CUBIC DOUBLE CENTRALIZERS AND CUBIC MULTIPLIERS

Keun Young Lee and Jung Rye Lee*

Abstract. In this paper, we establish the stability of cubic double centralizers and cubic multipliers on Banach algebras. We also prove the superstability of cubic double centralizers on Banach algebras which are cubic commutative and cubic without order.

1. Introduction

Let A be a complex Banach algebra. Recall that $A_l(A) := \{a \in A : aA = \{0\}\}$ is the left annihilator ideal and $A_r(A) := \{a \in A : Aa = \{0\}\}$ is the right annihilator ideal on A. A Banach algebra A is said to be *strongly without order* if $A_r(A) = A_l(A) = \{0\}$. We say that a Banach algebra A is *cubic without order* if $\{r \in A : \{ra^3; a \in A\} = \{0\}\} = \{0\} = \{r \in A : \{a^3r; a \in A\} = \{0\}\}$. It is easy to see that if A is cubic without order then A is strongly without order.

A linear mapping $L : A \to A$ is said to be *left centralizer* on A if $L(ab) = L(a)b$ for all $a, b \in A$. Similarly, a linear mapping $R : A \to A$ that $R(ab) = aR(b)$ for all $a, b \in A$ is called *right centralizer* on A. A *double centralizer* on A is a pair (L, R), where L is a left centralizer, R is a right centralizer and $aL(b) = R(a)b$ for all $a, b \in A$. For example, (L_c, R_c) is a double centralizer, where $L_c(a) := ca$ and $R_c(a) := ac$. The set $D(A)$ of all double centralizers equipped with the multiplication $(L_1, R_1) \cdot (L_2, R_2) = (L_1 L_2, R_1 R_2)$ is an algebra. The notion of double centralizer was introduced by Hochschild [7] and by Johnson [10].

An operator $T : A \to A$ is said to be a *multiplier* if $aT(b) = T(a)b$ for all $a, b \in A$.

A Banach algebra A is said to be *cubic commutative* if $(ab)^3 = a^3 b^3$ for all $a, b \in A$. We can show that there is a Banach algebra cubic commutative that is not commutative (see Example 2.4 of the present paper).

The functional equation is called stable if any function satisfying that functional equation "approximately" is near to a true solution of functional equation. We say that a functional equation is *superstable* if every approximate solution is an exact solution of it (see [2]).

In 1940, Ulam [16] proposed the following question concerning stability of group homomorphisms: *under what condition does there is an additive mapping near an approximately additive mapping?* Hyers [8] answered the problem of Ulam for the case where G_1 and G_2 are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mapping was given by Th. M. Rassias [15]. Since then, the stability problems of various functional equation have been extensively investigated by a number of authors (for instances, [1, 4, 5, 12, 13, 14]). In particular, one of the important functional equations is the following functional equations

(1.1) $f(x + y) + f(x - y) = 2f(x) + 2f(y),$

which is called a quadratic functional equation. The function $f(x) = bx^2$ is a solution of this functional equation. Every solution of functional equation (1.1) is said to be a *quadratic mapping*.

In [11], Jun and Kim considered the following cubic functional equation

(1.2) $f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x).$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.2), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

Moslehian, Rahbarnia and Sahoo [14] established the stability of double centralizers to Cauchy functional equations in the framework of Banach algebras. They also proved the superstability of double centralizers on Banach algebras which are strongly without order as follows.

THEOREM 1.1. *Let A be a strongly without order Banach algebra and let $L, R : A \to A$ be mappings for which there exist a positive real number r and a function $\psi : A \times A \to [0, \infty)$ satisfying either*

$$\lim_{n \to \infty} r^{-n} \psi(r^n a, b) = \lim_{n \to \infty} r^{-n} \psi(a, r^n b) = 0 \quad (a, b \in A)$$

or

$$\lim_{n\to\infty} r^n \psi(r^{-n}a, b) = \lim_{n\to\infty} r^n \psi(a, r^{-n}b) = 0 \quad (a, b \in A)$$

such that

$$\|aL(b) - R(a)b\| \leq \psi(a, b)$$

for all $a, b \in A$. Then (L, R) is a double centralizer on A.

Recently, Gordji, Ebadian, Ramezani and Park [6] introduced the quadratic double centralizers, and they established the stability as follows.

THEOREM 1.2. *Let A be a Banach algebra. Suppose that $s \in \{-1, 1\}$ and that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exist a mapping $g : A \to A$ with $g(0) = 0$ and functions $\phi_j, \psi_i : A \times A \to [0, \infty)$ $(1 \leq j \leq 2,\ 1 \leq i \leq 3)$ such that*

$$\tilde{\phi}_j(a, b) := \sum_{k=0}^{\infty} \frac{\phi_j(2^{sk}a, 2^{sk}b)}{4^{sk}} < \infty \ (1 \leq j \leq 2),$$

$$\lim_{n\to\infty} \frac{\psi_i(2^{sn}a, b)}{4^{sn}} = 0 = \lim_{n\to\infty} \frac{\psi_i(a, 2^{sn}b)}{4^{sn}} \ (1 \leq i \leq 3),$$

$$\|f(\lambda a + \lambda b) + f(\lambda a - \lambda b) - 2\lambda^2 f(a) - 2\lambda^2 f(b)\| \leq \phi_1(a, b)$$

$$\|g(\lambda a + \lambda b) + g(\lambda a - \lambda b) - 2\lambda^2 g(a) - 2\lambda^2 g(b)\| \leq \phi_2(a, b)$$

$$\|f(ab) - f(a)b^2\| \leq \psi_1(a, b)$$
$$\|g(ab) - a^2 g(b)\| \leq \psi_2(a, b)$$
$$\|a^2 f(b) - g(a)b^2\| \leq \psi_3(a, b)$$

for all $a, b \in A$ and all $\lambda \in \mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Also, if for each fixed $a \in A$ the mappings $t \to f(ta)$ and $t \to g(ta)$ from $\mathbb{R}$ to A are continuous, then there exists a unique quadratic double centralizer (L, R) on A satisfying

$$\|f(a) - L(a)\| \leq \frac{1}{4}\ \tilde{\phi}_1(a, a),$$

$$\|g(a) - R(a)\| \leq \frac{1}{4}\ \tilde{\phi}_2(a, a)$$

for all $a \in A$.

In this paper, we introduce the cubic double centralizers and cubic multipliers on Banach algebras, and we establish the stability of both of them. We also prove the superstability of cubic double centralizers on Banach algebras which are cubic without order and cubic commutative.

2. Stability of cubic double centralizers

In this section, let A be a complex Banach algebra. We establish the stability of cubic double centralizers.

DEFINITION 2.1. A mapping $L : A \to A$ is a cubic left centralizer if L satisfies the following properties:
1) L is a cubic mapping,
2) L is cubic homogeneous, that is, $L(\lambda a) = \lambda^3 L(a)$ for all $a \in A$ and $\lambda \in \mathbb{C}$,
3) $L(ab) = L(a)b^3$ for all $a, b \in A$.

DEFINITION 2.2. A mapping $R : A \to A$ is a cubic right centralizer if R satisfies the following properties:
1) R is a cubic mapping,
2) R is cubic homogeneous,
3) $R(ab) = a^3 R(b)$ for all $a, b \in A$.

DEFINITION 2.3. A cubic double centralizer of an algebra A is a pair (L, R), where L is a cubic left centralizer, R is a cubic right centralizer and $a^3 L(b) = R(a)b^3$ for all $a, b \in A$.

The following example introduces a cubic double centralizer.

EXAMPLE 2.4. Let $(A, \|.\|)$ be a Banach algebra. Let $T = A \times A \times A \times A$. We define $\|\|a\|\| = \|a_1\| + \|a_2\| + \|a_3\| + \|a_4\|$ for all $a = (a_1, a_2, a_3, a_4)$ in B. It is not hard to see that $(B, \|\|.\|\|))$ is a Banach space. For arbitrarily elements $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$ in B, we define $ab = (0, a_1 b_4, a_2 b_3, 0)$. Since A is a Banach algebra, we conclude that B is a Banach algebra. It is easy to see that $B^4 = \{abcd : a, b, c, d \in B\} = \{0\}$. But $B^3 = \{abc : a, b, c \in B\}$ is not zero. Now we consider the mapping $T : A \to A$ defined by

$$T(a) = a^3 \quad (a \in A).$$

Then T is a cubic mapping and cubic homogeneous. Since $B^4 = \{0\}$, we get

$$T(ab) = (ab)^3 = 0 = a^3 b^3 = T(a)b^3 = a^3 T(b)$$

and

$$a^3 T(b) = a^3 b^3 = 0 = T(a)b^3$$

for all $a, b \in B$. Hence (T, T) is a cubic double centralizer of B.

In the above example, B is a cubic commutative algebra, but it is not commutative.

THEOREM 2.5. *Suppose that $s \in \{-1, 1\}$ and that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exist a mapping $g : A \to A$ with $g(0) = 0$ and functions $\phi_j, \psi_i : A \times A \to [0, \infty)$ $(1 \le j \le 2,\ 1 \le i \le 3)$ such that*

(2.1) $$\tilde{\phi}_j(a, b) := \sum_{k=0}^{\infty} \frac{\phi_j(2^{sk}a, 2^{sk}b)}{8^{sk}} < \infty \ (1 \le j \le 2),$$

$$\lim_{n \to \infty} \frac{\psi_i(2^{sn}a, b)}{8^{sn}} = 0 = \lim_{n \to \infty} \frac{\psi_i(a, 2^{sn}b)}{8^{sn}} \ (1 \le i \le 3),$$

(2.2) $$\|f(2\lambda a + \lambda b) + f(2\lambda a - \lambda b) - 2\lambda^3 f(a + b)$$
$$- 2\lambda^3 f(a - b) - 12\lambda^3 f(a)\| \le \phi_1(a, b)$$

$$\|g(2\lambda a + \lambda b) + g(2\lambda a - \lambda b) - 2\lambda^3 g(a + b)$$
$$- 2\lambda^3 g(a + b) - 12\lambda^3 g(a)\| \le \phi_2(a, b)$$

(2.3) $$\|f(ab) - f(a)b^3\| \le \psi_1(a, b)$$

$$\|g(ab) - a^3 g(b)\| \le \psi_2(a, b)$$

(2.4) $$\|a^3 f(b) - g(a)b^3\| \le \psi_3(a, b)$$

for all $a, b \in A$ and all $\lambda \in \mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Also, if for each fixed $a \in A$ the mappings $t \to f(ta)$ and $t \to g(ta)$ from $\mathbb{R}$ to A are continuous, then there exists a unique cubic double centralizer (L, R) on A satisfying

(2.5) $$\|f(a) - L(a)\| \le \frac{1}{16} \, \tilde{\phi}_1(a, a),$$

(2.6) $$\|g(a) - R(a)\| \le \frac{1}{16} \, \tilde{\phi}_2(a, a)$$

for all $a \in A$.

Proof. Let $s = 1$. Putting $b = 0$ and $\lambda = 1$ in (2.2), we have

$$\|f(2a) - 8f(a)\| \leq \frac{1}{2}\phi_1(a, a)$$

for all $a \in A$. One can use induction to show that

$$(2.7) \qquad \|\frac{f(2^n a)}{8^n} - \frac{f(2^m a)}{8^m}\| \leq \frac{1}{16}\sum_{k=m}^{n-1}\frac{\phi_1(2^k a, 2^k a)}{8^k}$$

for all $n > m \geq 0$ and all $a \in A$. It follows from (2.7) and (2.1) that sequence $\{\frac{f(2^n a)}{8^n}\}$ is Cauchy. Since A is a Banach algebra, this sequence is convergent. Define

$$(2.8) \qquad\qquad L(a) := \lim_{n\to\infty}\frac{f(2^n a)}{8^n}.$$

Replacing a and b by $2^n a$ and $2^n b$, respectively, in (2.2), we get

$$\|\frac{f(2^n(2\lambda a + \lambda b))}{8^n} + \frac{f(2^n(2\lambda a - \lambda b))}{4^n} - 2\lambda^3\frac{f(2^n(a+b))}{8^n}$$
$$- 2\lambda^2\frac{f(2^n(a-b))}{8^n} - 12\lambda^3\frac{f(2^n a)}{8^n}\| \leq \frac{\phi_1(2^n a, 2^n b)}{8^n}$$

Taking the limit as $n \to \infty$, we obtain

$$(2.9) \qquad L(2\lambda a + \lambda b) + L(2\lambda a - \lambda b)$$
$$= 2\lambda^3 L(a+b) + 2\lambda^3 L(a-b) + 12\lambda^3 L(a)$$

for all $a, b \in A$ and all $\lambda \in \mathbb{T}$. Putting $\lambda = 1$ in (2.9), we obtain that L is a cubic mapping. Setting $b := 0$ in (2.9), we get

$$L(2\lambda a) = 8\lambda^3 L(a)$$

for all $a \in A$, $\lambda \in \mathbb{T}$. Since L is a cubic mapping, we obtain

$$L(\lambda a) = \lambda^3 L(a)$$

for all $a \in A$ and all $\lambda \in \mathbb{T}$. Under the assumption that $f(ta)$ is continuous in $t \in \mathbb{R}$ for each fixed $a \in A$, by the same reasoning as in the proof of [4], $L(\lambda a) = \lambda^3 L(a)$ for all $a \in A$ and all $\lambda \in \mathbb{R}$. Hence

$$L(\lambda a) = L(\frac{\lambda}{|\lambda|}\ |\lambda|a) = \frac{\lambda^3}{|\lambda|^3}L(|\lambda|a) = \frac{\lambda^3}{|\lambda|^3}|\lambda|^3 L(a)$$

for all $a \in A$ and $\lambda \in \mathbb{C} (\lambda \neq 0)$. This means that L is cubic homogeneous. It follows from (2.3) that

$$\|L(ab) - L(a)b^3\| = \lim_{n \to \infty} \frac{1}{8^n} \|f(2^n ab) - f(2^n a)b^3\| \leq \lim_{n \to \infty} \frac{\psi_1(2^n a, b)}{8^n} = 0$$

for all $a, b \in A$. Hence L is a cubic left centralizer on A. Applying (2.7) with $m = 0$, we get $\|L(a) - f(a)\| \leq \frac{1}{16} \tilde{\phi}_1(a, a)$ for all $a \in A$. Then the cubic mapping L satisfying (2.5) is unique.
A similar argument gives us a unique cubic right centralizer R defined by $R(a) := \lim_{n \to \infty} \frac{g(2^n a)}{8^n}$ which satisfies (2.6). Now we let $a, b \in A$ arbitrarily. Since L is cubic homogeneous, it follows from (2.4) and (2.5) that

$$\|a^3 L(b) - R(a)b^3\| = \frac{1}{8^n} \|a^3 L(2^n b) - 8^n R(a)b^3\|$$

$$\leq \frac{1}{8^n} [\|a^3 L(2^n b) - a^3 f(2^n b)\| + \|a^3 f(2^n b) - g(a)(8^n b^3)\|$$

$$+ \|8^n g(a)b^3 - 8^n R(a)b^3\|]$$

$$\leq \frac{1}{8^{n+1}} \tilde{\phi}_1(2^n b, 2^n b) \|a\|^2 + \frac{\psi_3(a, 2^n b)}{8^n} + \|g(a) - R(a)\| \|b\|^3.$$

The right hand side of the last inequality tends to $\|g(a) - R(a)\| \|b\|^3$ as $n \to \infty$. By (2.6), we obtain

$$\|a^3 L(b) - R(a)b^3\| \leq \frac{1}{16} \tilde{\phi}_2(a, a) \|b\|^3.$$

Since R is a cubic mapping, we obtain

$$\|a^3 L(b) - R(a)b^3\| = \frac{1}{8^n} \|8^n a^3 L(b) - R(2^n a)b^3\|$$

$$\leq \frac{1}{16} \tilde{\phi}_2(2^n a, 2^n a) \|b\|^3$$

$$= \frac{1}{16} \sum_{k=n}^{\infty} \frac{\phi_2(2^k a, 2^k a)}{8^k} \|b\|^3.$$

Passing to the limit as $n \to \infty$, we conclude $a^3 L(b) = R(a)b^3$. Thus (L, R) is a cubic double centralizer.

The proof for $s = -1$ is similar to $s = 1$. $\qquad \square$

COROLLARY 2.6. *Suppose that* $f : A \to A$ *is a mapping for which there exist a mapping* $g : A \to A$ *and constants* $\epsilon > 0$ *and* $0 < p < 3$ *such that*

$$\|f(2\lambda a + \lambda b) + f(2\lambda a - \lambda b) - 2\lambda^3 f(a+b) - 2\lambda^3 f(a-b) - 12\lambda^3 f(a)\|$$
$$\leq \epsilon(\|a\|^p + \|b\|^p),$$

$$\|g(2\lambda a + \lambda b) + g(2\lambda a - \lambda b) - 2\lambda^3 g(a+b) - 2\lambda^3 g(a-b) - 12\lambda^3 g(a)\|$$
$$\leq \epsilon(\|a\|^p + \|b\|^p),$$

$$\|f(ab) - f(a)b^3\| \leq \epsilon \, \|a\|^p \, \|b\|^p,$$

$$\|g(ab) - a^3 g(b)\| \leq \epsilon \, \|a\|^p \, \|b\|^p,$$

$$\|a^3 f(b) - g(a)b^3\| \leq \epsilon \, \|a\|^p \, \|b\|^p$$

for all $a, b \in A$ *and all* $\lambda \in \mathbb{T}$. *Also, if for each fixed* $a \in A$ *the mappings* $t \to f(ta)$ *and* $t \to g(ta)$ *from* $\mathbb{R}$ *to* A *are continuous, then there exists a unique quadratic double centralizer* (L, R) *on* A *satisfying*

$$\|f(a) - L(a)\| \leq \frac{\epsilon}{|8 - 2^p|} \, \|a\|^p,$$

$$\|g(a) - R(a)\| \leq \frac{\epsilon}{|8 - 2^p|} \, \|a\|^p$$

for all $a \in A$.

Proof. For $j = 1, 2$, putting $\phi_j(a, b) = \epsilon(\|a\|^p + \|b\|^p)$ and for $i = 1, 2, 3$, putting $\psi_i(a, b) = \epsilon \, \|a\|^p \, \|b\|^p$ in Theorem 2.5, we get the desired results. $\square$

3. Stability of cubic multipliers

Throughout this section, assume that A is a complex Banach algebra.

DEFINITION 3.1. *We say that a mapping* $T : A \to A$ *is a cubic multiplier if* T *satisfies the following properties:*
1) T *is a cubic mapping,*
2) T *is cubic homogeneous,*
3) $a^3 T(b) = T(a)b^3$ *for all* $a, b \in A$.

Now, we investigate the stability of cubic multipliers.

THEOREM 3.2. *Suppose that $s \in \{-1, 1\}$ and that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exist functions $\phi, \psi : A \times A \to [0, \infty)$ such that*

$$(3.1) \qquad \tilde{\phi}(a, b) := \sum_{k=0}^{\infty} \frac{\phi(2^{sk}a, 2^{sk}b)}{8^{sk}} < \infty,$$

$$\lim_{n \to \infty} \frac{\psi(2^{sn}a, b)}{8^{sn}} = 0 = \lim_{n \to \infty} \frac{\psi(a, 2^{sn}b)}{8^{sn}},$$

$$\|f(2\lambda a + \lambda b) + f(2\lambda a - \lambda b) - 2\lambda^3 f(a+b) - 2\lambda^3 f(a-b) - 12\lambda^3 f(a)\| \leq \phi(a, b),$$

$$(3.2) \qquad \|a^3 f(b) - f(a)b^3\| \leq \psi(a, b)$$

for all $a, b \in A$ and all $\lambda \in \mathbb{T}$. Also, if for each fixed $a \in A$ the mapping $t \to f(ta)$ from $\mathbb{R}$ to A is continuous, then there exists a unique cubic multiplier T on A satisfying

$$(3.3) \qquad \|f(a) - T(a)\| \leq \frac{1}{16} \tilde{\phi}(a, a)$$

for all $a \in A$.

Proof. Let $s = 1$. By the same reasoning as in the proof of Theorem 2.5, there exists a unique cubic mapping $T : A \to A$ defined by

$$T(a) := \lim_{n \to \infty} \frac{f(2^n a)}{8^n}$$

with satisfying $T(\lambda a) = \lambda^3 T(a)$ for all $a \in A$ and all $\lambda \in \mathbb{C}$. Also, $\|f(a) - T(a)\| \leq \dfrac{1}{16} \tilde{\phi}(a, a)$ for all $a \in A$. Let $a, b \in A$ be arbitrarily. Then T is cubic homogeneous. By using (3.2) and (3.3), we have

$$\|a^3 T(b) - T(a)b^3\| = \frac{1}{8^n} \|a^3 T(2^n b) - 8^n T(a)b^3\|$$

$$\leq \frac{1}{8^n} [\|a^3 T(2^n b) - a^3 f(2^n b)\| + \|a^3 f(2^n b) - f(a)(8^n b^3)\|$$

$$+ \|8^n f(a)b^3 - 8^n T(a)b^3\|]$$

$$\leq \frac{1}{8^{n+1}} \tilde{\phi}(2^n b, 2^n b) \|a\|^3 + \frac{\psi(a, 2^n b)}{8^n} + \frac{1}{16} \tilde{\phi}(a, a) \|b\|^3.$$

It follows from (3.1) that

$$\|a^3 T(b) - T(a)b^3\| \leq \frac{1}{16} \tilde{\phi}(a, a) \|b\|^3.$$

Finally, we obtain

$$\|a^3 T(b) - T(a)b^3\| = \frac{1}{8^n}\|8^n a^3 T(b) - T(2^n a)b^3\|$$

$$\leq \frac{1}{16}\,\tilde{\phi}(2^n a, 2^n a)\,\|b\|^3$$

$$= \frac{1}{16}\sum_{k=n}^{\infty}\frac{\phi(2^k a, 2^k a)}{8^k}\,\|b\|^3$$

$$\to 0 \qquad as \quad n \to \infty.$$

So $a^3 T(b) = T(a)b^3$. Hence T is a cubic multiplier.

The proof for $s = -1$ is similar. □

COROLLARY 3.3. *Suppose that* $f : A \to A$ *is a mapping for which there exist nonnegative real numbers* ϵ *and* p *with* $p \neq 2$ *such that*

$$\|f(2\lambda a + \lambda b) + f(2\lambda a - \lambda b) - 2\lambda^3 f(a + b) - 2\lambda^3 f(a - b) + 12\lambda^3 f(a)\|$$
$$\leq \epsilon(\|a\|^p + \|b\|^p),$$

$$\|a^3 f(b) - f(a)b^3\| \leq \epsilon\,\|a\|^p\,\|b\|^p$$

for all $a, b \in A$ *and all* $\lambda \in \mathbb{T}$. *Also, if for each fixed* $a \in A$ *the mapping* $t \to f(ta)$ *from* $\mathbb{R}$ *to* A *is continuous, then there exists a unique cubic multiplier* T *on* A *satisfying*

$$\|f(a) - T(a)\| \leq \frac{\epsilon}{|8 - 2^p|}\,\|a\|^p$$

for all $a \in A$.

Proof. Putting $\phi(a, b) = \epsilon(\|a\|^p + \|b\|^p)$ and $\psi(a, b) = \epsilon\,\|a\|^p\,\|b\|^p$ in Theorem 3.2, we get the result. □

4. Superstability of cubic double centralizers

In this section, we prove the superstability of cubic double centralizers on Banach algebras which are cubic without order and cubic commutative.

THEOREM 4.1. *Suppose that A is a Banach algebra cubic without order and cubic commutative and $s \in \{-1, 1\}$. Let $L, R : A \to A$ are mappings for which there exists a function $\psi : A \times A \to [0, \infty)$ such that*

$$\lim_{n \to \infty} n^{-3s} \psi(n^s x, y) = 0 = \lim_{n \to \infty} n^{-3s} \psi(x, n^s y),$$

$$\|x^3 L(y) - R(x)y^3\| \leq \psi(x, y)$$

for all $x, y \in A$. Then (L, R) is a cubic double centralizer.

Proof. We first show that L is cubic homogeneous. To do this, choose $\lambda \in \mathbb{C}$ and $x, z \in A$. We have

$$\|n^{3s}z^3(L(\lambda x) - \lambda^3 L(x))\| = \|n^{3s}z^3 L(\lambda x) - \lambda^3 n^{3s}z^3 L(x)\|$$

$$\leq \|n^{3s}z^3 L(\lambda x) - R(n^s z)(\lambda x)^3\| + \|\lambda^3 R(n^s z)x^3 - \lambda^3 n^{3s}z^3 L(x)\|$$

$$\leq \psi(n^s z, \lambda x) + |\lambda|^3 \psi(n^s z, x).$$

So

$$\|z^3(L(\lambda x) - \lambda^3 L(x))\| \leq n^{-3s}\psi(n^s z, \lambda x) + |\lambda|^3 n^{-3s}\psi(n^s z, x).$$

Since A is cubic without order, we conclude that $L(\lambda x) = \lambda^3 L(x)$. The cubicity of L follows from

$$\|z^3(L(2x + y) + L(2x - y) - 2L(x + y) - 2L(x - y) - 12L(x))\|$$

$$= n^{-3s}\|n^{3s}z^3 L(2x + y) + n^3 z^3 L(2x - y) - 2n^{3s}z^3 L(x + y)$$

$$- 2n^{3s}z^3 L(x - y) - 12n^{3s}z^3 L(x)\|$$

$$\leq n^{-3s}[\|n^{3s}z^3 L(2x + y) - R(n^s z)(2x + y)^3\|$$

$$+ \|n^{3s}z^3 L(2x - y) - R(n^s z)(2x - y)^3\|2\|R(n^s z)(x + y)^3 - n^{3s}z^3 L(x + y)\|$$

$$+ 2\|R(n^s z)(x - y)^3 - n^{3s}z^3 L(x - y)\| + 12\|R(n^s z)x^3 - n^{3s}z^3 L(x)\|]$$

$$\leq n^{-2s}[\psi(n^s z, 2x + y) + \psi(n^s z, 2x - y) + 2\psi(n^s z, x + y)$$

$$+ 2\psi(n^s z, x - y) + 12\psi(n^s z, x)]$$

for all $x, y \in A$.

Finally, since A is a cubic commutative Banach algebra, we have

$$\|z^3(L(xy) - L(x)y^3)\| = n^{-3s}\|n^{3s}z^3 L(xy) - n^{3s}z^3 L(x)y^3\|$$

$$\leq n^{-3s}[\|n^{3s}z^3 L(xy) - R(n^s z)(xy)^3\|$$

$$+ \|R(n^s z)x^3 y^3 - n^{3s}z^3 L(x)y^3\|]$$

$$\leq n^{-3s}[\psi(n^s z, xy) + \psi(n^s z, x)\|y\|^3]$$

for all $x, y, z \in A$. So $L(xy) = L(x)y^3$. Thus L is a cubic left centralizer. We can similarly prove that R is a cubic centralizer. Since L is cubic homogeneous, $L(x) = n^{-3s}L(n^s x)$ for all $n \in \mathbb{N}$ and $x \in A$. Thus

$$\|x^3 L(y) - R(x)y^3\| = n^{-3s}\|x^3 L(n^s y) - R(x)(n^{2s}y^3)\|$$
$$\leq n^{-3s}\psi(x, n^s y)$$

and hence by (4.1) we infer that $x^3 L(y) = R(x)y^3$ for all $x, y \in A$. Thus (L, R) is a cubic centralizer. $\square$

COROLLARY 4.2. *Suppose A is a Banach algebra cubic without order and cubic commutative and L, $R : A \to A$ are mappings for which there exist a nonnegative real number ϵ and a real number p either greater than 3 or less than 3, such that*

$$\|x^3 L(y) - R(x)y^3\| \leq \epsilon \|x\|^p \|y\|^p$$

for all $x, y \in A$. Then (L, R) is a cubic double centralizer.

Proof. Using Theorem 4.1 with $\psi(x, y) = \epsilon \|x\|^p \|y\|^p$, we get the desired result. $\square$

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Department of Applied Mathematics
Sejong University
Seoul 143-747, South Korea
E-mail: northstar21@hanmail.net

Department of Mathematics
Daejin University
Kyeonggi 487-711, South Korea
E-mail: jrlee@daejin.ac.kr