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EXISTENCE AND MULTIPLICITY RESULTS FOR SOME FOURTH ORDER SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. We prove the existence and multiplicity of nontrivial solutions for a fourth order problem $\Delta^2 u + c\Delta u = \alpha u - \beta(u+1)^-$ in Ω, $\Delta u = 0$ and $u = 0$ on $\partial\Omega$, where $\lambda_1 \le c \le \lambda_2$ (where $(\lambda_i)_{i\ge 1}$ is the sequence of the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$) and Ω is a bounded open set in R^N with smooth boundary $\partial\Omega$. The results are proved by applying minimax arguments and linking theory.

1. Introduction

In this paper we get some results about the number of nontrivial solutions for the fourth order semilinear elliptic boundary value problem:

$$\begin{aligned} &\Delta^2 u + c\Delta u = \alpha u - \beta(u+1)^- &&\text{in } \Omega, \\ &u = 0, \qquad \Delta u = 0 &&\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Δ^2 denote the biharmonic operator, Δ is the Laplacian on R^N, $u^- = \max\{-u, 0\}$. Throughout this paper Ω will denote a bounded and smooth domain of R^N, $N \ge 1$, $\alpha, c, \beta > 0 \in R$.

In section 2 we recall Mountain pass theorem and a Linking Theorem which will play a crucial role in our argument. In section 3 we define a Banach space H spanned by eigenfunctions of $\Delta^2 + c\Delta$ with Dirichlet boundary condition which can be applied in the linking theorem. In section 4 we prove main Theorem. The proofs are based on a variational approach where the associated functional is indefinite (it is unbounded

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from below and from above). First we prove the existence of a point of local minimum of the functional at a strictly negative level (which is the minimum of the functional on the negative functions cone). Then we succeed to apply the mountain pass lemma and to verify that this last critical level is strictly negative. Finally, by using a Linking Theorem, we obtain a result of existence of two nontrivial solutions for the problem when $\Lambda_s \leq \alpha \leq \Lambda_{s+1}$

2. Variational setting

In section 2 to introduce a Mountain pass Theorem and a Variational Linking Theorem, we define the following sets.

Definition 2.1. *Let X be a Hilbert space, $Y \subset X$, $\rho > 0$, and $e \in X \setminus Y$, $e \neq 0$. Set*

- $B_\rho(Y) = \{x \in Y \mid \|x\|_X \leq \rho\}$,
- $S_\rho(Y) = \{x \in Y \mid \|x\|_X = \rho\}$,
- $\Delta_\rho(e, Y) = \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\|_X \leq \rho\}$,
- $\Sigma_\rho(e, Y) = \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\|_X = \rho\} \cup \{v \mid v \in Y,\ \|v\|_X \leq \rho\}$.

Theorem 2.2 (Mountain pass theorem). *Let E be a real Banach space and $I \in C^1(E, R)$ satisfying (PS). suppose $I(0) = 0$ and*

(I_1) there are constants $\rho, \alpha > 0$ such that $I|\partial B_\rho \geq \alpha$,and

(I_2) there is an $e \in E \setminus \underline{B}_\rho$ such that $I(e) \leq 0$.

Then I possesses a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{g\in\Gamma} \max_{u\in g([0,1])} I(u).$$

where

$$\Gamma = \{g \in C([0, 1], E)|g(0) = 0, g(1) = e\}.$$

We recall a theorem of existence of two critical levels for a functional which is a variation of linking theorem.

Theorem 2.3 (A variation of linking.). *Let X be a Hilbert space which is topological direct sum of the subspaces X_1, X_2. Let $f \in C^1(X, R)$. Moreover assume*

(a) $dim X_1 < +\infty$,

(b) there exist $\rho > 0$, $R > 0$ *and* $e \in X_1$, $e \neq 0$ *such that* $\rho < R$ *and* $\sup_{S_\rho(X_1)} f < \inf_{\Sigma_R(e,X_2)} f$,

(c)$-\infty < a = \inf_{\Delta_R(e,X_2)} f$,

(d) $(PS)_c$ *condition holds for any* $c \in [a,b]$ *where* $b = \sup_{B_\rho(X_1)} f$. *Then there exist at least two critical levels* c_1 *and* c_2 *for the functional* f *such that*

$$\inf_{\Delta_R(e,X_2)} f \leq c_1 \leq \sup_{S_\rho(X_1)} f < \inf_{\Sigma_R(e,X_2)} f \leq c_2 \leq \sup_{B_\rho(X_1)} f.$$

3. Variational formulation

We study problem (1.1) by using a variational approach. Let λ_k denote the eigenvalues and e_k the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem $\Delta u + \lambda u = 0$ in Ω, with Dirichlet boundary condition, where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots, \lambda_i \to +\infty$ and that $e_1 > 0$ for all $x \in \Omega$. The eigenvalue problem

$$\begin{aligned} &\Delta^2 u + c\Delta u = \lambda u \qquad \text{in } \Omega, \\ &u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

has infinitely many eigenvalues $\Lambda_k(c) = \lambda_k(\lambda_k - c)$, $k = 1, 2, \cdots$ and corresponding eigenfunctions e_k. Set $H_k = span\{e_1, \cdots, e_k\}$, $H_k^\perp = \{w \in H | (w,v)_H = 0, \ \ \forall v \in H_k\}$.

Let $H = H^2(\Omega) \cap H_0^1(\Omega)$ be the Hilbert space equipped with the inner product $(u,v)_H = \int \Delta u \Delta v + \int \nabla u \nabla v$. The functional corresponding to (1.1) given by $I : H \to R$

$$I(u) := \frac{1}{2}\left(\int (\Delta u)^2 - c\int |\nabla u|^2\right) dx - \frac{\alpha}{2}\int u^2 dx - \frac{\beta}{2}\int ((u+1)^-)^2 dx. \tag{3.2}$$

Let $C^1(H,R)$ denote the set of all functionals which are Fréchet differentiable and whose Fréchet derivatives are continuous on H. It is easy to prove that I is a C^1 functional and its critical points are weak solutions of problem (1.1).

To use a variational approach it is necessary to check the Palais-Smale condition.

Definition 3.1. *We say that I satisfies the Palais-Smale condition $[(PS_c)]$, if for every sequence $(u_n)_{n\in N}$ in H with $I(u_n)$ bounded and $\underset{n}{lim}\nabla I(u_n)=0$, there exists a convergent subsequence.*

Lemma 3.2. *Assume that $\alpha\neq\Lambda_i$.Then $I(u)$ satisfies the $(PS)_c$ condition for every $c\in R$*

Proof. Let (u_k) be a sequence in H with $DI(u_k)\to 0$ and $I(u_k)\to c$. It is enough to show that $||u_k||$ is bounded, since $\forall u\in H$

$$\nabla I(u)=u+i^*[(1+c)\Delta u-\alpha u+\beta(u+1)^-]. \tag{3.3}$$

where $i^*: L^2(\Omega)\to H$, the adjoint of the immersion $i: H\to L^2(\Omega)$ is a compact operator. In fact, if $\{u_k\}_{k=1}^{\infty}\subset H$, then u_k converses strongly in $L^k(\Omega)$ By contradiction we suppose that $\lim\limits_{k}\|u_k\|_H=+\infty$. Up to a subsequence we can assume that $\lim\limits_{k}\frac{u_k}{\|u_k\|_H}=u$ weakly in H, strongly in $L^2(\Omega)$ and pointwise in Ω. Note that dividing $I(u_n)$ by $||u_n||$ and passing to the limit, we get $\int u^-dx=0$, and so $u\geq 0$ a.e. in Ω and $u\not\equiv 0$. On the other hand from $\nabla I(u_k)\to 0 \quad in \quad H$, we get $\lim\limits_{k\to\infty}\frac{\nabla I(u_k)}{\|u_k\|_H}=0$ as $n\to\infty$. So the bounded sequence $\lim\{\frac{u_k}{\|u_k\|_H}\}_{k\in N}$ converges strongly in H. Hence $u-i^*[(1+c)\Delta u-\alpha u]=0$. Here $i^*: L^2(\Omega)\to H$ is a compact operator. This implies that $u\geq 0$ is a nontrivial solution of

$$\Delta^2u+c\Delta u=\alpha u, \tag{3.4}$$

which contradicts to the equation (3.4) $(\alpha\neq\Lambda_i(c),\alpha\neq 0)$ that has only the trivial solution. So we discovered that $\{u_k\}_{k=1}^{\infty}$ is bounded in H, hence there exists a subsequence $\{u_{kj}\}_{kj=1}^{\infty}$ and $u\in H$ with $u_{kj}\to u$ in H. □

4. Main result

In this section we prove the existence and multiplicity of solutions for problem (1.1). Now we study the existence of solutions for the problem

$$\begin{aligned}&\Delta^2u+c\Delta u=\alpha u^+-\beta(u+1)^- &&\text{in } \Omega,\\ &u=0, \qquad \Delta u=0 &&\text{on } \partial\Omega.\end{aligned} \tag{4.1}$$

Theorem 4.1. *Assume that $c<\lambda_1$ and $0<\Lambda_1<\alpha$. Then we have:*
(i)If $\beta<0$, then equation (4.1) has no solution.
(ii)If $\beta=0$, then equation (4.1) has only the trivial solution.

Proof. Assume $\beta \leq 0$. We rewrite (1.1) as

$$\{-\Delta^2 - c\Delta + \Lambda_1\}u + \{-\Lambda_1 + \alpha\}u^+\Lambda_1 u^- = \beta(u+1)^-.$$

Multiply across by e_1 and integrate over Ω. Since $(\{-\Delta^2 - c\Delta + \Lambda_1\}u, e_1) = 0$, we have

$$\int_\Omega [\{-\Lambda_1 + \alpha\}u^+ + \Lambda_1 u^-]e_1 = \beta \int_\Omega (u+1)^- e_1. \tag{4.2}$$

But $\{-\Lambda_1 + \alpha\}u^+ + \Lambda_1 u^- \geq 0$ for all real valued function u and $e_1(x) > 0$ for $x \in \Omega$. Therefore the left hand side of (4.2) is always greater than or equal to zero. Hence if $\beta < 0$, then there is no solution of (4.1) and if $\beta = 0$, then the only possibility is $u \equiv 0$. □

THEOREM 4.2. *Assume that $\lambda_1 < c$ and $\alpha < \Lambda_1 < 0$. Then we have:*
(i)If $\beta > 0$, then equation (4.1) has no solution.
(ii)If $\beta = 0$, then equation (4.1) has only the trivial solution.

Proof. Assume $\beta \geq 0$. We rewrite (4.1) as

$$\{\Delta^2 + c\Delta - \Lambda_1\}u + \{\Lambda_1 - \alpha\}u^+ - \Lambda_1 u^- = -\beta(u+1)^-.$$

Multiply across by e_1 and integrate over Ω. Since $(\{\Delta^2 + c\Delta - \Lambda_1\}u, e_1) = 0$, we have

$$\int_\Omega [\{\Lambda_1 - \alpha\}u - \Lambda_1 u^-]e_1 = -\beta \int_\Omega (u+1)^- e_1. \tag{4.3}$$

But $\{\Lambda_1 - \alpha\}u^+ - \Lambda_u^- \leq 0$ for all real valued function u and $e_1(x) > 0$ for $x \in \Omega$. Therefore the left hand side of (4.3) is always greater than or equal to zero. Hence if $\beta > 0$, then there is no solution of (4.1) and if $\beta = 0$, then the only possibility is $u \equiv 0$. □

REMARK 4.3. *We have* $\underset{||u||_H \to 0}{lim} \int \frac{((u+1)^-)^2}{||u||_H} dx = 0.$

From the above equation

$$\int ((u+1)^-)^2 dx \leq ||u||_H{}^2 \cdot o(||u||_H).$$

LEMMA 4.4. *Assume $\Lambda_s < \alpha < \Lambda_{s+1}$, then*

$$\underset{H_s}{sup} I = 0.$$

Proof. It is obvious, since $I(u) \leq \frac{\Lambda_s - \alpha}{2} \int u^2 dx$ for all u in H_s and $I(0) = 0$. □

REMARK 4.5. *If there exists $s \geq 1$ such that $e_1 \in H_s^\perp$, then there isn't any nontrivial nonnegative function in H_s. In fact for all u in H such that $u \geq 0$, $u \neq 0$, we get $\langle u, e_1 \rangle = \lambda_1^2 \int ue_1 dx > 0$.*

LEMMA 4.6. *Assume $\Lambda_s \leq \alpha < \Lambda_{s+1}$, $s \geq 1$. Then*

$$\sup_{u \in H_s, \sigma \geq 0, ||u - \sigma e_1|| \to \infty} I(u - \sigma e_1) = -\infty.$$

Proof. Suppose by contradiction that there exist u_n in H_s, $\sigma_n \geq 0$ and C in R such that $||u_n - \sigma_n e_1|| \to \infty$ and

$$I(u_n - \sigma_n e_1) \geq C. \tag{4.4}$$

Up to a subsequence we can suppose that $\frac{u_n - \sigma_n e_1}{||u_n - \sigma_n e_1||} \to u - \sigma e_1$ in H, in $L^2(\Omega)$ and a.e. in Ω, there $u \in H_s$ and $\sigma \geq 0$. Dividing both sides of inequality (3.4) we get $u - \sigma e_1 \geq 0$ in Ω. Since $\alpha < \Lambda_1$, $e_1 \in H_s^\perp$. In this way $\langle u, e_1 \rangle = 0$, and so $0 \leq \langle u - \sigma e_1, e_1 \rangle = -\sigma \lambda_1^2$, which is possible if and only if $\sigma = 0$. this implies $u \geq 0$ in H_s, which is impossible for above Remark. □

LEMMA 4.7. *Assume $\Lambda_s < \alpha < \Lambda_s + 1$,then there exists $C > 0$ such that*

$$\lim_{\rho \to 0} \frac{1}{\rho^2} \inf_{u \in H_s^\perp, ||u|| = \rho} I(u) \geq C.$$

Proof. Observe that, by Lemma 2.3.3, for any $\epsilon > 0$ there exists $\rho > 0$ such that, if $||u|| \leq \rho$ we have $I(u) \geq C||u||^2 - \epsilon ||u||^2$, where $C = \inf\limits_{n \geq s+1} \frac{\Lambda_n - \alpha}{\lambda_n^2}$. The thesis follows. □

THEOREM 4.8. *Assume that $\lambda_1 < c < \lambda_2$, $\alpha < \Lambda_1$. Then for all $\beta > 0$ the problem (1.1) has at least one nontrivial solution.*

Proof. Since $\lambda_1 < c < \lambda_2$, $\alpha < \Lambda_1(c)$, observe that $I(0) = 0$, $\lim\limits_{t \to +\infty} I(-te_1) = -\infty$ and that I is locally coercive. In fact in $H_1^\perp$ By Lemma there exist constant $a > 0$, we have $I(u) \geq \frac{a}{2}||u||^2$. So for every ϵ in $(0, \frac{a}{2}\beta)$ there exists $\rho > 0$ such that

$$\lim_{||u|| = \rho} f(u) > (\frac{a}{2} - \beta\epsilon)\rho^2.$$

In this way, by the Mountain pass theorem, there exists a critical point u with positive critical value. Hence problem (1.1) has at least two solutions, one of which is nontrivial. □

THEOREM 4.9. *Assume $\lambda_1 < c < \lambda_2$, $\Lambda_1 \leq \alpha \leq \Lambda_2$. Then for all $\beta > 0$ the problem (1.1) has at least two nontrivial solutions.*

Proof. If $\lambda_1 < c < \lambda_2$, $\Lambda_s < \alpha < \Lambda_{s+1}(c)$. Lemma implies that there exist $R > \rho > 0$ and

$$\inf_{S_\rho(H_s)} I(u) > 0 > \sup_{\Sigma_\rho(e_1,H_s)} I(u).$$

In this way the hypotheses of the Linking Theorem are satisfied, so there exists a critical point u_1 such that

$$0 < \inf_{S_\rho(H_s)} i(u) < I(u_1) < \sup_{\Delta_\rho(e_1,H_s)} I(u).$$

The Linking Theorem also provides the existence of another critical point u_2 with non positive critical value, so there exists two Linking type critical values c_1,c_2. Therefore (1.1) has at least two nontrivial solutions. This implies that (1.1) has at least three solutions. □

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