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THE STABILITY OF FUNCTIONAL INEQUALITIES WITH ADDITIVE MAPPINGS

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Abstract. In this paper, we prove the generalized Hyers–Ulam stability of the functional inequalities associated with additive functional mappings. Also, we find the solution of these inequalities which satisfy certain conditions.

1. Introduction

In 1940, S. M. Ulam [20] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

In 1941, D. H. Hyers [7] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

In 1978, Th. M. Rassias [16] provided a generalization of Hyers' theorem which allows the *Cauchy difference to be unbounded.*

Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

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for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$.
Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$.

In 1982, J. M. Rassias [15] followed the innovative approach of Th. M. Rassias' theorem [16] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. In 1990, Th. M. Rassias [17] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \ge 1$. In 1991, Z. Gajda [3] following the same approach as in Th. M. Rassias [16], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [3], as well as by Th. M. Rassias and P. Šemrl [18] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. The inequality (1.1) that was introduced for the first time by Th. M. Rassias [16] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept of stability is known as *generalized Hyers–Ulam stability* or *Hyers–Ulam–Rassias stability* of functional equations (cf. the books of P. Czerwik [1], D. H. Hyers, G. Isac and Th. M. Rassias [8]).

P. Găvruta [4] provided a further generalization of Th. M. Rassias' theorem. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam stability to a number of functional equations and mappings (see [9]-[14]).

Throughout this paper, let G be a 2-divisible abelian group. Assume that Y is a Banach space with norm $\|\cdot\|_Y$.

In [5], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \le \|f(xy)\|, \tag{1.3}$$

then f satisfies the Jordan–Von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [19]. Gilányi [6] and Fechner [2] proved the generalized Hyers–Ulam stability of the functional inequality (1.3).

Now, we consider the following functional inequalities

$$\|f(x) + f(y) + f(z)\| \le \|2f(\frac{x+y+z}{2})\| + \phi(x, y, z) \tag{1.4}$$

$$\|f(x) + f(y) + f(z)\| \le \|f(x + y + z)\| + \phi(x, y, z) \tag{1.5}$$

which are associated with Jordan–Von Neumann type Cauchy-Jensen additive functional equations.

In this paper, we investigate the generalized Hyers–Ulam stability of the functional inequalities (1.4) and (1.5), which improve the main results of Park et al [12]. Also we prove if f satisfies one of the inequalities (1.4) and (1.5) with certain conditions, then f is Cauchy additive.

2. Stability of functional inequality (1.4)

We prove the generalized Hyers–Ulam stability of a functional inequality (1.4) associated with a Jordan–Von Neumann type 3-variable Jensen additive functional equation.

Theorem 2.1. *Let $(G,+)$ be a 2-divisible abelian group and $(Y, \|\cdot\|)$ be a Banach space. Assume that a mapping $f : G \to Y$ satisfies the inequality*

$$\|f(x)+f(y)+f(z)\| \le \|2f(\frac{x+y+z}{2})\| + \phi(x,y,z) \tag{2.1}$$

and that the map $\phi : G\times G\times G \to [0,\infty)$ satisfies the condition

$$\psi(x,y,z) := \sum_{j=0}^{\infty} 2^j \phi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}) < \infty$$

for all $x,y,z \in G$. Then there exists a unique Cauchy additive mapping $A : G \to Y$ such that

$$\|A(x)-f(x)\| \le [\psi(x, -\frac{x}{2}, -\frac{x}{2}) + 2\psi(\frac{x}{2}, -\frac{x}{2}, 0)] \tag{2.2}$$

for all $x \in G$.

Proof. Letting $x := 2x$, $y := -x$ and $z := -x$ in (2.1), we get

$$\|f(2x)+2f(-x)\| \le \phi(2x,-x,-x) + \|2f(0)\| \tag{2.3}$$

for all $x \in G$. Also by letting $y := -x$ and $z := 0$ in (2.1), we get

$$\|f(x)+f(-x)\| \le \phi(x,-x,0) + 3\|f(0)\| \tag{2.4}$$

for all $x \in G$. Setting $x,y,z := 0$ in (2.1), we get $\|f(0)\| \le \phi(0,0,0)$. So by the condition of ϕ, we have $f(0) = 0$. Hence we get by (2.3) and (2.4)

$$\begin{aligned}
&\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})\| \\
&\le \sum_{j=l}^{m-1} \|2^j f(\frac{x}{2^j}) - 2^{j+1} f(\frac{x}{2^{j+1}})\| \\
&\le \sum_{j=l}^{m-1} [\|2^j f(\frac{x}{2^j}) + 2^{j+1} f(-\frac{x}{2^{j+1}})\| + \|2^{j+1} f(-\frac{x}{2^{j+1}}) + 2^{j+1} f(\frac{x}{2^{j+1}})\|] \\
&\le \sum_{j=l}^{m-1} [2^j \phi(\frac{x}{2^j}, -\frac{x}{2^{j+1}} - \frac{x}{2^{j+1}}) + 2^{j+1}\phi(\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}, 0)]
\end{aligned} \tag{2.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It tends to zero as $l \to \infty$ by condition of ϕ. It means that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in G$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define a mapping $A : G \to Y$ by $A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get (2.2).

Next, we claim that the mapping $A : G \longrightarrow Y$ is a Cauchy additive mapping. We obtain by (2.1) and condition of ϕ

$$\begin{aligned}\|A(x) + A(y) + A(z)\| &= \lim_{n\to\infty} 2^n \|f(\frac{x}{2^n}) + f(\frac{y}{2^n}) + f(\frac{z}{2^n})\| \\ &\le \lim_{n\to\infty} [2^n \|2f(\frac{x+y+z}{2\cdot 2^n})\| + \phi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n})] \\ &= \|2A(\frac{x+y+z}{2})\|.\end{aligned}$$

Thus the mapping $A : G \longrightarrow Y$ is Cauchy additive by Proposition 2.1 in [12].

Now, let $T : G \longrightarrow Y$ be another Cauchy additive mapping satisfying (2.2). Then we obtain

$$\begin{aligned}\|A(x) - T(x)\| &= 2^n \|A(\frac{x}{2^n}) - T(\frac{x}{2^n})\| \\ &\le 2^n(\|A(\frac{x}{2^n}) - f(\frac{x}{2^n})\| + \|T(\frac{x}{2^n}) - f(\frac{x}{2^n})\|) \\ &\le 2\cdot[\psi(x, -\frac{x}{2}, -\frac{x}{2}) + 2\psi(\frac{x}{2}, -\frac{x}{2}, 0)]\end{aligned}$$

which tends to zero as $n \to \infty$, because

$$\begin{aligned}&\lim_{n\to\infty} \quad [\psi(x, -\frac{x}{2}, -\frac{x}{2}) + 2\psi(\frac{x}{2}, -\frac{x}{2}, 0)] \\ = \ &\lim_{n\to\infty} \quad \sum_{j=n}^{\infty}[2^j\phi(\frac{x}{2^j}, -\frac{x}{2^{j+1}} - \frac{x}{2^{j+1}}) + 2^{j+1}\phi(\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}, 0)] = 0.\end{aligned}$$

So we can conclude that $A(x) = T(x)$ for all $x \in G$. This proves the uniqueness of A. Hence the mapping $A : G \to Y$ is a unique Cauchy additive mapping satisfying (2.2). □

Remark 2.2. Let $(G, +)$ be a 2-divisible abelian group and $(Y, \|\cdot\|)$ be a Banach space. Assume that a mapping $f : G \to Y$ satisfies the inequality (2.1) and that the map $\phi : G \times G \times G \to [0, \infty)$ satisfies the condition

$$\psi(x, y, z) := \sum_{j=0}^{\infty} 2^j \phi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}) < \infty$$

for all $x, y, z \in G$. Then there exists a unique Cauchy additive mapping $A : G \to Y$ such that

$$\|\frac{f(x) - f(-x)}{2} - A(x)\| \le \frac{1}{2}[\psi(-x, \frac{x}{2}, \frac{x}{2}) + \psi(x, -\frac{x}{2}, -\frac{x}{2})] \tag{2.6}$$

for all $x \in G$.

Proof. The proof is similar to that of Theorem 2.1. □

Theorem 2.3. *Let $(G,+)$ be a 2-divisible abelian group and $(Y,\|\cdot\|)$ be a Banach space. Assume that a mapping $f: G\to Y$ satisfies the inequality (2.1) and that the map $\phi: G\times G\times G\to[0,\infty)$ satisfies the condition*

$$\psi(x,y,z):=\sum_{j=0}^{\infty}\frac{1}{2^j}\phi(2^jx,2^jy,2^jz)<\infty$$

for all $x,y,z\in G$. Then there exists a unique Cauchy additive mapping $A: G\to Y$ such that

$$\|A(x)-f(x)\|\le\frac{1}{2}[\psi(-2x,x,x)+\psi(2x,-2x,0)]+5\|f(0)\| \tag{2.7}$$

for all $x\in G$.

Proof. Similarly, we get by (2.3) and (2.4)

$$\begin{aligned}
&\|\frac{1}{2^l}f(2^lx)-\frac{1}{2^m}f(2^mx)\|\\
&\le\sum_{j=l}^{m-1}\|\frac{1}{2^j}f(2^jx)-\frac{1}{2^{j+1}}f(2^{j+1}x)\|\\
&\le\sum_{j=l}^{m-1}[\|\frac{1}{2^j}f(2^jx)+\frac{1}{2^{j+1}}f(-2^{j+1}x))\|\\
&\quad+\|\frac{1}{2^{j+1}}f(2^{j+1}x)+\frac{1}{2^{j+1}}f(-2^{j+1}x))\|]\\
&\le\sum_{j=l}^{m-1}[\frac{1}{2^{j+1}}\phi(-2^{j+1}x,2^jx,2^jx)\\
&\quad+\frac{1}{2^{j+1}}\phi(2^{j+1}x,-2^{j+1}x,0)+\frac{5}{2^{j+1}}\|f(0)\|]
\end{aligned}\tag{2.8}$$

for all nonnegative integers m and l with $m>l$ and all $x\in G$. It tends to zero as $l\to\infty$ and condition of ϕ. It means that the sequence $\{\frac{1}{2^n}f(2^nx)\}$ is a Cauchy sequence for all $x\in G$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^nx)\}$ converges. So one can define a mapping $A: G\to Y$ by $A(x):=\lim_{n\to\infty}\frac{1}{2^n}f(2^nx)$ for all $x\in G$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (2.8), we get (2.7).

The rest of the proof is similar to that of Theorem 2.1. □

Remark 2.4. Let $(G,+)$ be a 2-divisible abelian group and $(Y,\|\cdot\|)$ be a Banach space. Assume that a mapping $f: G\to Y$ satisfies the inequality (2.1) and that the map $\phi: G\times G\times G\to[0,\infty)$ satisfies the condition

$$\psi(x,y,z):=\sum_{j=0}^{\infty}\frac{1}{2^j}\phi(2^jx,2^jy,2^jz)<\infty$$

for all $x, y, z \in G$. Then there exists a unique Cauchy additive mapping $A : G \to Y$ such that

$$(2.9) \quad \|\frac{f(x) - f(-x)}{2} - A(x)\| \le \frac{1}{4}[\psi(-2x, x, x) + \psi(2x, -x, -x)] + 2\|f(0)\|$$

for all $x \in G$.

Proof. The proof is similar to that of Theorem 2.3. □

Corollary 2.5. *Let $(G, +)$ be a 2-divisible abelian group and $(Y, \|\cdot\|)$ be a normed space. Assume that a mapping $f : G \to Y$ satisfies the inequality (2.1) and that the map $\phi : G \times G \times G \to [0, \infty)$ satisfies the conditions*

(1) $\phi(2x, -x, -x) = 0$ *or* $\phi(-x, 2x, -x) = 0$ *or* $\phi(-x, -x, 2x) = 0$,

(2) $\phi(x, -x, 0) = 0$ *or* $\phi(x, 0, -x) = 0$ *or* $\phi(0, x, -x) = 0$,

(3) $\lim_{n\to\infty} 2^n \phi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) = 0$ *or* $\lim_{n\to\infty} \frac{1}{2^n}\phi(2^n x, 2^n y, 2^n z) = 0$ *for all* $x, y, z \in G$. *Then f is Cauchy additive.*

Proof. Letting $x, y, z := 0$ in (2.1), we get $\|f(0)\| \le \phi(0, 0, 0) = 0$. So $f(0) = 0$. Setting $z := 0$ and $y := -x$ in (2.1) and by condition (2), we get

$$\|f(x) + f(-x)\| \le \phi(x, -x, 0) = 0$$

for all $x \in G$. So $f(-x) = -f(x)$.

And by letting $x := 2x$, $y := -x$ and $z := -x$ in (2.1) and by condition (1), we get

$$(2.10) \quad \|f(2x) + 2f(-x)\| \le \phi(2x, -x, -x) = 0, \quad f(2x) = 2f(x)$$

for all $x \in G$.

Next, we will consider two cases for condition (3) of ϕ.

Case I : Assume $\lim_{n\to\infty} 2^n \phi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) = 0$ for all $x, y, z \in G$.

We get by (2.10)

$$f(2x) = 2f(x), \quad f(x) = 2f(\frac{x}{2}) = 4f(\frac{x}{4}) = \cdots = 2^n f(\frac{x}{2^n}) = \cdots.$$

So we can define $f(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ for all $x \in G$.

It follows from (2.1) that

$$\begin{aligned} \|f(x) + f(y) + f(z)\| &= \lim_{n\to\infty} 2^n \|f(\frac{x}{2^n}) + f(\frac{y}{2^n}) + f(\frac{z}{2^n})\| \\ &\le \lim_{n\to\infty} [2^n \|2f(\frac{x+y+z}{2 \cdot 2^n})\| + 2^n \phi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n})] \\ &= \|2f(\frac{x+y+z}{2})\| \end{aligned}$$

for all $x, y, z \in G$. So $\|f(x) + f(y) + f(z)\| \le \|2f(\frac{x+y+z}{2})\|$ for all $x, y, z \in G$. Thus the mapping $f : G \longrightarrow Y$ is Cauchy additive by Proposition 2.1 in [12].

Case II : Assume $\lim_{n\to\infty} \frac{1}{2^n}\phi(2^n x, 2^n y, 2^n z) = 0$ for all $x, y, z \in G$.

We obtain by (2.10)

$$f(2x) = 2f(x), \quad f(x) = \frac{1}{2}f(2x) = \frac{1}{4}f(4x) = \cdots = \frac{1}{2^n}f(2^n x) = \cdots.$$

So we can define $f(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ for all $x \in G$. By (2.1), we obtain

$$\begin{aligned} \|f(x)+f(y)+f(z)\| &\leq \lim_{n\to\infty} \frac{1}{2^n}\|f(2^n x)+f(2^n y)+f(2^n(z))\| \\ &\leq \lim_{n\to\infty}[\frac{1}{2^n}\|2f(2^n\frac{x+y+z}{2})\| + \frac{1}{2^n}\phi(2^n x, 2^n y, 2^n z)] \\ &= \|2f(\frac{x+y+z}{2})\|. \end{aligned}$$

Thus the mapping $f : G \longrightarrow Y$ is Cauchy additive by Proposition 2.1 in [12]. □

Now we obtain the result of Park et al [12] in the following corollary

Corollary 2.6. *Let $f : G \to Y$ be a mapping such that*

$$\|f(x)+f(y)+f(z)\|_Y \leq \|2f(\frac{x+y+z}{2})\|_Y \tag{2.11}$$

for all $x, y, z \in G$. Then f is Cauchy additive.

Corollary 2.7. *Assume that X is a normed space with norm $\|\cdot\|_X$. Let $r \neq 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} \|f(x)+f(y)+f(z)\|_Y \leq\ & \|2f(\frac{x+y+z}{2})\|_Y \\ & +\theta(\|x\|_X^r + \|y\|_X^r + \|z\|_X^r) \end{aligned} \tag{2.12}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that

$$\|\frac{f(x)-f(-x)}{2} - h(x)\|_Y \leq \frac{2^r+2}{|\,2^r-2\,|}\theta\|x\|_X^r \tag{2.13}$$

for all $x \in X$.

Proof. When $r > 1$, we apply Theorem 2.1. When $r < 1$, we apply Theorem 2.3. □

3. Stability of functional inequality (1.5)

We prove the generalized Hyers–Ulam stability of a functional inequality (1.5) associated with a Jordan–Von Neumann type 3-variable Cauchy additive functional equation.

Theorem 3.1. *Let $(G, +)$ be a 2-divisible abelian group and $(Y, \|\cdot\|)$ be a Banach space. Assume that a mapping $f : G \to Y$ satisfies the inequality*

$$\|f(x)+f(y)+f(z)\| \leq \|f(x+y+z)\| + \phi(x,y,z) \tag{3.1}$$

and that the map $\phi : G\times G\times G \to [0,\infty)$ satisfies the conditions

(1) $\rho(x) := \sum_{j=0}^{\infty} 2^j\phi_1(\frac{x}{2^{j+1}}) + 2^{j+1}\phi_2(\frac{x}{2^{j+1}}) < \infty$ *for all $x \in G$,*

(2) $\lim_{n\to\infty} 2^n\phi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) = 0$ *for all $x, y, z \in G$.*

Here,

$$\begin{aligned} \phi_1(x) &:= \min\{\phi(2x,-x,-x),\phi(-x,2x,-x),\phi(-x,-x,2x)\}, \\ \phi_2(x) &:= \min\{\phi(x,-x,0),\phi(x,0,-x),\phi(x,-x,0)\}. \end{aligned}$$

Then there exists a unique Cauchy additive mapping $A : G \to Y$ such that

$$\|A(x) - f(x)\| \le \rho(x) \tag{3.2}$$

for all $x \in G$.

Proof. Letting $x := 2x$, $y := -x$ and $z := -x$ in (3.1), we get

$$\|f(2x) + 2f(-x)\| \le \phi_1(x) + \|f(0)\| \tag{3.3}$$

for all $x \in G$. Also by letting $z := 0$ and $y := -x$ in (3.1), we get

$$\|f(x) + f(-x)\| \le \phi_2(x) + 2\|f(0)\| \tag{3.4}$$

for all $x \in G$. Setting $x, y, z := 0$ in (3.1), we get $\|f(0)\| \le \frac{1}{2}\phi(0,0,0)$. By the condition (2), we have $f(0) = 0$. Hence we get by (3.3) and (3.4)

$$\begin{aligned} &\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})\| \\ &\le \sum_{j=l}^{m-1} \|2^j f(\frac{x}{2^j}) - 2^{j+1} f(\frac{x}{2^{j+1}})\| \\ &\le \sum_{j=l}^{m-1} [\|2^j f(\frac{x}{2^j}) + 2^{j+1} f(-\frac{x}{2^{j+1}})\| + \|2^{j+1} f(-\frac{x}{2^{j+1}}) + 2^{j+1} f(\frac{x}{2^{j+1}})\|] \\ &\le \sum_{j=l}^{m-1} 2^j \phi_1(\frac{x}{2^{j+1}}) + 2^{j+1}\phi_2(\frac{x}{2^{j+1}}) \end{aligned} \tag{3.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It tends to zero as $l \to \infty$ by condition of ϕ. It means that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in G$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define a mapping $A : G \to Y$ by $A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.1), we get (3.2).

Next, we claim that the mapping $A : G \longrightarrow Y$ is a Cauchy additive mapping. We obtain by (3.1) and condition of ϕ

$$\begin{aligned} &\|A(x) + A(y) - A(x+y)\| = \lim_{n\to\infty} 2^n \|f(\frac{x}{2^n}) + f(\frac{y}{2^n}) - f(\frac{x+y}{2^n})\| \\ &\le \lim_{n\to\infty} 2^n [\|f(\frac{x}{2^n}) + f(\frac{x}{2^n}) + f(\frac{-x-y}{2^n})\| + \|f(\frac{-x-y}{2^n}) + f(\frac{x+y}{2^n})\|] \\ &\le \lim_{n\to\infty} 2^n [\phi(\frac{x}{2^n}, \frac{x}{2^n}, \frac{-x-y}{2^n}) + \phi(\frac{x+y}{2^n}, 0, \frac{-x-y}{2^n})] = 0. \end{aligned}$$

So, we have $A(x+y) = A(x) + A(y)$.

Now, let $T : G \longrightarrow Y$ be another Cauchy additive mapping satisfying (3.1). Then we obtain

$$\begin{aligned}\|A(x) - T(x)\| &= 2^n\|A(\frac{x}{2^n}) - T(\frac{x}{2^n})\| \\ &\leq 2^n(\|A(\frac{x}{2^n}) - f(\frac{x}{2^n})\| + \|T(\frac{x}{2^n}) - f(\frac{x}{2^n})\|) \leq 2^{n+1}\rho_1(\frac{x}{2^n})\end{aligned}$$

which tends to zero as $n \to \infty$, because

$$\lim_{n\to\infty} 2^{n+1}\rho_1(\frac{x}{2^n}) = 2\lim_{n\to\infty}\sum_{j=n}^{\infty}[2^j\phi_1(\frac{x}{2^{j+1}}) + 2^{j+1}\phi_2(\frac{x}{2^{j+1}})] = 0.$$

So we can conclude that $A(x) = T(x)$ for all $x \in G$. This proves the uniqueness of A. Hence the mapping $A : G \to Y$ is a unique Cauchy additive mapping satisfying (3.1). □

Remark 3.2. Let $(G, +)$ be a 2-divisible abelian group and $(Y, \|\cdot\|)$ be a Banach space. Assume that a mapping $f : G \to Y$ satisfies the inequality (3.1) and that the map $\phi : G \times G \times G \to [0, \infty)$ satisfies the conditions

(1) $\rho(x) := \sum_{j=0}^{\infty} 2^{j-1}[\phi(-\frac{x}{2^j}, \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}) + \phi(\frac{x}{2^j}, -\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}})] < \infty$ for all $x \in G$,

(2) $\lim_{n\to\infty} 2^n\phi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) = 0$ for all $x, y, z \in G$. Then there exists a unique Cauchy additive mapping $A : G \to Y$ such that

$$\|\frac{f(x) - f(-x)}{2} - A(x)\| \leq \rho(x) \tag{3.6}$$

for all $x \in G$.

Proof. The proof is similar to that of Theorem 3.1 □

Theorem 3.3. *Let $(G, +)$ be a 2-divisible abelian group and $(Y, \|\cdot\|)$ be a Banach space. Assume that a mapping $f : G \to Y$ satisfies the inequality* (3.1) *and that the map $\phi : G \times G \times G \to [0, \infty)$ satisfies the conditions*

(1) $\rho(x) := \sum_{j=0}^{\infty} \frac{1}{2^{j+1}}[\phi_1(-2^{j+1}x) + \phi_2(2^{j+1}x)] < \infty$ *for all* $x \in G$,

(2) $\lim_{n\to\infty} \frac{1}{2^n}\phi(2^n x, 2^n y, 2^n z) = 0$ *for all* $x, y, z \in G$.

Here,

$$\begin{aligned}\phi_1(x) &:= \min\{\phi(2x, -x, -x), \phi(-x, 2x, -x), \phi(-x, -x, 2x)\}, \\ \phi_2(x) &:= \min\{\phi(x, -x, 0), \phi(x, 0, -x), \phi(x, -x, 0)\}.\end{aligned}$$

Then there exists a unique Cauchy additive mapping $A : G \to Y$ such that

$$\|A(x) - f(x)\| \leq \rho(x) + 3\|f(0)\| \tag{3.7}$$

for all $x \in G$.

Proof. We get by (3.3) and (3.4)

$$\begin{aligned}
&\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x))\| \\
&\leq \sum_{j=l}^{m-1} \|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x))\| \\
&\leq \sum_{j=l}^{m-1} [\|\frac{1}{2^j}f(2^j x) + \frac{1}{2^{j+1}}f(-2^{j+1}x))\| \qquad (3.8)\\
&\quad + \|\frac{1}{2^{j+1}}f(2^{j+1}x) + \frac{1}{2^{j+1}}f(-2^{j+1}x))\|] \\
&\leq \sum_{j=l}^{m-1} [\frac{1}{2^{j+1}}\phi_1(-2^{j+1}) + \frac{1}{2^{j+1}}\phi_2(2^{j+1}x) + \frac{3}{2^{j+1}}\|f(0)\|]
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It tends to zero as $l \to \infty$ by condition of ϕ. It means that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in G$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define a mapping $A : G \to Y$ by $A(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x)$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to that of Theorem 3.1. □

Remark 3.4. Let $(G, +)$ be a 2-divisible abelian group and $(Y, \|\cdot\|)$ be a Banach space. Assume that a mapping $f : G \to Y$ satisfies the inequality (3.1) and that the map $\phi : G \times G \times G \to [0, \infty)$ satisfies the conditions

(1) $\rho(x) := \sum_{j=0}^{\infty} \frac{1}{2^{j+3}}[\phi(-2^{j+1}x, 2^j x, 2^j x) + \phi(2^{j+1}x, -2^j x, -2^j x)] < \infty$ for all $x \in G$,

(2) $\lim_{n\to\infty} \frac{1}{2^n}\phi(2^n x, 2^n y, 2^n z) = 0$ for all $x, y, z \in G$.

Then there exists a unique Cauchy additive mapping $A : G \to Y$ such that

$$\|\frac{f(x) - f(-x)}{2} - A(x)\| \leq \rho(x) \qquad (3.9)$$

for all $x \in G$.

Proof. The proof is similar to that of Theorem 3.3. □

Corollary 3.5. *Let $(G, +)$ be a 2-divisible abelian group and $(Y, \|\cdot\|)$ be a normed space. Assume that a mapping $f : G \to Y$ satisfies the inequality* (3.1) *and that the map $\phi : G \times G \times G \to [0, \infty)$ satisfies the conditions*

(1) *$\phi(2x, -x, -x) = 0$ or $\phi(-x, 2x, -x) = 0$ or $\phi(-x, -x, 2x) = 0$,*

(2) *$\phi(x, -x, 0) = 0$ or $\phi(x, 0, -x) = 0$ or $\phi(0, x, -x) = 0$ for all $x \in G$,*

(3) *$\lim_{n\to\infty} 2^n\phi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) = 0$ or $\lim_{n\to\infty} \frac{1}{2^n}\phi(2^n x, 2^n y, 2^n z) = 0$ for all $x, y, z \in G$. Then f is Cauchy additive.*

Proof. Letting $x = y = z = 0$ in (3.1) and by condition, we get

$$\|f(0)\| \leq \frac{1}{2}\phi(0,0,0) = 0.$$

So $f(0) = 0$. And by letting $z = 0$ and $y = -x$ in (3.1) and by condition (2), we get

$$\|f(x) + f(-x)\| \leq \phi(x, -x, 0) = 0 \tag{3.10}$$

for all $x \in G$. Hence $f(-x) = -f(x)$. Setting $x = 2x$, $y = -x$ and $z = -x$ in (3.1) and by condition (1), we get

$$\|f(2x) + 2f(-x)\| \leq \phi(2x, -x, -x) = 0 \tag{3.11}$$

for all $x \in G$. So $f(2x) = 2f(x)$.

Next, we will consider cases for condition (3) of ϕ.

Case I : Assume $\lim_{n\to\infty} 2^n\phi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) = 0$ for all $x, y, z \in G$. We get by (3.11)

$$f(2x) = 2f(x), \quad f(x) = 2f(\frac{x}{2}) = 4f(\frac{x}{4}) = \cdots = 2^n f(\frac{x}{2^n}) = \cdots.$$

So we can define $f(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ for all $x \in G$. It follows from (3.1) that

$$\begin{aligned} \|f(x) + f(y) + f(z)\| &= \lim_{n\to\infty} 2^n \|f(\frac{x}{2^n}) + f(\frac{y}{2^n}) + f(\frac{z}{2^n})\| \\ &\leq \lim_{n\to\infty} 2^n [\|f(\frac{x+y+z}{2^n})\| + \phi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n})] \\ &= \|f(x+y+z)\| \end{aligned}$$

for all $x, y, z \in G$. So $\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\|$ for all $x, y, z \in G$. Thus the mapping $f : G \longrightarrow Y$ is Cauchy additive by Proposition 2.2 in [12].

Case II : Assume $\lim_{n\to\infty} \frac{1}{2^n}\phi(2^n x, 2^n y, 2^n z) = 0$ for all $x, y, z \in G$. We obtain by (3.11)

$$f(2x) = 2f(x), \quad f(x) = \frac{1}{2}f(2x) = \frac{1}{4}f(4x) = \cdots = \frac{1}{2^n}f(2^n x) = \cdots.$$

So we can define $f(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ for all $x \in G$. By (3.1), we obtain

$$\begin{aligned} \|f(x) + f(y) + f(z)\| &\leq \lim_{n\to\infty} \frac{1}{2^n} \|f(2^n x) + f(2^n y) + f(2^n(z))\| \\ &\leq \lim_{n\to\infty} \frac{1}{2^n} [\|f(2^n(x+y+z))\| + \phi(2^n x, 2^n y, 2^n z)] \\ &= \|f(x+y+z)\|. \end{aligned}$$

Thus the mapping $f : G \longrightarrow Y$ is Cauchy additive by Proposition 2.2 in [12]. $\square$

Now we obtain the result of Park et al [12] in the following corollary

Corollary 3.6. *Let $f : G \to Y$ be a mapping such that*

$$\|f(x) + f(y) + f(z)\|_Y \leq \|f(x + y + z)\|_Y \tag{3.12}$$

for all $x, y, z \in G$. Then f is Cauchy additive.

Corollary 3.7. *Assume that X is a normed space with norm $\|\cdot\|_X$. Let $r \neq 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} \|f(x) + f(y) + f(z)\|_Y \leq \|f(x + y + z)\|_Y \\ + \theta(\|x\|^r{}_X + \|y\|^r{}_X + \|z\|^r{}_X) \end{aligned} \tag{3.13}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that

$$\|\frac{f(x) - f(-x)}{2} - h(x)\|_Y \leq \frac{2^r + 2}{| 2^r - 2 |}\theta\|x\|^r{}_X$$

for all $x \in X$.

Proof. When $r > 1$, we apply Theorem 3.1. When $r < 1$, we apply Theorem 3.3. □

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