Korean J. Math. **17** (2009), No. 4, pp. 429–436

PROJECTIVE PROPERTIES OF REPRESENTATIONS OF A QUIVER OF THE FORM $Q = \bullet \rightrightarrows \bullet \to \bullet$

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Abstract. We define a projective representation $M_1 \rightrightarrows M_2 \to M_3$ of a quiver $Q = \bullet \rightrightarrows \bullet \to \bullet$ and consider their properties. Then we show that any projective representation $M_1 \rightrightarrows M_2 \to M_3$ of a quiver $Q = \bullet \rightrightarrows \bullet \to \bullet$ is isomorphic to the quotient of a direct sum of projective representations $0 \rightrightarrows 0 \to P$, $0 \rightrightarrows P \xrightarrow{id} P$ and $P \underset{e_2}{\overset{e_1}{\rightrightarrows}} P \oplus P \xrightarrow{id_{P\oplus P}} P \oplus P$, where $e_1(a) = (a, 0)$ and $e_2(a) = (0, a)$.

1. Introduction

A quiver is just a directed graph with vertices and edges (arrows) ([1]). We may consider many different types of quivers. We allow multiple edges between two vertices and multiple vertices possibly infinitely many vertices, and edges going from a vertex back to the same vertex called a loop. Originally a representation of quiver assigned a vector space to each vertex - and a linear map to each edge (or arrow) - with the linear map going from the vector space assigned to the initial vertex to the one assigned to the terminal vertex. For example, a representation of the quiver $Q = \bullet \to \bullet$ is $V_1 \xrightarrow{f} V_2$, V_1 and V_2 are vector spaces and f is a linear map (morphism). Then we can extend a vector space to an R-module and a linear map to an R-linear map in a representation of a quiver. Given a quiver $Q = \bullet \rightrightarrows \bullet$ we can define two linear maps $V_1 \underset{f_{12}}{\overset{f_{11}}{\rightrightarrows}} V_2$. Now we can define morphisms between these

Received September 25, 2009. Revised October 26, 2009.
2000 Mathematics Subject Classification: 16E30; 13C11, 16D80.
Key words and phrases: quiver, projective module, projective representation.
This study was supported by research funds from Dong-A University.
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two representations. A morphism of $V_1 \overset{f_{11}}{\underset{f_{12}}{\rightrightarrows}} V_2$ to $W_1 \overset{g_{11}}{\underset{g_{12}}{\rightrightarrows}} W_2$ is given by a commutative diagram

$$\begin{array}{ccc} V_1 & \overset{f_{11}}{\underset{f_{12}}{\rightrightarrows}} & V_2 \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle s_2} \\ W_1 & \overset{g_{11}}{\underset{g_{12}}{\rightrightarrows}} & W_2 \end{array}$$

with s_1, s_2 linear maps.

In ([3]) a homotopy of quiver was developed and in ([2]) cyclic quiver ring was studied. The theory of projective representations was developed in ([4]) and the theory of injective representation was studied in ([5]). Recently, in ([7]) injective covers and envelopes of representations of linear quivers were studied, and in ([6]) properties of multiple edges of quivers were studied.

A left R-module P is said to be projective if given any surjective left R-linear map $\sigma : M' \to M$ and any R-linear map $h : P \to M$, there is an R-linear map $g : P \to M'$ such that $\sigma g = h$. That is

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle g}\swarrow & \downarrow{\scriptstyle h} & & \\ M' & \xrightarrow{\sigma} & M & \longrightarrow & 0 \end{array}$$

can always be completed to a commutative diagram.

2. Projective properties of representations of a quiver $Q = \bullet \rightrightarrows \bullet \rightarrow \bullet$

Definition 2.1. A representation $P_1 \overset{f_{11}}{\underset{f_{12}}{\rightrightarrows}} P_2 \xrightarrow{f_2} P_3$ of a quiver $Q = \bullet \rightrightarrows \bullet \rightarrow \bullet$ is called a projective representation if every diagram of representations

$$
\begin{array}{ccccccc}
 & & (P_1 \underset{f_{12}}{\overset{f_{11}}{\rightrightarrows}} P_2 \xrightarrow{f_2} P_3) & & \\
 & & \downarrow \alpha \qquad \downarrow \beta \qquad \downarrow \gamma & & \\
(M_1 \underset{g_{12}}{\overset{g_{11}}{\rightrightarrows}} M_2 \xrightarrow{g_2} M_3) & \longrightarrow & (N_1 \underset{h_{12}}{\overset{h_{11}}{\rightrightarrows}} N_2 \xrightarrow{h_2} N_3) & \longrightarrow & (0 \rightrightarrows 0 \longrightarrow 0)
\end{array}
$$

can be completed to a commutative diagram as follows :

$$
\begin{array}{ccccccc}
 & & (P_1 \underset{f_{12}}{\overset{f_{11}}{\rightrightarrows}} P_2 \xrightarrow{f_2} P_3)) & & \\
 & s \quad t \quad u & \downarrow \alpha \qquad \downarrow \beta \qquad \downarrow \gamma & & \\
(M_1 \underset{g_{12}}{\overset{g_{11}}{\rightrightarrows}} M_2 \xrightarrow{g_2} M_3) & \longrightarrow & (N_1 \underset{h_{12}}{\overset{h_{11}}{\rightrightarrows}} N_2 \xrightarrow{h_2} N_3) & \longrightarrow & (0 \rightrightarrows 0 \longrightarrow 0)
\end{array}
$$

THEOREM 2.2. *If* $P_1 \underset{f_{12}}{\overset{f_{11}}{\rightrightarrows}} P_2 \xrightarrow{f_2} P_3$ *is a projective representation of a quiver* $Q = \bullet \rightrightarrows \bullet \rightarrow \bullet$, *then* P_1, P_2, *and* P_3 *are projective left* R*-modules.*

Proof. Let M, N be left R-modules and $\alpha : P_1 \rightarrow N$ be an R-linear map and $k : M \rightarrow N$ be an onto R-linear map. Then since $P_1 \underset{f_{12}}{\overset{f_{11}}{\rightrightarrows}} P_2 \xrightarrow{f_2} P_3$ is a projective representation we can complete the following diagram

$$
\begin{array}{ccccccc}
 & & (P_1 \underset{f_{12}}{\overset{f_{11}}{\rightrightarrows}} P_2 \xrightarrow{f_2} P_3) & & \\
 & & \downarrow \alpha \qquad \downarrow 0 \qquad \downarrow 0 & & \\
(M \rightrightarrows 0 \longrightarrow 0) & \longrightarrow & (N \rightrightarrows 0 \longrightarrow 0) & \longrightarrow & (0 \rightrightarrows 0 \longrightarrow 0)
\end{array}
$$

as a commutative diagram. Thus P_1 is a projective left R-module.

Let $\beta : P_2 \rightarrow N$ be an R-linear map and $k : M \rightarrow N$ be an onto R-linear map. Let $e_i : N \oplus N \longrightarrow N$ by $e_1(a, b) = a$ and $e_2(a, b) = b$, and $t_i : M \oplus M \longrightarrow M$ by $t_1(a, b) = a$ and $t_2(a, b) = b$. Define $G : P_1 \longrightarrow N \oplus N$ by $G(m) = (\beta f_{11}(m), \beta f_{12}(m)))$, then $\beta f_{11}(m) = e_1 G(m)$ and $\beta f_{12}(m) = e_2 G(m)$. Then since $P_1 \underset{f_{12}}{\overset{f_{11}}{\rightrightarrows}} P_2 \xrightarrow{f_2} P_3$ is a projective representation we can complete the following diagram

$$\begin{array}{ccccc} & & (P_1 \overset{f_{11}}{\underset{f_{12}}{\rightrightarrows}} P_2 \xrightarrow{f_2} P_3) & & \\ & & \downarrow G \qquad \downarrow \beta \qquad \downarrow 0 & & \\ ((M \oplus M) \overset{t_1}{\underset{t_2}{\rightrightarrows}} M \longrightarrow 0) & \longrightarrow & ((N \oplus N) \overset{e_1}{\underset{e_2}{\rightrightarrows}} N \longrightarrow 0) & \longrightarrow & (0 \rightrightarrows 0 \longrightarrow 0) \end{array}$$

as a commutative diagram. Thus P_2 is a projective left R-module.

Let $\gamma : P_3 \to N$ be an R-linear map and $k : M \to N$ be an onto R-linear map. Let $e_i : N \oplus N \longrightarrow N$ by $e_1(a,b) = a$ and $e_2(a,b) = b$, and $t_i : M \oplus M \longrightarrow M$ by $t_1(a,b) = a$ and $t_2(a,b) = b$. Define $G : P_1 \longrightarrow N \oplus N$ by $G(m) = (f_{12}\gamma f_2(m), f_2\gamma f_2(m)))$, then $f_{11}\gamma f_2(m) = e_1 G(m)$ and $f_{12}\gamma f_2(m) = e_2 G(m)$. Then since $P_1 \overset{f_{11}}{\underset{f_{12}}{\rightrightarrows}} P_2 \xrightarrow{f_2} P_3$ is a projective representation we can complete the following diagram

$$\begin{array}{ccccc} & & (P_1 \overset{f_{11}}{\underset{f_{12}}{\rightrightarrows}} P_2 \xrightarrow{f_2} P_3) & & \\ & & \downarrow G \qquad \downarrow \gamma f_2 \qquad \downarrow \gamma & & \\ ((M \oplus M) \overset{t_1}{\underset{t_2}{\rightrightarrows}} M \xrightarrow{id} M) & \longrightarrow & ((N \oplus N) \overset{e_1}{\underset{e_2}{\rightrightarrows}} N \xrightarrow{id} N) & \longrightarrow & (0 \rightrightarrows 0 \longrightarrow 0) \end{array}$$

as a commutative diagram. Thus P_3 is a projective left R-module. $\square$

Lemma 2.3. *If P is a projective left R-module, then a representation $0 \rightrightarrows 0 \longrightarrow P$ of a quiver $Q = \bullet \rightrightarrows \bullet \to \bullet$ is a projective representation.*

Proof. The lemma follows by completing the diagram

$$\begin{array}{ccccc} & & (0 \rightrightarrows 0 \longrightarrow P) & & \\ & & \downarrow \qquad \downarrow \qquad \downarrow \gamma & & \\ (M_1 \overset{g_{11}}{\underset{g_{12}}{\rightrightarrows}} M_2 \xrightarrow{g_2} M_3) & \longrightarrow & (N_1 \overset{h_{11}}{\underset{h_{12}}{\rightrightarrows}} N_2 \xrightarrow{h_2} N_3) & \longrightarrow & (0 \rightrightarrows 0 \longrightarrow 0) \end{array}$$

as a commutative diagram. $\square$

Lemma 2.4. *If P is a projective left R-module, then a representation $0 \rightrightarrows P \xrightarrow{id} P$ of a quiver $Q = \bullet \rightrightarrows \bullet \to \bullet$ is a projective representation.*

Proof. Let $\beta : P \longrightarrow N_2$ be an R-linear map and $k_2 : M_2 \longrightarrow N_2$ be an onto R-linear map and choose $h_2\beta : P \longrightarrow N_3$ as an R-linear map. Then since P is a projective left R-module, there exists $t : P \longrightarrow M_2$ such that $k_2 t = \beta$. Now choose $g_2 t : P \longrightarrow M_3$ as an R-linear map. Then t and $g_2 t$ complete the following diagram

$$\begin{array}{ccccccc} & (0 & \rightrightarrows & P & \xrightarrow{id} & P) & \\ & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle h_2\beta} & \\ (M_1 \underset{g_{12}}{\overset{g_{11}}{\rightrightarrows}} M_2 \xrightarrow{g_2} M_3) \longrightarrow & (N_1 & \underset{h_{12}}{\overset{h_{11}}{\rightrightarrows}} & N_2 & \xrightarrow{h_2} & N_3) & \longrightarrow (0 \rightrightarrows 0 \longrightarrow 0) \end{array}$$

as a commutative diagram. Therefore, $0 \rightrightarrows P \xrightarrow{id} P$ is a projective representation. □

THEOREM 2.5. *If P is a projective left R-module, then a representation*

$$P \underset{e_2}{\overset{e_1}{\rightrightarrows}} P \oplus P \xrightarrow{id_{P\oplus P}} P \oplus P$$

of a quiver $Q = \bullet \rightrightarrows \bullet \rightarrow \bullet$ is a projective representation, where $e_1(a) = (a, 0)$ and $e_2(a) = (0, a)$

Proof. Let $\alpha : P \longrightarrow N_1$ be an R-linear map and $k_1 : M_1 \longrightarrow N_1$ be an onto R-linear map and consider the following diagram

$$\begin{array}{ccccccc} & (P & \underset{e_2}{\overset{e_1}{\rightrightarrows}} & P \oplus P & \xrightarrow{id_{P\oplus P}} & P \oplus P & \\ & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow & \\ (M_1 \underset{g_{12}}{\overset{g_{11}}{\rightrightarrows}} M_2 \xrightarrow{g_2} M_3) \longrightarrow & (N_1 & \underset{h_{12}}{\overset{h_{11}}{\rightrightarrows}} & N_2 & \xrightarrow{h_2} & N_3) & \longrightarrow (0 \rightrightarrows 0 \longrightarrow 0) \end{array}$$

Since P is a projective left R-module we can complete the following diagram by s.

$$\begin{array}{ccccc} & & P & & \\ & \swarrow{\scriptstyle s} & \downarrow{\scriptstyle \alpha} & & \\ M_1 & \longrightarrow & N_1 & \longrightarrow & 0 \end{array}$$

Define $t : P \oplus P \longrightarrow M_2$ by $t((a_1, a_2)) = h_{11}(s(a_1)) + h_{12}(s(a_2))$. Then $h_{11}(s(a)) = t(e_1(a)))$ and $h_{12}(s(a)) = t(e_2(a))$. And define $u : P \oplus P \longrightarrow$

M_3 by $u((a_1, a_2)) = g_2(t(a_1, a_2))$ Then we can complete the following diagram

$$
\begin{array}{ccccccc}
 & & & (P \overset{e_1}{\underset{e_2}{\rightrightarrows}} P \oplus P \overset{id_{P\oplus P}}{\longrightarrow} P \oplus P) & & & \\
 & {\scriptstyle s} \swarrow \cdots & {\scriptstyle t} \swarrow \cdots & \downarrow \alpha \quad \downarrow \quad \downarrow & {\scriptstyle u} \swarrow \cdots & & \\
(M_1 \overset{g_{11}}{\underset{g_{12}}{\rightrightarrows}} M_2 \overset{g_2}{\longrightarrow} M_3) & \longrightarrow & (N_1 \overset{h_{11}}{\underset{h_{12}}{\rightrightarrows}} N_2 \overset{h_2}{\longrightarrow} N_3) & \longrightarrow & (0 \rightrightarrows 0 \longrightarrow 0) & &
\end{array}
$$

as a commutative diagram. Hence, $P \overset{e_1}{\underset{e_2}{\rightrightarrows}} P \oplus P \overset{id_{P\oplus P}}{\longrightarrow} P \oplus P$ is a projective representation. □

Remark 1. A representation $P \rightrightarrows 0 \longrightarrow 0$ of a quiver $Q = \bullet \rightrightarrows \bullet \rightarrow \bullet$ is not a projective representation if $P \neq 0$. Because we can not complete the following diagram

$$
\begin{array}{ccccc}
 & & (P \rightrightarrows 0 \longrightarrow 0 & & \\
 & & \downarrow id \qquad \downarrow \qquad \downarrow & & \\
(P \overset{id}{\underset{id}{\rightrightarrows}} P \longrightarrow 0) & \longrightarrow & (P \rightrightarrows 0 \longrightarrow 0) & \longrightarrow & (0 \rightrightarrows 0 \longrightarrow 0)
\end{array}
$$

as a commutative diagram.

Remark 2. A representation $P \overset{id}{\underset{id}{\rightrightarrows}} P \longrightarrow 0$ of a quiver $Q = \bullet \rightrightarrows \bullet \rightarrow \bullet$ is not a projective representation if $P \neq 0$. Because we can not complete the following diagram

$$
\begin{array}{ccccc}
 & & (P \overset{id}{\underset{id}{\rightrightarrows}} P \longrightarrow 0 & & \\
 & & \downarrow id \qquad \downarrow id \qquad \downarrow & & \\
(P \overset{id}{\underset{id}{\rightrightarrows}} P \overset{id}{\longrightarrow} P) & \longrightarrow & (P \overset{id}{\underset{id}{\rightrightarrows}} P \longrightarrow 0) & \longrightarrow & (0 \rightrightarrows 0 \longrightarrow 0)
\end{array}
$$

as a commutative diagram.

Remark 3. Similarly we see that $0 \rightrightarrows P \longrightarrow 0$ and $P \rightrightarrows 0 \longrightarrow P$ of a quiver $Q = \bullet \rightrightarrows \bullet \rightarrow \bullet$ are not projective representations if $P \neq 0$.

We now have our main Theorem.

THEOREM 2.6. *Any projective representation* $M_1 \rightrightarrows M_2 \longrightarrow M_3$ *of a quiver* $Q = \bullet \rightrightarrows \bullet \rightarrow \bullet$ *is isomorphic to the quotient of a direct sum of projective representations of the forms* $0 \rightrightarrows 0 \longrightarrow P$, $0 \rightrightarrows P \xrightarrow{id} P$ *and* $P \underset{e_2}{\overset{e_1}{\rightrightarrows}} P \oplus P \xrightarrow{id_{P\oplus P}} P \oplus P$, *where* $e_1(a) = (a, 0)$ *and* $e_2(a) = (0, a)$, *and* P *is a projective left* R*-module.*

Proof. Since $M_1 \rightrightarrows M_2 \longrightarrow M_3$ is a projective representation of a quiver $Q = \bullet \rightrightarrows \bullet \rightarrow \bullet$, the diagram

$$\begin{array}{ccccc} & & (M_1 \underset{f_{12}}{\overset{f_{11}}{\rightrightarrows}} M_2 \xrightarrow{f_2} M_3) & & \\ & & \downarrow id \qquad \downarrow \qquad \downarrow & & \\ (M_1 \underset{id}{\overset{id}{\rightrightarrows}} M_1 \longrightarrow 0) & \longrightarrow & (M_1 \rightrightarrows 0 \longrightarrow 0) & \longrightarrow & (0 \rightrightarrows 0 \longrightarrow 0) \end{array}$$

can be completed to a commutative diagram by $id : M_1 \longrightarrow M_1$, $t : M_2 \longrightarrow M_1$, $0 : M_3 \longrightarrow 0$. Then we can get $tf_{11} = id_{M_1}$ and $tf_{12} = id_{M_1}$ so that $M_2 \cong f_{11}(M_1) \oplus Ker(t)$ and $M_2 \cong f_{12}(M_1) \oplus Ker(t)$. Now the following diagram

$$\begin{array}{ccccc} & & (M_1 \underset{f_{12}}{\overset{f_{11}}{\rightrightarrows}} M_2 \xrightarrow{f_2} M_3) & & \\ & & \downarrow f_{11}\oplus f_{12} \quad \downarrow id \qquad \downarrow & & \\ (M_2 \underset{id}{\overset{id}{\rightrightarrows}} M_2 \xrightarrow{id} M_2) & \longrightarrow & (M_2 \underset{id}{\overset{id}{\rightrightarrows}} M_2 \longrightarrow 0) & \longrightarrow & (0 \rightrightarrows 0 \longrightarrow 0) \end{array}$$

can be completed to a commutative diagram by $f_{11} \oplus f_{12} : M_1 \longrightarrow M_2$, $id : M_2 \longrightarrow M_2$, $u : M_3 \longrightarrow M_2$. Then we can get $uf_2 = id_{M_2}$ so that $M_3 \cong M_2 \oplus Ker(u)$. Therefore,

$$M_3 \cong M_2 \oplus Ker(u) \cong f_{11}(M_1) \oplus Ker(t) \oplus Ker(u)$$

and

$$M_3 \cong M_2 \oplus Ker(u) \cong f_{12}(M_1) \oplus Ker(t) \oplus Ker(u).$$

This completes the proof.

□

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