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ON ELLIPTIC ANALOGUE OF THE HARDY SUMS

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Abstract. Main purpose of this paper is to define an elliptic analogue of the Hardy sums. Some results, which are related to elliptic analogue of the Hardy sums, are given.

1. Introduction, definitions and notations

We set

$$q = e^{\pi i \tau},\ \tau \in \mathbb{C},\ |q| < 1.$$

The classical theta functions, $\vartheta_n(0,q)$ $(n = 2,3,4)$ are defined as follows: ([14], [20], [30])

$$\vartheta_2(0,q) = 2q^{\frac{1}{4}} \sum_{n=0} q^{n(n+1)} = 2q^{\frac{1}{4}} \prod_{n=1}^{\infty} \left(1-q^{2n}\right)\left(1+q^{2n}\right)^2,$$

$$\vartheta_3(0,q) = 1 + 2\sum_{n=1} q^{n^2} = \prod_{n=1}^{\infty} \left(1-q^{2n}\right)\left(1+q^{2n-1}\right)^2,$$

$$\vartheta_4(0,q) = 1 + 2\sum_{n=1} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} \left(1-q^{2n}\right)\left(1-q^{2n-1}\right)^2.$$

In the remainder of our work, we shall write $\vartheta_2(0,q)$, $\vartheta_3(0,q)$, $\vartheta_4(0,q)$ as $\vartheta_2(q)$, $\vartheta_3(q)$ and $\vartheta_4(q)$, respectively [9]. The theta series $\frac{\vartheta_2(q)}{q^{1/4}}$, $\vartheta_3(q)$, $\vartheta_4(q)$ converge on the (open) unit disk

$$D = \left\{q = e^{\pi i \tau},\ \tau \in \mathbb{C},\ |q| < 1\right\}.$$

The Dedekind eta function η is defined by

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right),$$

where $z \in \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$. An elegant functional equation of η-function under the modular transformation is given by cf. [1]:

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Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma(1)$, modular group. Then

$$\log \eta(Az) = \log \eta(z) + \frac{\pi i(a+d)}{12c} - \pi i(s(d,c) - \frac{1}{4}) + \frac{1}{2}\log(cz+d), \tag{1.1}$$

where $s(d,c)$ is the Dedekind sum defined as follows:

$$s(d,c) = \sum_{n=1}^{c-1} ((\frac{n}{c}))((\frac{dn}{c})),$$

when

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & x \notin \mathbb{Z} \\ 0, & x \in \mathbb{Z}. \end{cases}$$

(For detail see [4], [6], [15], [26], [27], [28]).

Relations between theta functions and η-function are given by (see cf. [7], [18], [21], [22], [23])

$$\begin{aligned} \vartheta_2(z) &= \frac{2\eta^2(2z)}{\eta(z)}, \\ \vartheta_3(z) &= \frac{\eta^5(z)}{\eta^2(2z)\eta^2(\frac{z}{2})}, \\ \vartheta_4(z) &= \frac{\eta^2(\frac{z}{2})}{\eta(z)}. \end{aligned}$$

The following identities are obtained by taking logarithms of the above

$$\begin{aligned} \log \vartheta_2(z) &= \log 2 + 2\log \eta(2z) - \log \eta(z), \\ \log \vartheta_3(z) &= 5\log \eta(z) - 2\log \eta(2z) - 2\log \eta(\frac{z}{2}), \\ \log \vartheta_4(z) &= 2\log \eta(\frac{z}{2}) - \log \eta(z) \text{ cf. [25].} \end{aligned} \tag{1.2}$$

Relations between Hardy sums and theta functions were given Berndt [5], Simsek [25] and Berndt and Goldberg [6].

Let h, $k \in \mathbb{Z}$ with $k > 0$. Hardy sums (or Hardy-Berndt sums) are defined as follows:

$$\begin{aligned} S(h,k) &= \sum_{j=1}^{k-1} (-1)^{j+1+[\frac{hj}{k}]}, \\ s_1(h,k) &= \sum_{j=1}^{k} (-1)^{[\frac{hj}{k}]}((\frac{j}{k})), \\ s_2(h,k) &= \sum_{j=1}^{k} (-1)^{j}((\frac{j}{k}))((\frac{hj}{k})), \\ s_3(h,k) &= \sum_{j=1}^{k} (-1)^{j}((\frac{hj}{k})), \end{aligned}$$

$$s_4(h,k) = \sum_{j=1}^{k-1}(-1)^{[\frac{hj}{k}]},$$

$$s_5(h,k) = \sum_{j=1}^{k}(-1)^{j+[\frac{hj}{k}]}((\frac{j}{k}))$$

(For detail see [5], [6], [10], [11], [19], [26], [24], [25] [29]).

Relations between Hardy sums and theta functions are given as follows:

Theorem 1.1. *$A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(2)$, $c > 0$. If c is even and $(c,d) = 1$, then*

$$\log\vartheta_2(Az) = \log\vartheta_2(z) + \frac{1}{2}\log(cz+d) - \frac{\pi i}{4} + \pi i\left(\frac{a+d}{4c}\right) - \pi i s_2(d,c),$$

where $s_2(d,c)$ is a Hardy's sum.

Theorem 1.2. *$A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma^0(2)$, $c > 0$. If d is odd and $(c,d) = 1$, then*

$$\log\vartheta_4(Az) = \log\vartheta_4(z) + \frac{1}{2}\log(cz+d) - \frac{\pi i}{4} - \frac{\pi i}{4}s_4(d,c),$$

where $s_4(d,c)$ is a Hardy's sum.

The proof of the Theorem 1.1 and Theorem 1.2 have been given by Berndt [5] and the first author [25].

Let $z = \frac{w_1}{w_2} = x + iy$ and $s = \sigma + it$ with x, y, σ, t real. For any complex number z, we choose the branch of $\log z$ with $-\pi \leq \arg z < \pi$. Let a_1 and a_2 be arbitrary real numbers. For $z \in \mathbb{H}$ and $\sigma > 2$, define Eisenstein series $G(z,s,a_1,a_2)$ as follows;

$$G(z,s,r_1,r_2) = \sum_{(r_1,r_2)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{((m+r_1)z+n+r_2)^s}.$$

Let $r_1 = r_2 = 0$. The above equation reduces to $G(z,s)$, which is given as follows;

$$G(z,s) = \sum_{0\neq(m,n)\in\mathbb{Z}^2} \frac{1}{(m+nz)^s} \tag{1.3}$$

(For details see [16], [17]).

An alternate way of defining the normalized Eisenstein series is to sum only over relatively prime pairs m, n in (1.3):

$$E(z,k) = \frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)=1}} \frac{1}{(m+nz)^k},$$

where (m,n) stands for the greatest common divisor and

$$E(z,k) = \frac{1}{2\zeta(k)}G(z,k) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n z},$$

where B_k is the Bernoulli number (cf. [1], [14], [13], [23], [30]).

2. Theorems on elliptic analogue of the multiple Hardy sums

The multiple Dedekind sum investigated by many prominent mathematicians is a natural generalization of the classical Dedekind sums. In this work, we consider papers of Zagier [31] and Egami [12]. In this section we introduce elliptic analogue of Hardy sums. Our method is similar to Zagier and Egami's expect the use of an elliptic function in the place of the cotangent function which appeared there. By a limiting procedure we can recover the corresponding results on multiple Dedekind, cotangent sums.

Let $\tau \in \mathbb{H}$ and L_τ (resp. $L_\tau^{'}$) denotes the lattice $2\pi i(2\mathbb{Z}\tau + \mathbb{Z})$ of the complex plane. If $w_1, w_2 \in L_\tau$, then $\frac{w_1}{w_2} \notin \mathbb{R}$.

$$\wp(\tau, z) = \frac{1}{z^2} + \sum_w^* \left(\frac{1}{(z+w)^2} - \frac{1}{w^2} \right).$$

From (1.3)

$$\begin{aligned} g_2(\tau) &= \sum_w^* \frac{1}{w^4}, \\ g_3(\tau) &= \sum_w^* \frac{1}{w^6}, \end{aligned}$$

and

$$e_1(\tau) = \wp(\tau, \pi i),$$

where $\sum^*$ denotes the summation over all $w \in L_\tau$ except the origin.

In [12], we recall that $g_2(\tau)$ and $g_3(\tau)$ are modular forms for $\Gamma(1)$, while $e_1(\tau)$ is a modular form for $\Gamma_0(2) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma(1) : c \equiv 0(2) \right\}$.

Now we introduce the function $\varphi(\tau, z)$ as follows:

$$\varphi(\tau, z) = \sqrt{\wp(\tau, z) - e_1(\tau)}, \text{ (cf. [12])}$$

which plays the principal role throughout this paper, where $e_1(\tau) = \wp(\tau, \pi i)$. The function $\varphi(\tau, z)$ is a meromorphic Jacobi form for $\Gamma_0(2)$ of weight 1 and index 0 by periodicity for $L_\tau^{'}$, and

$$\varphi(A\tau, \frac{z}{cz+d}) = (cz+d)\varphi(\tau, z)$$

for $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(2)$ (cf. [12]). Furthermore $\varphi(\tau, z)$ has the following q-expansion:

$$\varphi(\tau, z) = \frac{1}{2} \frac{\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}}}{\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}} \prod_{n=1}^{\infty} \frac{(1+q^n\zeta)(1+q^n\zeta^{-1})(1-q^n)^2}{(1-q^n\zeta)(1-q^n\zeta^{-1})(1+q^n)^2}, \tag{2.1}$$

where $\zeta = e^z$ and $q = e^{\pi i \tau}$.

Definition 2.1 (Multiple elliptic Dedekind sums [12]). Let p be a natural number and $a_1, \ldots, a_j$ be integers coprime to p such that $p + a_1 + \cdots + a_j$ is even. For $\mathrm{Im}(\tau) > 0$ a multiple elliptic Dedekind sums is defined by

$$D_\tau(p; a_1, \ldots, a_j) = \sum_{\substack{m,n=0 \\ (m,n)\neq(0,0)}}^{p-1} (-1)^m \prod_{k=1}^{r} \varphi(\tau, \frac{2\pi i a_k(m\tau + n)}{p}).$$

One can easily see that $D_\tau(p; a_1, \ldots, a_j)$ is a modular form of level p and weight j. The reciprocity law of the sum $D_\tau(p; a_1, \ldots, a_j)$ is given by Egami [12].

Zagier [31] defined the following multiple Dedekind sums:

$$d(p; a_1, \ldots, a_j) = (-1)^{\frac{j}{2}} \sum_{m=1}^{p-1} \cot(\frac{\pi m a_1}{p}) \cdots (\frac{\pi m a_j}{p}).$$

The sum $d(p; a_1, \ldots, a_j)$ vanishes identically when j is odd. The connection between $d(p; a_1, \ldots, a_j)$ and $D_\tau(p; a_1, \ldots, a_j)$ is given as follows:

Theorem 2.2 ([12]). *Let p be a natural number and $a_1, \ldots, a_j$ be integers coprime to p. If $p + a_1 + \cdots + a_j$ is even, then*

$$\lim_{\mathrm{Im}\tau \to \infty} D_\tau(p; a_1, \ldots, a_j) = \frac{1}{2^j} \left\{ d(p; a_1, \ldots, a_j) + (-1)^j p \sum_{v=1}^{p-1} (-1)^{j + [\frac{a_1 v}{p}] + \cdots + [\frac{a_j v}{p}]} \right\}.$$

By using the above theorem, we give some useful results in the following (for detail see [12]):

Remark 2.3. (i) Substituting $0 \leq x < 1$, $z = 2\pi i(x\tau + y)$ into (2.1) the infinite product of q-expansion (2.1) tends to 1 as $\mathrm{Im}\tau \to \infty$.

(ii)

$$\frac{\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}}}{\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}} \to \begin{cases} -1, & x > 0 \\ i\cot(\pi y), & x = 0. \end{cases}$$

(iii) By using (i) and (ii), we have

$$\lim_{\mathrm{Im}\tau \to \infty} \varphi(\tau, 2\pi i(x\tau + y)) = \begin{cases} \frac{1}{2}(-1)^{1+[y]}, & x \notin \mathbb{Z} \\ \frac{1}{2i}\cot(\pi y), & x \in \mathbb{Z}. \end{cases} \tag{2.2}$$

For $x \notin \mathbb{Z}$, Fourier expansion of $\lim_{\mathrm{Im}\tau \to \infty} \varphi(\tau, 2\pi i(x\tau + y))$ is given by the following lemma:

Lemma 2.4. *If $x \notin \mathbb{Z}$, then*

$$\lim_{\mathrm{Im}\tau \to \infty} \varphi(\tau, 2\pi i(x\tau + y)) = -\frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi y)}{2n-1}.$$

Proof. Substituting

$$(-1)^{[x]} = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x)}{2n-1}$$

into (2.2), we have the desired result. □

By using Lemma 2.4, we define new sum related to functions $\varphi(\tau, 2\pi i(x\tau+y))$ as follows:

Definition 2.5. Let a, b be coprime natural numbers. For $\mathrm{Im}\tau > 0$, the sum $B(a,b)$ is defined by

$$B(a,b) = \lim_{\mathrm{Im}\tau\to\infty} \sum_{u=1}^{b-1} \varphi(\tau, 2\pi i(x\tau + \frac{au}{b})).$$

Theorem 2.6. *Let a, b be coprime natural numbers. If $x \notin \mathbb{Z}$, then*

$$\begin{aligned} s_4(a,b) &= 2 \lim_{\mathrm{Im}\tau\to\infty} \sum_{u=1}^{b-1} \varphi(\tau, 2\pi i(x\tau + \frac{au}{b})) \\ &= 2B(a,b), \end{aligned}$$

where $s_4(a,b)$ stands for Hardy's sums.

Proof. Substituting $y = \frac{an}{b}$ with $u = 1, 2, \ldots, b-1$ and $(a,b) = 1$ into (2.2) with $x \notin \mathbb{Z}$ we obtain by Lemma 2.4

$$\sum_{u=1}^{b-1} (\lim_{\mathrm{Im}\tau\to\infty} \varphi(\tau, 2\pi i(x\tau + \frac{au}{b}))) = \frac{1}{2} \sum_{u=1}^{b-1} (-1)^{1+[\frac{au}{b}]}.$$

By the definition of Hardy sum $s_4(a,b)$ in the above, we have the desired result. □

By using Definition 2.1, we define the sum $B(p,a,b)$ as follows:

Let p be natural number and a, b be integers coprime to p. Let $p+a+b$ be even. We set

$$B(p,a,b) = \lim_{\mathrm{Im}\tau\to\infty} D_\tau(p;a,b) = \frac{1}{4}\left\{ d(p;a,b) + p\sum_{v=1}^{p-1}(-1)^{j+[\frac{av}{p}]+[\frac{bv}{p}]} \right\}. \tag{2.3}$$

In the following, we give relations between $B(p,a,b)$, $r_p(a,b)$ and three-term relation for Hardy sums. In what follows, we assume that a, b and p are pairwise coprime positive integers and a', b' and p' satisfy $aa' \equiv 1 \pmod b$, $bb' \equiv 1 \pmod p$ and $pp' \equiv 1 \pmod a$.

Lemma 2.7 ([19]). *Let a, b and p be positive integers with $a+b+p = 0 \pmod 2$, $(a,p) = 1$ and $aa' \equiv 1 \pmod p$. Further let $aa' \equiv 1 + p\delta$, $a' + \delta = 1 \pmod 2$, and for $1 \le i < p$, let $j \equiv a'i \pmod p$. Then*

$$i + [\frac{ba'i}{p}] \equiv j + [\frac{bj}{p}] + [\frac{aj}{p}] \pmod 2.$$

Now to define three-term relation for Hardy sums $S(ab',p)$, we note that a, b, and b' are odd and consequently $a+b+p \equiv 0 \pmod 2$. If $bb' = 1+pk$, then k is even and so $b'+k \equiv 1 \pmod 2$. Hence by Lemma 3 (for details, see [19], [24])

$$S(ab',p) = \sum_{j=1}^{p-1}(-1)^{j+1+[\frac{ab'j}{p}]} = \sum_{m=1}^{p-1}(-1)^{m+1+[\frac{ma}{p}]+[\frac{mb}{p}]}. \tag{2.4}$$

In [8], Berndt and Yeap defined the following sums, which are related to Dedekind sums: Let h and k be coprime positive integers, and set $h+k=\mu c$, where μ and c are positive integers. Define

$$r_\mu(h,k) = \sum_{j=1}^{k-1}\cot(\frac{\pi\mu j}{k})\cot(\frac{\pi h j}{k}). \tag{2.5}$$

Note that when $\mu = 1$, $r_1(h,k) = 4ks(h,k)$, which is a Dedekind sum.

Theorem 2.8. *Let a, b and p be natural number and a, b be coprime to p. Under the same conditions of Lemma 2.7, we have*

$$B(p,a,b) = -\frac{1}{12}(\frac{b}{a}+\frac{a}{b}+\frac{p^2}{ab})+\frac{p}{4}S(ab',p)+\frac{p}{4}+\frac{p}{4a}r_p(b,a)+\frac{p}{4b}r_p(a,b). \tag{2.6}$$

If $a+b=pc$, where p and c are positive integers, then we have

$$\begin{aligned} B(p,a,b) &= \frac{p}{4}S(ab',p)+\frac{1}{4}\sum_{j=1}^{\mu-1}\cot^2(\frac{\pi h j}{\mu}) \\ &= \frac{p}{4}S(ab',p)+\frac{(p-1)(p-2)}{12}, \end{aligned}$$

where $S(ab',p)$ is the three-term relation for Hardy sums and $r_p(a,b)$ is the Berndt and Yeap sum.

For a proof of the above theorem we need the following relation:

The mathematical literature contains many evaluations of finite trigonometric sums of the sort ([8])

$$\sum_{j=1}^{k-1}\cot^2(\frac{\pi j}{k}) = \frac{(k-1)(k-2)}{3}. \tag{2.7}$$

Now, by (2.5), recall that $h+k=\mu c$. Thus by the fact that h and k are coprime and (2.7), we get

$$\begin{aligned} \sum_{j=1}^{\mu-1}\cot(\frac{\pi h j}{\mu})\cot(\frac{\pi k j}{\mu}) &= -\sum_{j=1}^{\mu-1}\cot^2(\frac{\pi h j}{\mu}) \\ &= -\sum_{j=1}^{\mu-1}\cot^2(\frac{\pi j}{\mu}) = -\frac{(\mu-1)(\mu-2)}{3}. \end{aligned} \tag{2.8}$$

And we see from the equation (2.23) in [8] that have

$$\begin{aligned}(2.9)\quad &\sum_{j=1}^{\mu-1}\cot(\frac{\pi hj}{\mu})\cot(\frac{\pi kj}{\mu})\\ &=\frac{1}{3}(\frac{k}{h}+\frac{h}{k}+\frac{\mu^2}{hk})-\mu(1+\frac{1}{k}r_\mu(h,k)+\frac{1}{h}r_\mu(h,k)).\end{aligned}$$

Proof of theorem 2.8. By using Theorem 2.2 and (2.3), we have

$$B(p,a,b)=\frac{1}{4}d(p;a,b)+\frac{p}{4}\sum_{v=1}^{p-1}(-1)^{j+[\frac{av}{p}]+[\frac{bv}{p}]}.$$

Now, the definition of $d(p;a,b)$ in the above yields

$$B(p,a,b)=\frac{1}{4}\sum_{j=1}^{p-1}\cot(\frac{\pi aj}{p})\cot(\frac{\pi bj}{p})+\frac{p}{4}\sum_{v=1}^{p-1}(-1)^{j+[\frac{av}{p}]+[\frac{bv}{p}]},$$

where $a+b+p$ is even. Here, we note that a, b, and b' are odd and consequently $a+b+p\equiv 0 \pmod 2$. If $bb'\equiv 1+pk$, then k is even and so $b'+k\equiv 1 \pmod 2$. Hence by Lemma 2.7, we obtain

$$B(p,a,b)=\frac{1}{4}\sum_{j=1}^{p-1}\cot(\frac{\pi aj}{p})\cot(\frac{\pi bj}{p})+\frac{p}{4}S(ab',p).$$

By substituting (2.8) and (2.9) into the above equation and after some elementary calculations, we arrive at the result. □

Remark 2.9. In [26] the first author gave behaviors of the Weierstrass $\wp$-function under the modular transformations. By using this relations, he gave between $\wp$-function and Hardy sums. In [2], [3], Bayad studied on the Jacobi form and Dedekind sums. He gave many applications related to Dedekind sums and Jacobi form.

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References

[1] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer-Verlag, 1976.

[2] A. Bayad, *Applications aux sommes elliptiques d'Apostol-Dedekind-Zagier*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 8, 529–532.

[3] ———, *Sommes elliptiques multiples d'Apostol-Dedekind-Zagier*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 7, 457–462.

[4] B. C. Berndt, *Generalized Dedekind eta-functions and generalized Dedekind sums*, Trans. Amer. Math. Soc. **178** (1973), 495–508.

[5] ———, *Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan*, J. Reine Angew. Math. **303/304** (1978), 332–365.
[6] B. C. Berndt and L. A. Goldberg, *Analytic properties of arithmetic sums arising in the theory of the classical theta functions*, SIAM J. Math. Anal. **15** (1984), no. 1, 143–150.
[7] B. C. Berndt and L. Schoenfeld, *Periodic analogues of the Euler-Maclaurin and Poisson summation formulas with applications to number theory*, Acta Arith. **28** (1975/76), no. 1, 23–68.
[8] B. C. Berndt and B. P. Yeap, *Explicit evaluations and reciprocity theorems for finite trigonometric sums*, Adv. in Appl. Math. **29** (2002), no. 3, 358–385.
[9] D. Bertrand, *Theta functions and transcendence*, Ramanujan J. **1** (1997), no. 4, 339–350.
[10] M. Can, M. Cenkci, and V. Kurt, *Generalized Hardy-Berndt sums*, Proc. Jangjeon Math. Soc. **9** (2006), no. 1, 19–38.
[11] U. Dieter, *Cotangent sums, a further generalization of Dedekind sums*, J. Number Theory **18** (1984), no. 3, 289–305.
[12] S. Egami, *An elliptic analogue of the multiple Dedekind sums*, Compositio Math. **99** (1995), no. 1, 99–103.
[13] M. I. Knoop, *Modular Functions in Analytic Number Theory*, Markham Publishing Company, Chicago, 1970.
[14] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer-Verlag, New York, 1993.
[15] V. Kurt, *On Dedekind sums*, Indian J. Pure Appl. Math. **21** (1990), no. 10, 893–896.
[16] J. Lewittes, *Analytic continuation of the series* $\sum(m+nz)^{-s}$, Trans. Amer. Math. Soc. **159** (1971), 505–509.
[17] ———, *Analytic continuation of Eisenstein series*, Trans. Amer. Math. Soc. **171** (1972), 469–490.
[18] S. J. Patterson, *An Introduction to the Theory of the Riemann Zeta-Function*, Cambridge University Press, Cambridge, 1988.
[19] M. R. Pettet and R. Sitaramachandrarao, *Three-term relations for Hardy sums*, J. Number Theory **25** (1987), no. 3, 328–339.
[20] H. Rademacher, *Zur Theorie der Modulfunktionen*, J. Reine Angew. Math. **167** (1932), 312–336.
[21] ———, *Topics in Analytic Number Theory*, Springer-Verlag, 1988.
[22] R. A. Rankin, *Elementary proofs of relations between Eisenstein series*, Proc. Roy. Soc. Edinburgh Sect. A **76** (1976/77), no. 2, 107–117.
[23] B. Schoneneberg, *Zur Theorie der verallgemeinerten Dedekindschen Modulfunktionen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1969), 119–128.
[24] Y. Simsek, *Theorems on three-term relations for Hardy sums*, Turkish J. Math. **22** (1998), no. 2, 153–162.
[25] ———, *Relations between theta-functions Hardy sums Eisenstein and Lambert series in the transformation formula of* $log_{g,h}(z)$, J. Number Theory **99** (2003), no. 2, 338–360.
[26] ———, *On Weierstrass* $\wp(z)$ *function Hardy sums and Eisenstein series*, Proc. Jangjeon Math. Soc. **7** (2004), no. 2, 99–108.
[27] ———, *Generalized Dedekind sums associated with the Abel sum and the Eisenstein and Lambert series*, Adv. Stud. Contemp. Math. (Kyungshang) **9** (2004), no. 2, 125–137.
[28] Y. Simsek and S. Yang, *Transformation of four Titchmarsh-type infinite integrals and generalized Dedekind sums associated with Lambert series*, Adv. Stud. Contemp. Math. (Kyungshang) **9** (2004), no. 2, 195–202.
[29] R. Sitaramachandrarao, *Dedekind and Hardy sums*, Acta Arith. **XLVIII** (1978), 325–340.
[30] M. Waldschmidt, P. Moussa, J. M. Luck, and C. Itzykson, *From Number Theory to Physics*, Springer-Verlag, 1995.
[31] D. Zagier, *Higher dimensional Dedekind sums*, Math. Ann. **202** (1973), 149–172.

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