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MULTIPLE SOLUTIONS FOR THE NONLINEAR HAMILTONIAN SYSTEM

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Abstract. We give a theorem of the existence of the multiple solutions of the Hamiltonian system with the square growth nonlinearity. We show the existence of m solutions of the Hamiltonian system when the square growth nonlinearity satisfies some given conditions. We use critical point theory induced from the invariant function and invariant linear subspace.

1. Introduction and statement of main result

Let $H(z(t))$ be a C^2 function defined on R^{2n} which is $2\pi-$periodic with respect to the variable t. Let $z = (p, q)$, $p = (z_1, \cdots, z_n)$, $q = (z_{n+1}, \cdots, z_{2n})$. In this paper we investigate the existence and the multiplicity of $2\pi-$periodic solutions of the following Hamiltonian system

$$\begin{aligned} \dot{p} &= -H_q(p, q), \qquad (1.1) \\ \dot{q} &= H_p(p, q). \end{aligned}$$

Letting J be the standard symplectic structure on R^{2n}, i.e.,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

I_n is the $n \times n$ identity matrix on R^n, system (1.1) can be written in a compact version

$$-J\dot{z} = H_z(z), \qquad (1.2)$$

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where $z : R \to R^{2n}$, $\dot{z} = \frac{dz}{dt}$ and H_z is the gradient of H. We define

$$F(z) = H_z(z(t))$$

Let $L^2(S^1, R^{2n})$ denote the set of $2n$-tuples of 2π periodic functions which are square integrable. If $z \in L^2(S^1, R^{2n})$, it has a Fourier expansion $z = \sum_{k \in Z} a_k e^{ikt}$, where $a_k = \frac{1}{2\pi}\int_0^{2\pi} z(t)e^{-ikt}dt \in C^{2n}$, $a_{-k} = \bar{a_k}$ and $\sum_{k \in Z} |a_k|^2 < \infty$. Let

$$E = W^{\frac{1}{2},2}(S^1, R^{2n}) = \{z \in L^2(S^1, R^{2n}) | \sum_{k \in Z}(1 + |k|)|a_k|^2 < \infty\}$$

and let

$$\|z\| = \|z\|_{L^{\frac{1}{2},2}} = (\sum_{k \in Z}(1 + |k|)|a_k|^2)^{\frac{1}{2}}. \qquad (1.3)$$

The space E endowed with this norm is a real Hilbert space continuously embedded in $L^2(S^1, R^{2n})$. Let $a \cdot b$ and $|\cdot|$ denote the usual inner product and norm on R^{2n}. We assume that the Hamiltonian function H satisfies the following conditions:

(H1) $H \in C^2(R^{2n}, R)$, $H(z) = o(|z|^2)$ as $|z| \to 0$.

(H2) There exist constants α and β such that α, $\beta \neq Z$, $\alpha < \beta$ and

$$\alpha|z|^2 < H_z(z) \cdot z < \beta|z|^2 \qquad \forall z \in R^{2n}.$$

(H3) There exist integers j_1, j_2,. . ., j_{2m} in $[\alpha, \beta]$.

(H4) There exist γ and C such that $j_{2m} < \gamma < \beta$ and

$$H(z) \geq \frac{1}{2}\gamma\|z\|^2 - C \qquad \forall z \in R^{2n}.$$

(H5) H is $2\pi-$periodic with respect to t.

By (H2), there is a constant $C > 0$ such that

$$\|F_z(z)\|_{L^2(R^{2n})} \leq C,$$

then

$$\Phi(z) = \int_0^{2\pi} H(z(t))dt \in C^1(E, R).$$

In this paper we are looking for the weak solutions $z \in E$ of (1.2); that is, $z \in E$ satisfies

$$\int_0^{2\pi} (\dot{z} - J(H_z(z))) \cdot Jwdt = 0 \qquad \forall w \in E.$$

We observe that by Proposition 2.1, the weak solutions of (1.2) coincide with the critical points of the corresponding functional

$$f(z) = \frac{1}{2}A(z) - \int_0^{2\pi} H(z(t))dt,$$

where $A(z) = \int_0^{2\pi} \dot{z} \cdot Jzdt$. Here $f(z) \in C^1(E, R)$.

Our main result is the following:

THEOREM 1.1. *Assume that H satisfies the conditions $(H1)$-$(H5)$. Then (1.2) has at least m weak solutions, which are geometrically distinct and nonconstant. Moreover, if H is of class C^k, these solutions are of class $C^k(S^1, R^{2n})$.*

Our proof use the ideas in [1]. In section 2, we introduce a closed invariant linear subspace X of E which is invariant under A, the invariant subspaces of X and the invariant function on X. We obtain some results on the norm $\|\cdot\|$ and the functional $f(z)$, and recall a critical point theory in terms of the S^1-invariant functional and S^1-invariant subspaces which plays a crucial role for the proof of the main result. In section 3, we show that the functional f satisfies the conditions for the multiple solution theorem, and prove Theorem 1.1.

2. Some results on $\|\cdot\|$, f

Let $E = W^{\frac{1}{2},2}(S^1, R^{2n})$. The scalar product in L^2 naturally extends as the duality pairing between E and $E' = W^{-\frac{1}{2},2}(S^1, R^{2n})$. For smooth $z = (p, q) \in E$, where p and q are each $n-$ tuples, we can check that

$$\|Az\| \le (\sum_{j\in Z} |j||p_j|^2)^{\frac{1}{2}} (\sum_{j\in Z} |j||q_j|^2)^{\frac{1}{2}} \le \|z\|_E^2.$$

Therefore A extends to all of E as a continuous quadratic form. This extension will still be denoted by A. Let $e_1, \cdots, e_{2n}$ denote the usual bases in R^{2n} and set

$$E^0 = span\{e_1, \cdots, e_{2n}\},$$

$$E^+ = span\{(\sin jt)e_k - (\cos jt)e_{k+n}, (\cos jt)e_k + (\sin jt)e_{k+n}, \\ \mid j \in N, 1 \le k \le n\},$$

$$E^- = span\{(\sin jt)e_k + (\cos jt)e_{k+n}, (\cos jt)e_k - (\sin jt)e_{k+n} \mid j \in N, 1 \le k \le n\}.$$

Then $E = E^0 \oplus E^+ \oplus E^-$ and E^+, E^-, E^0 are the subspaces of E on which A is positive definite, negative definite, and null, and these spaces E^+, E^- and E^0 are mutually orthogonal in $L^2(S^1, R^{2n})$. If $z = z^0 + z^+ + z^- \in E$, $A_+ = A|_{E^+}$ and $A_- = A|_{E^-}$, then

$$\|z\|^2 = |z^0|^2 + A_+(z^+) - A_-(z^-)$$

which serves as an equivalent norm on E. The space E with this norm $\|\cdot\|$ is a Hilbert space.
We need the following facts in [4]:

Proposition 2.1. *For each $s \in [1, \infty)$, E is compactly embedded in $L^s(S^1, R^{2n})$. In particular, there is an $\alpha_s > 0$ such that*

$$\|z\|_{L^s} \le \alpha_s \|z\|$$

for all $z \in E$.

By the following proposition which was proved in Proposition 2 of [3], $f \in C^1$ and *Frèchet* differentiable in E, hence the weak solutions of (1.2) coincide with the critical points of the functional $f(z)$.

Proposition 2.2. *Assume that H satisfies the conditions $(H1)$-$(H5)$. Then $f(z)$ is C^1, that is, $f(z)$ is continuous and Frèchet differentiable in E with Frèchet derivative*

$$\begin{aligned} \nabla f(z)\omega &= \int_0^{2\pi} (\dot{z} - J(H_z)) \cdot J\omega \\ &= \int_0^{2\pi} [(\dot{p} + H_q(z)) \cdot \psi - (\dot{q} - H_p(z)) \cdot \phi] dt, \end{aligned}$$

where $z = (p, q)$ and $\omega = (\phi, \psi) \in E$. Moreover the functional $z \mapsto \int_0^{2\pi} H(z)dt$ is C^1.

Let us define some notations and concepts on S^1-invariant set and S^1-invariant function: Let X be a real Hilbert space on which the compact Lie group S^1 acts by means of time translations, hence by orthogonal transformations; for $z \in X$ and $\theta \in [0, 2\pi]$, we define an S^1-action on X by

$$(T_\theta z)(t) = z(t + \theta), \qquad \text{for all } t \in [0, 2\pi].$$

Let $Fix\{T_\theta\}$ be the set of fixed points of the action, i.e.,

$$Fix\{T_\theta\} = \{z \in X \mid T_\theta z = z,\ \forall \theta \in [0, 2\pi]\}.$$

We say a subset B of X an S^1-invariant set if for all $z \in B$ and $\theta \in [0, 2\pi]$, $T_\theta z \in B$. A function $f : X \to R^1$ is called S^1-invariant, if $f(T_\theta z) = f(z)$, $\forall z \in X$, for all $\theta \in [0, 2\pi]$. Let $C(B, X)$ be the set of continuous functions from B into X. If B is an invariant set we say $h \in C(B, X)$ is an equivariant map if $h(T_\theta z) = T_\theta h(z)$ for all $\theta \in [0, 2\pi]$ and $z \in B$.

Now we define the subspaces X, X^+, X^-, and X^0 of E as follows: Let us denote, for $i = \sqrt{-1}$ and $k \in N$,

$$\begin{aligned}
\check{\phi}_k &= (\sin kt)e_k - (\cos kt)e_{k+n},\\
i\check{\phi}_k &= (\cos kt)e_k + (\sin kt)e_{k+n},\\
\check{\psi}_k &= (\sin kt)e_k + (\cos kt)e_{k+n},\\
i\check{\psi}_k &= (\cos kt)e_k - (\sin kt)e_{k+n}.
\end{aligned}$$

Let z be a function of $W^{\frac{1}{2},2}(S^1, R^{2n})$; there exists one and only one function of $W^{\frac{1}{2},2}(R, R^{2n})$ which is 2π periodic in t and equals z on S^1; we shall denote this function by z. Let X be the closed subspace of E defined by

$$X = \{z \in E \mid a_k = 0 \text{ if } k \text{ is even }\}.$$

Then X is a closed invariant linear subspace of E compactly embedded in $L^2(S^1, R^{2n})$. Moreover $A(X) \subseteq X$, $A : X \to X$ is an isomorphism and $\nabla f(X) \subseteq X$. Therefore constrained critical points on X are in fact free critical points on E. Moreover, distinct critical orbits give rise to geometrically distinct solutions. From now on f will denote the restriction of f to X. Let

$$\begin{aligned}
X^+_{m,l} &= \{z \mid z \in X,\ z \in \operatorname{span}\{\check{\phi}_k, i\check{\phi}_k \mid m \le k \le l\}\},\\
X^-_{m,l} &= \{z \mid z \in X,\ z \in \operatorname{span}\{\check{\psi}_k, i\check{\psi}_k \mid m \le k \le l\}\},\\
X^+ &= \{z \mid z \in X,\ z \in \operatorname{span}\{\check{\phi}_k, i\check{\phi}_k \mid n+1 \le k < \infty\}\},\\
X^- &= \{z \mid z \in X,\ z \in \operatorname{span}\{\check{\psi}_k, i\check{\psi}_k \mid n+1 \le k < \infty\}\},\\
X^0 &= \{z \mid z \in X,\ z \in \operatorname{span}\{\check{\phi}_k, i\check{\phi}_k \mid 1 \le k \le n\}\}.
\end{aligned}$$

Then $X = X^+ \oplus X^- \oplus X^0$ and $A(z)$ is positive definite, negative definite and null on X^+, X^-, X^0, respectively. For

$$z = z^+ + z^- + z^0 \in X^+ \oplus X^- \oplus X^0 = X,$$

we take a norm for X

$$\|z\|_X^2 = A(z^+) - A(z^-) + |z^0|^2.$$

With this norm, X become a Hilbert space and X^+, X^-, X^0 are orthogonal subspaces of X with respect to the inner product associated with this norm, as well, as the L^2 inner product. We note that by Proposition 2.2, $f(z) \in C^1(X, R)$. We have the following lemma:

LEMMA 2.1. *Assume that H satisfies the conditions (H1)-(H5). Let $z \in Fix\{T_\theta\}$ and z be a critical point of the functional of f, i.e., $\nabla f(z) = 0$. Then $f(z) = 0$.*

Proof. Let $\rho : R \to R$ be a Borel function defined by

$$\rho(z) = \begin{cases} \frac{H_z(z)\cdot z}{|z|^2} & if\ z \neq (0, \dots, 0), \\ \alpha + \frac{\beta - \alpha}{2} & if\ z = (0, \dots, 0). \end{cases}$$

Since $\nabla f(z)z = 0$, we have that

$$\int_0^{2\pi} \dot{z}Jw - \rho(z)z \cdot w dt = 0 \qquad \text{for } \ w \in E. \tag{2.1}$$

Let us set $z = z_1 + z_2$, z_1, $z_2 \in E$, such that $\int_0^{2\pi} \dot{z}_1 J z_1 dt \leq \alpha \int_0^{2\pi} z_1^2 dt$ and $\int_0^{2\pi} \dot{z}_2 J z_2 dt \geq \beta \int_0^{2\pi} z_2^2 dt$. Putting $z = z_1 + z_2$ and $w = z_2 - z_1$ into (2.1), we have

$$\int_0^{2\pi} (\rho(z) - \alpha) z_1^2 dt + \int_0^{2\pi} (\beta - \rho(z)) z_2^2 dt \leq 0.$$

Thus we have $(\rho(z) - \alpha)z_1^2 = (\beta - \rho(z))z_2^2$=0 a.e. in $]0, 2\pi[$. Since ρ is continuous on $R\backslash 0$, $\rho(z) \in \{\alpha, \beta\}$ if $z \neq (0, \dots, 0)$. In any case we have that

$$H_z(z) \cdot z = 2H(z).$$

It follows that

$$f(z) = \int_0^{2\pi} [\frac{1}{2} H_z(z) \cdot z - H(z)] dt = 0.$$

Thus we prove the lemma. □

Now we recall the critical point theory in terms of the S^1−invariant subspace and S^1−invariant function in Theorem 4.1 of [1] which plays a crucial role for the proof of Theorem 1.1: Let S_r be the sphere centered at the origin of radius r. Let $f : X \to R$ be a functional of the form

$$f(z) = L(z) - \psi(z), \tag{2.2}$$

where $L : X \to R$ is linear, continuous, symmetric and equivariant, $\psi : X \to R$ is of class C^1 and invariant and $D\psi : X \to X$ is compact.

THEOREM 2.1. *Assume that $f \in C^1(X, R^1)$ is S^1-invariant and there exist two closed invariant linear subspaces V, W of X and $r > 0$ with the following properties:*
(a) $V + W$ is closed and of finite codimension in X;
(b) $Fix\{T_\theta\} \subseteq V + W$;
(c) $L(W) \subseteq W$;
(d) $\sup_{S_r \cap V} f < +\infty$ and $\inf_W f > -\infty$;
(e) $u \notin Fix\{T_\theta\}$ whenever $Df(z) = 0$ and

$$\inf_W f \leq f(z) \leq \sup_{S_r \cap V} f;$$

(f) f satisfies $(P.S.)_c$ condition whenever $\inf_W f \leq c \leq \sup_{S_r \cap V} f$.
Then f possesses at least

$$\frac{1}{2}(dim(V \cap W) - codim_X(V + W))$$

distinct critical orbits in $f^{-1}([\inf_W f, \sup_{S_r \cap V} f])$.

3. Proof of theorem 1.1

From now on we shall prove Theorem 1.1 by applying the multiplicity result of Theorem 2.1. We assume that H satisfies the conditions (H1)-(H5). Let $c_0 = \frac{\alpha + j_1}{2}$ and let $L_0 : X \to X$ be the linear operator such that

$$L_0(z) = A(z) - c_0 \int_0^{2\pi} z^2 dt.$$

Then L_0 is symmetric, bijective and equivariant. Let $X^-(L_0)$ be the negative space of L_0 and $X^+(L_0)$ be the positive space of L_0. Then

$$X = X^-(L_0) \oplus X^+(L_0).$$

Moreover, we have

$$\forall z \in X^-(L_0) : L_0(z) \leq ((j_1 - 1) - c_0) \int_0^{2\pi} z^2 dt, \tag{3.1}$$

$$\forall z \in X^+(L_\infty) : L_0(z) \geq (j_1 - c_0) \int_0^{2\pi} z^2 dt.$$

Thus there exists $b > 0$ such that

$$L_0(z) \leq -b\|z\|_X^2, \qquad \forall z \in X^-(L_0),$$

$$L_0(z) \geq b\|z\|_X^2, \qquad \forall z \in X^+(L_0).$$

Then

$$f(z) = \frac{1}{2} L_0(z) - \psi_0(z), \tag{3.2}$$

where

$$\psi_0(z) = \int_0^{2\pi} [H(z) - \frac{1}{2} c_0 z^2] dt.$$

Since X is compactly embedded in $L^2(S^1, R^{2n})$, the map $D\psi_0 : X \to X$ is compact.

LEMMA 3.1. *Assume that H satisfies the conditions (H1)-(H5). Then the functional $f(z)$ is bounded from above on $X^-(L_0)$ and from below on $X^+(L_0)$. That is,*

$$-\infty < \inf_{z \in X^+(L_0)} f(z) \qquad \textit{and} \qquad \sup_{z \in X^-(L_0)} f(z) < \infty.$$

Proof. Let us take a, $\bar{a} \in R$ with

$$\alpha < a < \inf_{z \in X \setminus \{0\}} \frac{H_z(z) \cdot z}{|z|^2} < \bar{a} < j_1 \tag{3.3}$$

and set $H_0(z) = H(z) - \frac{1}{2} c_0 z^2$. Then there exists constant $c \geq 0$ such that $H_0(z) \leq \frac{1}{2} \tau z^2 + c$, where $\tau = \bar{a} - c_0 < \frac{j_1 - \alpha}{2}$. We have that for $z \in X^+(L_0)$,

$$L_0(z) \geq (j_1 - c_0) \int_0^{2\pi} z^2 dt = \frac{j_1 - \alpha}{2} \int_0^{2\pi} z^2 dt,$$

$$\psi_0(z) = \int_0^{2\pi} H_0(z) dt \leq \frac{1}{2} \tau \int_0^{2\pi} z^2 dt + 2\pi c.$$

Thus we have

$$f(z) = \frac{1}{2} L_0(z) - \psi_0(z) \geq \frac{1}{2} (\frac{j_1 - \alpha}{2} - \tau) \int_0^{2\pi} z^2 dt - 2\pi c > -\infty.$$

Similarly, from (3.3), there exists constant $\bar{c} > 0$ such that $H_0(z) \geq \frac{1}{2}\bar{\tau}z^2 - \bar{c}$, where $\bar{\tau} = a - c_0 > \frac{\alpha - j_1}{2}$. Then we have that for $z \in X^-(L_0)$,

$$L_0(z) \leq (\alpha - c_0)\int_0^{2\pi} z^2 dt = -\frac{j_1 - \alpha}{2}\int_0^{2\pi} z^2 dt,$$

$$\psi_0(z) = \int_0^{2\pi} H_0(z)dt \geq \frac{1}{2}\bar{\tau}\int_0^{2\pi} z^2 dt - 2\pi\bar{c}.$$

Thus we have

$$f(z) = \frac{1}{2}L_0(z) - \psi_0(z) \leq (\frac{\alpha - j_1}{4} - \frac{1}{2}\bar{\tau})\int_0^{2\pi} z^2 dt + 2\pi\bar{c} < \infty.$$

□

LEMMA 3.2. *Assume that H satisfies the conditions (H1)-(H5). Then the functional f satisfies $(P.S.)_c$ condition for every $c \in R$.*

Proof. We shall use the finite dimensional reduction method. Let us define

$$P_0 = \int_\alpha^\beta dE_\lambda, \qquad P_+ = \int_\beta^{+\infty} dE_\lambda, \qquad P_- = \int_{-\infty}^\alpha dE_\lambda,$$

where $\{E_\lambda\}$ is the spectral resolution of the map:$z(t) \mapsto -J\dot{z(t)}$, and let

$$H_0 = P_0 H, \qquad H_\pm = P_\pm H.$$

Let us define the finite dimensional reduction functional

$$\tilde{f}(z) = \frac{1}{2}\int_0^{2\pi} \dot{u(z)}Ju(z)dt - \int_0^{2\pi} H(u(z))dt,$$

where $u(z) = z + u_+(z) + u_-(z)$, $z \in H_0$, and $u_\pm(z) \in H_\pm$. Let us set $w = z + u_-(z)$, Then we have

$$\tilde{f}(z) = \frac{1}{2}\int_0^{2\pi} \dot{w}Jwdt - \int_0^{2\pi} H(w)dt$$

$$+\{\frac{1}{2}[\int_0^{2\pi} \dot{u(z)}Ju(z)dt - \int_0^{2\pi} \dot{w}Jwdt] - \int_0^{2\pi} [H(u(z)) - H(w)]dt\}.$$

We have that

$$\frac{1}{2}[\int_0^{2\pi} \dot{u(z)}Ju(z)dt - \int_0^{2\pi} \dot{w}Jwdt] - \int_0^{2\pi} [H(u(z)) - H(w)]dt$$

$$= \frac{1}{2}\int_0^{2\pi} \dot{u(z)}Ju_+dt - \int_0^{2\pi} (H_z(su_+ - w), u_+)ds$$

$$= \int_0^{2\pi}\int_0^{2\pi}(d_z^2H(su_+ + w)u_+, u_+)sdsdt - \frac{1}{2}\int_0^{2\pi}\dot{u_+}Ju_+ \le 0$$

by condition (H2). By condition (H3) and condition (H4), we have

$$\tilde{f}(z) \le \frac{1}{2}\int_0^{2\pi}\dot{w}Jwdt - \int_0^{2\pi}H(w)dt$$

$$\le \frac{1}{2}(2\pi j_m - \gamma)\|w\|^2 + C \longrightarrow -\infty \quad \text{as} \quad \|z\| \to \infty$$

Thus the functional $\tilde{f}$ is bounded from below and satisfies the $(P.S.)$ condition, so the function f satisfies the $(P.S.)$ condition. □

Let $c_1 = \frac{j_m+\beta}{2}$ and let $L_1 : X \to X$ be the linear operator such that

$$L_1(z) = A(z) - c_1\int_0^{2\pi} z^2dt. \tag{3.4}$$

Then L_1 is symmetric, bijective and equivariant. Let $X^-(L_1)$ be the negative space of L_1 and $X^+(L_1)$ be the positive space of L_1. Then

$$X = X^-(L_1) \oplus X^+(L_1).$$

Moreover, we have

$$\forall z \in X^-(L_1) : L_1(z) \le (j_m - c_1)\int_0^{2\pi} z^2dt, \tag{3.5}$$

$$\forall z \in X^+(L_1) : L_1(z) \ge \{\beta - c_1\}\int_0^{2\pi} z^2dt.$$

Thus there exists $d > 0$ such that

$$L_1(z) \le -d\|z\|_X^2, \qquad \forall z \in X^-(L_1),$$

$$L_1(z) \ge d\|z\|_X^2, \qquad \forall z \in X^+(L_1).$$

Hence

$$f(z) = \frac{1}{2}L_1(z) - \psi_1(z), \tag{3.6}$$

where

$$\psi_1(z) = \int_0^{2\pi}[H(z) - \frac{1}{2}c_1z^2]dt.$$

Since X is compactly embedded in $L^2(S^1, R^{2n})$, the map $D\psi_1 : X \to X$ is compact.

LEMMA 3.3. *Assume that H satisfies the conditions (H1)-(H5). Let*

$$H_1(z) = H(z) - \frac{1}{2}c_1 z^2$$

Then

$$\inf_{z\in X} \frac{H_1(z)}{1+z^2} > -\infty, \qquad \lim_{|z|\to 0} \inf \frac{H_1(z)}{z^2} \geq 0 \tag{3.7}$$

and

$$\lim_{\substack{|z|\to 0\\ z\in X}} \inf \frac{\int_0^{2\pi} H_1(z)dt}{\|z\|_X^2} \geq 0. \tag{3.8}$$

Proof. Let us take e, $\bar{e}$ with

$$j_{2m} < e < \sup_{z\in X\setminus\{0\}} \frac{H_z(z)\cdot z}{|z|^2} < \bar{e} < \beta. \tag{3.9}$$

Then there exists $h > 0$ such that $H_1(z) \geq \frac{1}{2}\theta z^2 - h$, where $\theta = e - c_1 > \frac{j_{2m}-\beta}{2}$. Thus $\frac{H_1(z)}{1+z^2} \geq \frac{\frac{1}{2}\theta z^2 - h}{1+z^2} > \frac{\frac{j_{2m}-\beta}{4}z^2 - h}{1+z^2} > -\infty$ because $h > 0$ and $\frac{j_{2m}-\beta}{4}z^2 < 0$. Next we will prove (3.8). Let

$$\gamma_1(s) = \begin{cases} \left(\frac{H_1(s)}{s^2}\right)^- & \text{if } |s| \neq 0,\\ 0 & \text{if } s = (0,\ldots,0). \end{cases}$$

Then $\gamma_1 : R^{2n} \to R$ is bounded, continuous, with $\gamma_1(0,\ldots,0) = 0$ and $H_1(s) \geq -\gamma_1(s)s^2$. If (z_n) is a sequence in X with $z_n \to (0,\ldots,0)$, then, up to a subsequence, $z_n \to (0,\ldots,0)$ a.e. and $w_n = \frac{z_n}{\|z_n\|_X}$ is strongly convergent in $L^2(S^1)$. Since

$$\frac{\int_0^{2\pi} H_1(z_n)dt}{\|z_n\|_X^2} \geq -\int_0^{2\pi} \gamma_1(z_n)w_n^2 dt.$$

Thus $\lim_{\substack{|z|\to 0\\ z\in X}} \inf \frac{\int_0^{2\pi} H_1(z)dt}{\|z\|_X^2} \geq 0$. Thus we prove the lemma. □

Let us set

$$S_r = \{z \in X^-(L_1) |\ \|z\|_X = r\}.$$

LEMMA 3.4. *Assume that H satisfies the conditions (H1)-(H5). Then there exists a neighborhood $S_r \subset X^-(L_1)$ of 0 with radius $r > 0$ such that*

$$\sup_{z\in S_r\cap X^-(L_1)} f(z) < 0.$$

Proof. From (3.6), we have

$$f(z) = \frac{1}{2}L_1(z) - \int_0^{2\pi} H_1(z)(z)dt.$$

From (3.5), there exists $d > 0$ such that $L_1(z) \leq -d\|z\|_X^2$, $\forall z \in X^-(L_1)$. By Lemma 3.3, $\lim_{\substack{|z|\to 0\\ z\in X}} \inf \frac{\int_0^{2\pi} H_1(z)dt}{\|z\|_X^2} \geq 0$.

$$\frac{f(z)}{\|z\|_X^2} \leq -d - \lim_{\substack{|z|\to 0\\ z\in X}} \inf \frac{\int_0^{2\pi} H_1(z)dt}{\|z\|_X^2} < 0.$$

If $V = X^-(L_1)$, then

$$\lim_{\substack{|z|\to 0\\ z\in V}} \sup \frac{f(z)}{\|z\|_X^2} < 0.$$

There exists $r > 0$ such that if $z \in S_r \cap V$, then

$$\sup_{z\in S_r\cap V} f(z) < 0.$$

□

Proof of Theorem 1.1

If we set $V = X^-(L_1)$ and $W = X^+(L_0)$, then V and W are closed invariant subspaces of X with $V+W = X$, $L(W) \subseteq W$, $\mathrm{codim}(V+W) = 0$ and $Fix\{T_\theta\} \subset V + W$. By Proposition 2.2, f is $C^1(X, R^1)$ and by Lemma 3.2, $f(z)$ satisfies the $(P.S.)_c$ condition for any $c \in R$. By Lemma 3.1 and Lemma 3.4, assumption (d) of Theorem 2.1 is satisfied. By Lemma 2.1, the condition (e) of Theorem 2.1 is satisfied. Thus by Theorem 2.1, (1.2) has at least $\frac{1}{2}\dim(V \cap W) = m$ nontrivial solutions.

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