 0$ in (2.2), we get

$$f_0(0) + f_1(0) + f_2(0) = 0.$$

So by letting $\mu = 1$, $y = -x$, and $z = t = 0$ in (2.2), we get

$$f(x) = f_0(-x) - f_0(0)$$

for all $x \in A$. Similarly, we have

$$f(x) = f_1(-x) - f_1(0) = f_2(-x) - f_2(0)$$

for all $x \in A$. So $f_0(w), f_2(w) \in B_0$ for all $w \in A_0$.

It follows from (2.2) that

$$\begin{aligned}&\|\mu f(x+y) - f(\mu x) - f(\mu y)\|_B \\ &= \|\mu f(x+y) - f_0(-\mu x) - f_1(-\mu y) - f_2(0)\|_B = 0\end{aligned}$$

for all $x, y \in A$ and all $\mu \in \mathbb{T}^1$. Therefore, the mapping $f : A \to B$ is additive and $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{T}^1$. By the same reasoning as in the proof of Theorem 2.1 of [11], the mapping $f : A \to B$ is $\mathbb{C}$-linear. If at least $w_0 = 0$ or $x = 0$ or $w_2 = 0$, then $f(\{w_0, x, w_2\}) = f(0) = 0$, and hence by (2.1) and (2.3), we have

$$\lim_{n\to\infty} \frac{1}{2^n} \big\|\{f_0(2^n w_0), f_1(x), f_2(w_2)\}\big\|_B = 0.$$

Also, we have

$$f(x) = f_i(0) - f_i(x)$$

for all $x \in A$ and all $0 \le i \le 2$. So (2.3) implies that

$$\begin{aligned}&\big\|f(\{w_0, x, w_2\}) - \{f(w_0), f(x), f(w_2)\}\big\|_B \\ &= \lim_{n\to\infty} \frac{1}{2^n}\big\|f(2^n\{w_0, x, w_2\}) - \{f(2^n w_0), f(x), f(w_2)\}\big\|_B \\ &= \lim_{n\to\infty} \frac{1}{2^n}\big\|f(2^n\{w_0, x, w_2\}) + \{f_0(2^n w_0), f_1(x), f_2(w_2)\}\big\|_B \\ &\le \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n w_0, x, w_2) = 0\end{aligned}$$

for all $w_0, w_2 \in A_0$ and all $x \in A$. So

$$f(\{w_0, x, w_2\}) = \{f(w_0), f(x), f(w_2)\}$$

for all $w_0, w_2 \in A_0$ and all $x \in A$. Since $f(w) \in B_0$ for all $w \in A_0$, the mapping $f : A \to B$ is a proper JCQ^*-triple homomorphism, as desired. □

Remark 2.2. We can formulate a similar theorem if we replace the condition (2.1) by one of the following conditions

- $\lim_{n\to\infty} \frac{1}{2^n}\varphi(w_0, 2^n x, w_2) = 0$;
- $\lim_{n\to\infty} \frac{1}{2^n}\varphi(w_0, x, 2^n w_2) = 0$;
- $\lim_{n\to\infty} \frac{1}{8^n}\varphi(2^n w_0, 2^n x, 2^n w_2) = 0$

for all $w_0, w_2 \in A_0$ and all $x \in A$.

Remark 2.3. We can formulate a similar theorem if we replace the condition (2.1) by one of the following conditions

- $\lim_{n\to\infty} 2^n \varphi(\frac{w_0}{2^n}, x, w_2) = 0$;
- $\lim_{n\to\infty} 2^n \varphi(w_0, \frac{x}{2^n}, w_2) = 0$;
- $\lim_{n\to\infty} 2^n \varphi(w_0, x, \frac{w_2}{2^n}) = 0$;
- $\lim_{n\to\infty} 8^n \varphi(\frac{w_0}{2^n}, \frac{x}{2^n}, \frac{w_2}{2^n}) = 0$

for all $w_0, w_2 \in A_0$ and all $x \in A$.

Corollary 2.4. *Let θ, r_i $(0 \le i \le 2)$ be non-negative real numbers such that $r_0 + r_1 + r_2 \ne 3$ or $r_j \ne 1$ for some $0 \le j \le 2$. Suppose that $f, f_i : A \to B$ $(0 \le i \le 2)$ are mappings satisfying (2.2) with $f(0) = 0$ and $f(w), f_0(0), f_2(0) \in B_0$ for all $w \in A_0$. Let*

$$\|f(\{w_0, x, w_2\}) + \{f_0(w_0), f_1(x), f_2(w_2)\}\|_B \le \theta \|w_0\|_A^{r_0} \|x\|_A^{r_1} \|w_2\|_A^{r_2}$$

for all $w_0, w_2 \in A_0$ and all $x \in A$ (by putting $\|\cdot\|_A^0 = 1$). Then the mapping $f : A \to B$ is a proper JCQ^*-triple homomorphism. Moreover,

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x) = f_2(0) - f_2(x)$$

for all $x \in A$.

Proof. It follows from Theorem 2.1 and Remarks 2.2 and 2.3. □

Corollary 2.5. *Let θ, r_i $(0 \le i \le 2)$ be non-negative real numbers such that $r_j \in [0, 1)$ for some $0 \le j \le 2$ or $r_i < 3$ (respectively, $r_i > 3$) for all $0 \le i \le 2$. Suppose that $f, f_i : A \to B$ $(0 \le i \le 2)$ are mappings satisfying (2.2) with $f(0) = 0$ and $f(w), f_0(0), f_2(0) \in B_0$ for all $w \in A_0$. Let*

$$\|f(\{w_0, x, w_2\}) + \{f_0(w_0), f_1(x), f_2(w_2)\}\|_B \le \theta\big(\|w_0\|_A^{r_0} + \|x\|_A^{r_1} + \|w_2\|_A^{r_2}\big)$$

for all $w_0, w_2 \in A_0$ and all $x \in A$ (by putting $\|\cdot\|_A^0 = 1$). Then the mapping $f : A \to B$ is a proper JCQ^*-triple homomorphism. Moreover,

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x) = f_2(0) - f_2(x)$$

for all $x \in A$.

Proof. The result follows from Theorem 2.1 and Remarks 2.2 and 2.3. □

3. Derivations on proper JCQ^*-triples

Throughout this section, assume that$(A, A_0, \{.,.,.\})$ is a proper JCQ^*-triple with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$.

We investigate derivations on proper JCQ^*-triples.

Theorem 3.1. *Let $\varphi : A_0 \times A_0 \times A_0 \to [0, +\infty)$ be a function satisfying (2.1) for all $x, w_0, w_2 \in A_0$. Assume that $f, f_i : A_0 \to A$ $(0 \le i \le 2)$ are mappings with $f(0) = 0$. Let*

$$\|\mu f(x) - f_0(w_0) - f_1(w_1) - f_2(w_2)\|_A \le \Big\|kf\big(\frac{\mu x + w_0 + w_1 + w_2}{k}\big)\Big\|_A, \tag{3.1}$$

$$\begin{aligned}(3.2)\qquad &\Big\|f(\{w_0,w_1,w_2\})+\{f_0(w_0),w_1,w_2\}+\{w_0,f_1(w_1),w_2\}\\ &\qquad+\{w_0,w_1,f_2(w_2)\}\Big\|_A\le\varphi(w_0,w_1,w_2)\end{aligned}$$

for all $x, w_0, w_1, w_2 \in A_0$. *Then the mapping* $f : A_0 \to A$ *is a proper* JCQ^*-*triple derivation. Moreover,*

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x) = f_2(0) - f_2(x)$$

for all $x \in A_0$.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A_0 \to A$ is $\mathbb{C}$-linear and

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x) = f_2(0) - f_2(x)$$

for all $x \in A_0$. If at least $w_0 = 0$ or $w_1 = 0$ or $w_2 = 0$, then $f(\{w_0, w_1, w_2\}) = f(0) = 0$ and by (2.1) and (3.2), we have

$$\lim_{n\to\infty}\frac{1}{2^n}\Big\|\{f_0(2^n w_0),w_1,w_2\}+\{2^n w_0,f_1(w_1),w_2\}+\{2^n w_0,w_1,f_2(w_2)\}\Big\|_A=0.$$

So (3.2) implies that

$$\begin{aligned}&\big\|f(\{w_0,w_1,w_2\})-\{f(w_0),w_1,w_2\}-\{w_0,f(w_1),w_2\}-\{w_0,w_1,f(w_2)\}\big\|_A\\ &=\lim_{n\to\infty}\frac{1}{2^n}\Big\|f(2^n\{w_0,w_1,w_2\})+\{f_0(2^n w_0),w_1,w_2\}+\{2^n w_0,f_1(w_1),w_2\}\\ &\qquad+\{2^n w_0,w_1,f_2(w_2)\}\Big\|_A\\ &\le\lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n w_0,w_1,w_2)=0\end{aligned}$$

for all $w_0, w_1, w_2 \in A_0$. Hence

$$f(\{w_0,w_1,w_2\})=\{f(w_0),w_1,w_2\}+\{w_0,f(w_1),w_2\}+\{w_0,w_1,f(w_2)\}$$

for all $w_0, w_1, w_2 \in A_0$.

Therefore the mapping $f : A_0 \to A$ is a proper JCQ^*-triple derivation. □

Remark 3.2. We can formulate a similar theorem if we replace the condition (2.1) by one of the following conditions

- $\lim_{n\to\infty}\frac{1}{2^n}\varphi(w_0, 2^n w_1, w_2) = 0$;
- $\lim_{n\to\infty}\frac{1}{2^n}\varphi(w_0, w_1, 2^n w_2) = 0$;
- $\lim_{n\to\infty}\frac{1}{8^n}\varphi(2^n w_0, 2^n w_1, 2^n w_2) = 0$

for all $w_0, w_1, w_2 \in A_0$.

Remark 3.3. We can formulate a similar theorem if we replace the condition (2.1) by one of the following conditions

- $\lim_{n\to\infty} 2^n\varphi(\frac{w_0}{2^n}, w_1, w_2) = 0$;
- $\lim_{n\to\infty} 2^n\varphi(w_0, \frac{w_1}{2^n}, w_2) = 0$;
- $\lim_{n\to\infty} 2^n\varphi(w_0, w_1, \frac{w_2}{2^n}) = 0$;
- $\lim_{n\to\infty} 8^n\varphi(\frac{w_0}{2^n}, \frac{w_1}{2^n}, \frac{w_2}{2^n}) = 0$

for all $w_0, w_1, w_2 \in A_0$.

Corollary 3.4. *Let θ, r_i $(0 \le i \le 2)$ be non-negative real numbers such that $r_0 + r_1 + r_2 \neq 3$ or $r_j \neq 1$ for some $0 \le j \le 2$. Suppose that $f, f_i : A_0 \to A$ $(0 \le i \le 2)$ are mappings satisfying (3.1) with $f(0) = 0$. Let*

$$\big\|f(\{w_0, w_1, w_2\}) + \{f_0(w_0), w_1, w_2\} + \{w_0, f_1(w_1), w_2\} + \{w_0, w_1, f_2(w_2)\}\big\|_A \le \theta \|w_0\|_{A_0}^{r_0} \|w_1\|_{A_0}^{r_1} \|w_2\|_{A_0}^{r_2}$$

for all $w_0, w_1, w_2 \in A_0$ (by putting $\|\cdot\|_{A_0}^0 = 1$). Then the mapping $f : A_0 \to A$ is a proper JCQ^*-triple derivation. Moreover,

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x) = f_2(0) - f_2(x)$$

for all $x \in A$.

Proof. It follows from Theorem 3.1 and Remarks 3.2 and 3.3. □

Corollary 3.5. *Let θ, r_i $(0 \le i \le 2)$ be non-negative real numbers such that $r_j \in [0, 1)$ for some $0 \le j \le 2$ or $r_i < 3$ (respectively, $r_i > 3$) for all $0 \le i \le 2$. Suppose that $f, f_i : A_0 \to A$ $(0 \le i \le 2)$ are mappings satisfying (3.1) with $f(0) = 0$. Let*

$$\big\|f(\{w_0, w_1, w_2\}) + \{f_0(w_0), w_1, w_2\} + \{w_0, f_1(w_1), w_2\} + \{w_0, w_1, f_2(w_2)\}\big\|_A \le \theta\big(\|w_0\|_{A_0}^{r_0} + \|w_1\|_{A_0}^{r_1} + \|w_2\|_{A_0}^{r_2}\big)$$

for all $w_0, w_1, w_2 \in A_0$ (by putting $\|\cdot\|_{A_0}^0 = 1$). Then the mapping $f : A_0 \to A$ is a proper JCQ^*-triple derivation. Moreover,

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x) = f_2(0) - f_2(x)$$

for all $x \in A_0$.

Proof. It follows from Theorem 3.1 and Remarks 3.2 and 3.3. □

4. Stability of homomorphisms on proper JCQ^*-triples

In this section, by using an idea of Găvruta [6], we prove the generalized Hyers-Ulam stability of homomorphisms in proper JCQ^*-triples.

Theorem 4.1. *Let $\varphi : A \times A \times A \to [0, +\infty)$ be a function such that $\varphi(0, 0, 0) = 0$ and*

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) = 0, \tag{4.1}$$

$$\widetilde{\varphi}(x) := \sum_{i=0}^{\infty} \frac{1}{2^i}\Big[\varphi(2^i x, 2^i x, 0) + \varphi(2^i x, 0, 0) + \varphi(0, 2^i x, 0)\Big] < \infty \tag{4.2}$$

for all $x, y, z \in A$. Suppose that $f, f_i : A \to B$ $(0 \le i \le 2)$ are mappings satisfying $f(0) = 0$ and $f(w), f_i(w) \in B_0$ for all $w \in A_0$ and all $0 \le i \le 2$. Let

$$\|f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z)\|_B \le \varphi(x, y, z), \tag{4.3}$$

$$\|f(w_0 + w_1 + w_2) - f_0(w_0) - f_1(w_1) - f_2(w_2)\|_{B_0} \le \varphi(w_0, w_1, w_2), \tag{4.4}$$

$$\|f(\{w_0,x,w_1\})-\{f_0(w_0),f_1(x),f_2(w_2)\}\|_B\le\varphi(w_0,x,w_2) \tag{4.5}$$

for all $\mu\in\mathbb{T}^1$, all $w_0,w_1,w_2\in A_0$ and all $x,y,z\in A$. Then there exists a unique proper JCQ^*-triple homomorphism $H:A\to B$ such that

$$\begin{aligned}\|f(x)-H(x)\|_B&\le\frac{1}{2}\widetilde{\varphi}(x),\\ \|f_0(x)-f_0(0)-H(x)\|_B&\le\frac{1}{2}\widetilde{\varphi}(x)+\varphi(x,0,0),\\ \|f_1(x)-f_1(0)-H(x)\|_B&\le\frac{1}{2}\widetilde{\varphi}(x)+\varphi(0,x,0),\\ \|f_2(x)-f_2(0)-H(x)\|_B&\le\frac{1}{2}\widetilde{\varphi}(x)+\varphi(0,0,x)\end{aligned} \tag{4.6}$$

for all $x\in A$.

Proof. Letting $y=z=0$ and $\mu=1$ in (4.3), we get

$$\|f(x)-f_0(x)-f_1(0)-f_2(0)\|_B\le\varphi(x,0,0) \tag{4.7}$$

for all $x\in A$. Similarly, we get

$$\|f(y)-f_1(y)-f_0(0)-f_2(0)\|_B\le\varphi(0,y,0), \tag{4.8}$$

$$\|f(z)-f_2(z)-f_0(0)-f_1(0)\|_B\le\varphi(0,0,z) \tag{4.9}$$

for all $y,z\in A$. Since $f_0(0)+f_1(0)+f_2(0)=0$, by using (4.3), (4.7), (4.8) and (4.9), we get

$$\|f(\mu x+\mu y+\mu z)-\mu f(x)-\mu f(y)-\mu f(z)\|_B\le\psi(x,y,z), \tag{4.10}$$

where

$$\psi(x,y,z):=\varphi(x,y,z)+\varphi(x,0,0)+\varphi(0,y,0)+\varphi(0,0,z)$$

for all $\mu\in\mathbb{T}^1$ and all $x,y,z\in A$. Letting $y=x,z=0$ and $\mu=1$ in (4.10), we get

$$\|f(2x)-2f(x)\|_B\le\psi(x,x,0) \tag{4.11}$$

for all $x\in A$. Replacing x by 2^nx in (4.11) and dividing both sides of (4.11) by 2^{n+1}, we get

$$\Big\|\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^nx)}{2^n}\Big\|_B\le\frac{1}{2^{n+1}}\psi(2^nx,2^nx,0) \tag{4.12}$$

for all $x\in A$ and all non-negative integers n. By (4.12), we have

$$\Big\|\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^mx)}{2^m}\Big\|_B\le\frac{1}{2}\sum_{i=m}^{n}\frac{\psi(2^ix,2^ix,0)}{2^i} \tag{4.13}$$

for all $x\in A$ and all non-negative integers n and m with $n\ge m$. Thus we conclude from (4.2) and (4.13) that the sequence $\{\frac{1}{2^n}f(2^nx)\}$ is a Cauchy sequence in B for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges in B for all $x \in A$. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = \lim_{n\to\infty} \frac{1}{2^n} f_i(2^n x) \quad (i = 0, 1, 2) \tag{4.14}$$

for all $x \in A$. Letting $m = 0$ and passing the limit when $n \to \infty$ in (4.13), we get (4.6). It follows from (4.1), (4.3) and (4.14) that

$$\begin{aligned} &\Big\|H(\mu x + \mu y) - \mu H(x) - \mu H(y)\Big\|_B \\ &= \lim_{n\to\infty} \frac{1}{2^n}\Big\|f(2^n \mu x + 2^n \mu y) - \mu f_0(2^n x) - \mu f_1(2^n y)\Big\|_B \\ &\le \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y, 0) = 0 \end{aligned}$$

for all $\mu \in \mathbb{T}^1$and all $x, y \in A$. Hence

$$H(\mu x + \mu y) = \mu H(x) + \mu H(y)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. By the same reasoning as in the proof of Theorem 2.1 of [11], the mapping $H : A \to B$ is $\mathbb{C}$-linear. It follows from (4.4) that the sequence $\{\frac{1}{2^n}f(2^n w)\}$ is a Cauchy sequence in B_0 for all $w \in A_0$. So $H(w) \in B_0$ for all $w \in A_0$. It follows from (4.1), (4.5) and (4.14) that

$$\begin{aligned} &\|H(\{w_0, x, w_2\}) - \{H(w_0), H(x), H(w_2)\}\|_B \\ &\le \lim_{n\to\infty} \frac{1}{8^n}\Big\|f(8^n\{w_0, x, w_2\}) - \{f_0(2^n w_0), f_1(2^n x), f_2(2^n w_2)\}\Big\|_B \\ &\le \lim_{n\to\infty} \frac{1}{8^n}\varphi(2^n w_0, 2^n x, 2^n w_2) = 0 \end{aligned}$$

for all $w_0, w_2 \in A_0$ and all $x \in A$. Hence

$$H(\{w_0, x, w_1\}) = \{H(w_0), H(x), H(w_2)\}$$

for all $w_0, w_2 \in A_0$ and all $x \in A$. So $H : A \to B$ is a proper JCQ^*-triple homomorphism. Now, we show that H is unique. Let $T : A \to B$ be another proper JCQ^*-triple homomorphism satisfying (4.6). It follows from (4.2), (4.6) and (4.14) that

$$\begin{aligned} \|H(x) - T(x)\|_B &= \lim_{n\to\infty} \frac{1}{2^n}\|f(2^n x) - T(2^n x)\|_B \\ &\le \frac{1}{2}\lim_{n\to\infty} \frac{1}{2^n}\widetilde{\varphi}(2^n x) = 0 \end{aligned}$$

for all $x \in A$. So $H = T$. □

Theorem 4.2. *Let $\phi : A \times A \times A \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} 8^n \phi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) = 0, \tag{4.15}$$

$$\widetilde{\phi}(x) := \sum_{i=1}^{\infty} 2^i \Big[\phi(\frac{x}{2^i}, \frac{x}{2^i}, 0) + \phi(\frac{x}{2^i}, 0, 0) + \phi(0, \frac{x}{2^i}, 0)\Big] < \infty \tag{4.16}$$

for all $x, y, z \in A$. Suppose that $f, f_i : A \to B$ $(0 \le i \le 2)$ are mappings satisfying $f(0) = f_i(0) = 0$ and $f(w), f_i(w) \in B_0$ for all $w \in A_0$ and all $0 \le i \le 2$. Let

$$\|f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z)\|_B \le \phi(x, y, z), \tag{4.17}$$

$$\begin{aligned} &\|f(w_0 + w_1 + w_2) - f_0(w_0) - f_1(w_1) - f_2(w_2)\|_{B_0} \le \phi(w_0, w_1, w_2), \\ &\|f(\{w_0, x, w_1\}) - \{f_0(w_0), f_1(x), f_2(w_2)\}\|_B \le \phi(w_0, x, w_2) \end{aligned} \tag{4.18}$$

for all $\mu \in \mathbb{T}^1$, all $w_0, w_1, w_2 \in A_0$ and all $x, y, z \in A$. Then there exists a unique proper JCQ^*-triple homomorphism $H : A \to B$ such that

$$\begin{aligned} &\|f(x) - H(x)\|_B \le \frac{1}{2}\widetilde{\phi}(x), \\ &\|f_0(x) - H(x)\|_B \le \frac{1}{2}\widetilde{\phi}(x) + \phi(x, 0, 0), \\ &\|f_1(x) - H(x)\|_B \le \frac{1}{2}\widetilde{\phi}(x) + \phi(0, x, 0), \\ &\|f_2(x) - H(x)\|_B \le \frac{1}{2}\widetilde{\phi}(x) + \phi(0, 0, x) \end{aligned} \tag{4.19}$$

for all $x \in A$.

Proof. Similar to Theorem 4.1, we get

$$\|f(\mu x + \mu y + \mu z) - \mu f(x) - \mu f(y) - \mu f(z)\|_B \le \Psi(x, y, z), \tag{4.20}$$

where

$$\Psi(x, y, z) := \phi(x, y, z) + \phi(x, 0, 0) + \phi(0, y, 0) + \phi(0, 0, z)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Letting $y = x, z = 0$ and $\mu = 1$ in (4.20), we get

$$\|f(2x) - 2f(x)\|_B \le \Psi(x, x, 0) \tag{4.21}$$

for all $x \in A$. Replacing x by $\frac{x}{2^{n+1}}$ in (4.21) and multiplying both sides of (4.21) to 2^n, we get

$$\left\|2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right)\right\|_B \le 2^n \Psi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, 0\right) \tag{4.22}$$

for all $x \in A$ and all non-negative integers n. By (4.22), we get

$$\left\|2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_B \le \sum_{i=m}^{n} 2^i \Psi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, 0\right) \tag{4.23}$$

for all $x \in A$ and all non-negative integers n and m with $n \ge m$. Thus we conclude from (4.16) and (4.23) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in B for all $x \in A$. Since B is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges in B for all $x \in A$. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = \lim_{n\to\infty} 2^n f_i\left(\frac{x}{2^n}\right) \quad (i = 0, 1, 2)$$

for all $x \in A$. Letting $m = 0$ and passing the limit when $n \to \infty$ in (4.23), we get (4.19).

The rest of proof is similar to the proof of Theorem 4.1. □

Corollary 4.3. *Let θ, r_i $(0 \le i \le 2)$ be non-negative real numbers such that $0 < r_i < 1$ (respectively, $r_i > 3$) for all $0 \le i \le 2$. Suppose that $f, f_i : A \to B$ $(0 \le i \le 2)$ are mappings with $f(0) = f_i(0) = 0$ and $f(w), f_i(w) \in B_0$ for all $w \in A_0$ and all $0 \le i \le 2$. Let*

$$\begin{aligned}&\|f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z)\|_B \\ \le\ & \theta(\|x\|_A^{r_0} + \|y\|_A^{r_1} + \|z\|_A^{r_2}), \\ &\|f(w_0 + w_1 + w_2) - f_0(w_0) - f_1(w_1) - f_2(w_2)\|_{B_0} \\ \le\ & \theta(\|w_0\|_A^{r_0} + \|w_1\|_A^{r_1} + \|w_2\|_A^{r_2}), \\ &\|f(\{w_0, x, w_1\}) - \{f_0(w_0), f_1(x), f_2(w_2)\}\|_B \\ \le\ & \theta(\|w_0\|_A^{r_0} + \|x\|_A^{r_1} + \|w_2\|_A^{r_2})\end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $w_0, w_1, w_2 \in A_0$ and all $x, y, z \in A$. Then there exists a unique proper JCQ^*-triple homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le 2\theta\Big[\frac{\|x\|_A^{r_0}}{|2 - 2^{r_0}|} + \frac{\|x\|_A^{r_1}}{|2 - 2^{r_1}|}\Big],$$

$$\|f_i(x) - H(x)\|_B \le 2\theta\Big[\frac{\|x\|_A^{r_0}}{|2 - 2^{r_0}|} + \frac{\|x\|_A^{r_1}}{|2 - 2^{r_1}|}\Big] + \theta\|x\|_A^{r_i}$$

for all $x \in A$ and all $0 \le i \le 2$.

Theorem 4.4. *Let θ, r_i $(0 \le i \le 2)$ be non-negative real numbers such that $r_0 + r_1 + r_2 < 3$ and $0 < r_i < 1$ for some $0 \le i \le 2$. Suppose that $f, f_i : A \to B$ $(0 \le i \le 2)$ are mappings with $f(0) = 0$ and $f(w), f_0(w), f_2(w) \in B_0$ for all $w \in A_0$. Let*

$$\|f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z)\|_B \le \theta\|x\|_A^{r_0}\|y\|_A^{r_1}\|z\|_A^{r_2}, \tag{4.24}$$

$$\|f(\{w_0, x, w_2\}) - \{f_0(w_0), f_1(x), f_2(w_2)\}\|_B \le \theta\|w_0\|_A^{r_0}\|x\|_A^{r_1}\|w_2\|_A^{r_2} \tag{4.25}$$

for all $\mu \in \mathbb{T}^1$, all $w_0, w_2 \in A_0$ and all $x, y, z \in A$ (by putting $\|\cdot\|_A^0 = 1$). Then the mapping $f : A \to B$ is a proper JCQ^*-triple homomorphism. Moreover, if $r_i, r_j > 0$ for some $0 \le i < j \le 2$, then

$$f(x) = f_0(x) - f_0(0) = f_1(x) - f_1(0) = f_2(x) - f_2(0)$$

for all $x \in A$.

Proof. Without loss of generality, we may assume that $0 < r_2 < 1$. It is clear that $f_0(0) + f_1(0) + f_2(0) = 0$. By letting $y = z = 0$ and $\mu = 1$ in (4.24), we get

$$f(x) = f_0(x) - f_0(0) \tag{4.26}$$

for all $x \in A$. Similarly, we have

$$f(x) = f_1(x) - f_1(0) \tag{4.27}$$

for all $x \in A$. We have two cases:

Case I. $r_0 = r_1 = 0$. We infer from (4.24) that

$$\begin{aligned}
&\|f(\mu x + \mu y + \mu z) - \mu f(x) - \mu f(y) - \mu f(z)\|_B \\
&\leq \|f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z)\|_B \\
&\quad + \|f(z) - f_0(0) - f_1(0) - f_2(z)\|_B \leq 2\theta \|z\|_A^{r_2}
\end{aligned} \tag{4.28}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. By letting $z = 0$ in (4.28), we get

$$f(\mu x + \mu y) = \mu f(x) + \mu f(y)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. By the same reasoning as in the proof of Theorem 2.1 of [11], the mapping $f : A \to B$ is $\mathbb{C}$-linear. It follows from (4.24), (4.26), and (4.27) that

$$f(x) = \lim_{n\to\infty} \frac{1}{2^n} f_0(2^n x) = \lim_{n\to\infty} \frac{1}{2^n} f_1(2^n x) = \lim_{n\to\infty} \frac{1}{2^n} f_2(2^n x)$$

for all $x \in A$. So (4.25) implies that

$$\begin{aligned}
&\left\|f(\{w_0, x, w_2\}) - \{f(w_0), f(x), f(w_2)\}\right\|_B \\
&= \lim_{n\to\infty} \frac{1}{8^n} \left\|f(\{2^n w_0, 2^n x, 2^n w_2\}) - \{f_0(2^n w_0), f_1(2^n x), f_2(2^n w_2)\}\right\|_B \\
&\leq \theta \lim_{n\to\infty} \frac{2^{nr_2}}{8^n} \|w_2\|_A^{r_2} = 0
\end{aligned}$$

for all $w_0, w_2 \in A_0$ and all $x \in A$. Therefore,

$$f(\{w_0, x, w_1\}) = \{f(w_0), f(x), f(w_2)\}$$

for all $w_0, w_2 \in A_0$ and $x \in A$. So the mapping $f : A \to B$ is a proper JCQ^*-triple homomorphism.

Case II. $r_0 > 0$ or $r_1 > 0$. Without loss of generality, we may assume that $r_1 > 0$. Letting $x = y = 0$ and $\mu = 1$ in (4.24), we get that $f(z) = f_2(z) - f_2(0)$ for all $z \in A$. It follows from (4.24) that

$$\begin{aligned}
&\|f(\mu x + \mu y + \mu z) - \mu f(x) - \mu f(y) - \mu f(z)\|_B \\
&= \|f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z)\|_B \leq \theta \|x\|_A^{r_0} \|y\|_A^{r_1} \|z\|_A^{r_2}
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. By putting $z = 0$ in the last inequality, we infer that the mapping f is $\mathbb{C}$-linear. The rest of the proof is similar to the proof of Case I. □

The following theorem is an alternative result of Theorem 4.4 and its proof is similar to the proof of Theorem 4.4.

Theorem 4.5. *Let θ, r_i $(0 \leq i \leq 2)$ be non-negative real numbers such that $r_i > 3$ for some $0 \leq i \leq 2$. Suppose that $f, f_i : A \to B$ $(0 \leq i \leq 2)$ are mappings satisfying* (4.24) *and* (4.25) *(by putting $\|\cdot\|_A^0 = 1$) with $f(0) = f_i(0) = 0$ and $f(w), f_0(w), f_2(w) \in B_0$ for all $w \in A_0$. Then the mapping $f : A \to B$*

is a proper JCQ^*-triple homomorphism. Moreover, if $r_i, r_j > 0$ for some $0 \le i < j \le 2$, then

$$f(x) = f_i(x)$$

for all $x \in A$ and all $0 \le i \le 2$.

For $r_0 = r_1 = r_2 = 0$, we have the following theorem.

Theorem 4.6. *Let θ be non-negative real number and let $f, f_i : A \to B$ $(0 \le i \le 2)$ be mappings such that $f(w), f_i(w) \in B_0$ $(0 \le i \le 2)$ for all $w \in A_0$ and*

$$\|f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z)\|_B \le \theta,$$
$$\|f(w_0 + w_1 + w_2) - f_0(w_0) - f_1(w_1) - f_2(w_2)\|_{B_0} \le \theta,$$
$$\|f(\{w_0, x, w_1\}) - \{f_0(w_0), f_1(x), f_2(w_2)\}\|_B \le \theta$$

for all $\mu \in \mathbb{T}^1$, all $w_0, w_1, w_2 \in A_0$ and all $x, y, z \in A$. Then there exists a unique proper JCQ^*-triple homomorphism $H : A \to B$ such that

$$\|f(x) + f(0) - H(x)\|_B \le 4\theta + 2M,$$
$$\|f_i(x) - f_i(0) - H(x)\|_B \le 6\theta + 4M \qquad (i = 0, 1, 2)$$

for all $x \in A$, where $M = \|f_0(0) + f_1(0) + f_2(0)\|_B$.

Proof. Similar to the proof of Theorem 4.1, we have

$$\|f(\mu x + \mu y + \mu z) - \mu f(x) - \mu f(y) - \mu f(z)\|_B \le 4\theta + 2M$$

for all $x, y, z \in A$ and all $\mu \in \mathbb{T}^1$, where $M = \|f_0(0) + f_1(0) + f_2(0)\|_B$. Using the same proof as in Theorem 4.1, we infer that

$$\left\|\frac{1}{2^{n+1}} f(2^{n+1}x) - \frac{1}{2^m} f(2^m x) - \sum_{i=m}^{n} \frac{1}{2^{i+1}} f(0)\right\|_B \le (2\theta + M) \sum_{i=m}^{n} \frac{1}{2^i} \tag{4.29}$$

for all $x \in A$ and all non-negative integers n and m with $n \ge m$. Thus we conclude from (4.29) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence in B for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges in B for all $x \in A$. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = \lim_{n\to\infty} \frac{1}{2^n} f_i(2^n x) \quad (i = 0, 1, 2)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 4.1. □

Theorem 4.7. *Let $\theta \ge 0, r_0, r_1, r_2$ be real numbers such that $r_0 + r_1 > 0$ and $r_2 < 0$. Assume that $f, f_i : A \to B$ $(0 \le i \le 2)$ are mappings with $f_0(0) = f_1(0) = 0$ and $f(w), f_0(w), f_2(w) \in B_0$ for all $w \in A_0$ and satisfying*

$$\|f(x + y + z) - f_0(x) - f_1(y) - f_2(z)\|_B \le \theta \|x\|_A^{r_0} \|y\|_A^{r_1} \|z\|_A^{r_2}, \tag{4.30}$$
$$\|f(\{w_0, x, w_2\}) - \{f_0(w_0), f_1(x), f_2(w_2)\}\|_B \le \theta \|w_0\|_A^{r_0} \|x\|_A^{r_1} \|w_2\|_A^{r_2} \tag{4.31}$$

for all $w_2 \in A_0 \setminus \{0\}$, all $w_0 \in A_0$ ($w_0 \in A_0 \setminus \{0\}$ if $r_0 < 0$) and all $x, y \in A$ ($x \in A \setminus \{0\}$ if $r_0 < 0$ and $y \in A \setminus \{0\}$ if $r_1 < 0$), $z \in A \setminus \{0\}$. If the *mappings* $t \to f(tx)$ *and* $t \to f_i(tx)$ $(0 \le i \le 2)$ *are continuous in* $0 \in \mathbb{R}$ *for each fixed* $x \in A$, *then*

(i) $f = f_0 = f_1 = f_2$,

(ii) *the mapping* $f : A \to B$ *is a proper* JCQ^**-triple homomorphism.*

Proof. Without loss of generality, we may assume that $r_1 > 0$. Letting $y = 0$ in (4.30) and $x = 0$ in (4.31), we get

$$f(0) = 0, \qquad f(x+z) = f_0(x) + f_2(z), \qquad (x, z \neq 0). \tag{4.32}$$

Replacing x and z by $\frac{x}{n}$ and $\frac{z}{n}$, respectively, in (4.32) and letting $n \to \infty$, we get that $f_2(0) = 0$. Letting $y = -x$ in (4.30) and using (4.32), we get

$$\|f(z) - f(x+z) - f_1(-x)\|_B \le \theta \|x\|_A^{r_0+r_1} \|z\|_A^{r_2}, \qquad (x, z \neq 0). \tag{4.33}$$

Therefore

$$\lim_{n\to\infty} f\big(\frac{x}{n} + z\big) = f(z), \qquad (x, z \neq 0). \tag{4.34}$$

Since $f(0) = 0$, (4.34) holds for all $x, z \in A$. It follows from (4.32) and (4.34) that $f = f_2$. So by replacing z by $\frac{z}{n}$ in (4.32) and letting $n \to \infty$ and using (4.34), we get $f = f_0$. Hence (4.32) implies that the mapping f is additive. Thus (4.30) means that

$$\|f(y) - f_1(y)\|_B \le \theta \|x\|_A^{r_0} \|y\|_A^{r_1} \|z\|_A^{r_2}, \qquad (y \in A, x, z \neq 0).$$

So $f = f_1$, and this proves (i).

To prove (ii), since the mapping f is additive, by the same reasoning as in the proof of the main result of [13], the mapping $f : A \to B$ is $\mathbb{C}$-linear. Now, (4.31) implies that

$$\begin{aligned} &\|f(\{w_0, x, w_2\}) - \{f(w_0), f(x), f(w_2)\}\|_B \\ = &\lim_{n\to\infty} \frac{1}{n} \big\|f(\{w_0, x, nw_2\}) - \{f(w_0), f(x), f(nw_2)\}\big\|_B \\ \le &\,\theta \lim_{n\to\infty} n^{r_2-1} \|w_0\|_A^{r_0} \|x\|_A^{r_1} \|w_2\|_A^{r_2} = 0 \end{aligned}$$

for all $x \in A$ and all $w_0, w_2 \in A_0 \setminus \{0\}$. Since $f(0) = 0$, therefore

$$f(\{w_0, x, w_2\}) = \{f(w_0), f(x), f(w_2)\}$$

for all $x \in A$ and all $w_0, w_2 \in A_0$. So the mapping $f : A \to B$ is a proper JCQ^*-triple homomorphism. □

Theorem 4.8. *Let* $\theta \ge 0$ *and* $r_0, r_1 < 0$ *be real numbers and let* s_0, s_1, s_2 *be real numbers such that* $s_j \neq 1$ *for some* $0 \le j \le 2$. *Assume that* $f : A \to B$ *is a mapping with* $f(0) = 0$ *and* $f(w) \in B_0$ *for all* $w \in A_0$ *and*

$$\|f(x+y) - f(x) - f(y)\|_B \le \theta \|x\|_A^{r_0} \|y\|_A^{r_1}, \tag{4.35}$$

$$\|f(\{w_0, x, w_1\}) - \{f(w_0), f(x), f(w_2)\}\|_B \le \theta \|w_0\|_A^{s_0} \|x\|_A^{s_1} \|w_2\|_A^{s_2} \tag{4.36}$$

for all $w_0, w_2 \in A_0 \setminus \{0\}$ and all $x, y \in A \setminus \{0\}$. If the mapping $t \to f(tx)$ is continuous in $0 \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to B$ is a proper JCQ^*-triple homomorphism.

Proof. Let $y \in A \setminus \{0\}$. Replacing x and y in (4.35) by $\frac{y}{2} + ny$ and $\frac{y}{2} - ny$, respectively, we get

$$f(y) = \lim_{n\to\infty} [f(\frac{y}{2} + ny) + f(\frac{y}{2} - ny)] \tag{4.37}$$

for all $y \in A \setminus \{0\}$. Since $f(0) = 0$, (4.37) holds for all $y \in A$. Let $x, y \in A \setminus \{0\}$. It follows from (4.35) and (4.37) that

$$\begin{aligned}
&\|f(x+y) - f(x) - f(y)\|_B \\
&= \lim_{n\to\infty} \Big\| f(\frac{x+y}{2} + n(x+y)) + f(\frac{x+y}{2} - n(x+y)) \\
&\quad - f(\frac{x}{2} + nx) - f(\frac{x}{2} - nx) - f(\frac{y}{2} + ny) - f(\frac{y}{2} - ny)\Big\|_B \\
&\le \limsup_{n\to\infty} \Big\| f(\frac{x+y}{2} + n(x+y)) - f(\frac{x}{2} + nx) - f(\frac{y}{2} + ny)\Big\|_B \\
&\quad + \limsup_{n\to\infty} \Big\| f(\frac{x+y}{2} - n(x+y)) - f(\frac{x}{2} - nx) - f(\frac{y}{2} - ny)\Big\|_B \\
&\le \theta\Big[\lim_{n\to\infty} \big(\frac{1}{2} + n\big)^{r_0+r_1} + \lim_{n\to\infty} \big(n - \frac{1}{2}\big)^{r_0+r_1}\Big] \|x\|_A^{r_0} \|y\|_A^{r_1} = 0.
\end{aligned}$$

So we have $f(x+y) = f(x) + f(y)$ for all $x, y \in A \setminus \{0\}$. Since $f(0) = 0$, we get that the mapping f is additive. By the same reasoning as in the proof of the main result of [13], the mapping $f : A \to B$ is $\mathbb{C}$-linear. Without loss of generality we may assume that $s_0 \neq 1$. Let $s_0 < 1$ (we have similar proof when $s_0 > 1$). It follows from (4.36) that

$$\begin{aligned}
&\|f(\{w_0, x, w_2\}) - \{f(w_0), f(x), f(w_2)\}\|_B \\
&= \lim_{n\to\infty} \frac{1}{n} \|f(\{nw_0, x, w_2\}) - \{f(nw_0), f(x), f(w_2)\}\|_B \\
&\le \theta \lim_{n\to\infty} n^{s_0-1} \|w_0\|_A^{s_0} \|x\|_A^{s_1} \|w_2\|_A^{s_2} = 0
\end{aligned}$$

for all $x \in A \setminus \{0\}$ and all $w_0, w_2 \in A_0 \setminus \{0\}$. Since $f(0) = 0$, we get that

$$f(\{w_0, x, w_2\}) = \{f(w_0), f(x), f(w_2)\}$$

for all $x \in A$ and all $w_0, w_2 \in A_0$. So the mapping $f : A \to B$ is a proper JCQ^*-triple homomorphism. □

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