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THE COEFFICIENTS OF BELL DOMAINS AND THE CRITICAL POINTS OF CORRESPONDING FUNCTIONS

Moonja Jeong

Abstract. In this note, we determine the properties of the coefficients of Bell domains in the plane and find some coefficients to consist of Bell domain.

1. Introduction

In this paper, a non-degenerate finitely connected domain in the plane is a domain such that no boundary component is a point. To calculate the Bergman kernel associated to the given domain explicitly is possible only for a few special domains. It is well known that the Bergman kernel can be rational only for simply connected domains (see [1]). Conditions for checking whether the Bergman kernel associated to a given domain is algebraic are as follows (see [2], [3]).

Proposition 1.1. *Suppose Ω is a non-degenerate finitely connected domain in the plane. The following conditions are equivalent.*

(1) *The Bergman kernel associated to Ω is algebraic.*

(2) *The Szegö kernel associated to Ω is algebraic.*

(3) *There is a single proper holomorphic mapping of Ω onto the unit disc which is algebraic.*

(4) *Every proper holomorphic mapping of Ω onto the unit disc is algebraic.*

So in order to know that the Bergman kernel associated to Ω is algebraic, it is enough to find a proper holomorphic map of Ω onto the unit disc which is algebraic. Also, to find such a domain with algebraic

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proper map from the given domain onto the unit disc is an interesting problem.

A domain $D = \{z \in \mathbb{C} : |z+1/z| < r\}$ with $r > 2$ is doubly connected and the function f defined by

$$f(z) = \frac{1}{r}(z + \frac{1}{z})$$

is an algebraic proper map from the given domain D onto the unit disc. So the Bergman kernel associated with D is algebraic.

We are seeking for n-connected domains satisfying similar equation. We know that every non-degenerate n-connected domain in the plane has a canonical representation as in the following theorem in [6], which is called a *Bell domain* of it.

THEOREM 1.2. *Every non-degenerate n-connected planar domain with $n \geq 2$ is mapped biholomorphically onto a domain $W_{\mathbf{a},\mathbf{b}}$ defined by*

$$\left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

with suitable complex numbers a_k and b_k where $\mathbf{a} = (a_1, a_2, \cdots, a_{n-1})$ and $\mathbf{b} = (b_1, b_2, \cdots, b_{n-1})$.

Bell domain is important in the sense that every Bell domain $W_{\mathbf{a},\mathbf{b}}$ has the algebraic Bergman kernel. That is, the function $f_{\mathbf{a},\mathbf{b}}$ defined by

$$f_{\mathbf{a},\mathbf{b}}(z) = z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}$$

is an algebraic proper holomorphic mapping from $W_{\mathbf{a},\mathbf{b}}$ onto the unit disc.

Therefore, the above theorem implies the following corollary.

COROLLARY 1.3. *Every non-degenerate n-connected domain in the plane is biholomorphic to a domain with the algebraic Bergman kernel.*

In this paper we study the set of coefficients $(\mathbf{a}, \mathbf{b})$ which correspond to Bell domains representing non-degenerate n-connected domains in the plane.

2. The coefficient body of Bell domains

To find the property of the coefficients, we define the following.

DEFINITION 2.1. For every $n \geq 2$, let $\mathbf{B}_n$ be the set of all complex vectors $(\mathbf{a}, \mathbf{b})$ in $\mathbb{C}^{2n-2}$ such that the corresponding domains

$$W_{\mathbf{a},\mathbf{b}} = \left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

are non-degenerate n-connected domains in the plane.

We call $\mathbf{B}_n$ the *coefficient body* for non-degenerate n-connected canonical domains.

The analysis of $\mathbf{B}_2$ can be seen in [7] as follows.

PROPOSITION 2.2. *For a complex number a, let a' be a complex number such that $(a')^2 = a$. Then $\mathbf{B}_2 = \{(a, b) \in \mathbb{C}^2 : a \neq 0, |b + 2a'| < 1, |b - 2a'| < 1\}$.*

Note that it is independent of the choice of a'. The following lemma for the condition of $\mathbf{B}_n$ with $n \geq 2$ is in [8].

LEMMA 2.3. *The coefficient body $\mathbf{B}_n$ is the set of all $(\mathbf{a}, \mathbf{b})$ such that*

$$f'_{\mathbf{a},\mathbf{b}}(z) = 0$$

has $2n - 2$ solutions $c_1, \cdots, c_{2n-2}$ counted with multiplicities such that

$$|f_{\mathbf{a},\mathbf{b}}(c_j)| < 1$$

for every j.

In particular, $\mathbf{B}_n$ is an open subset of $\mathbb{C}^{2n-2}$.

Now we seek for a condition for $\mathbf{B}_3$. Let

$$f_1(z) = f_{a,a,b,-b}(z) = z + \frac{a}{z - b} + \frac{a}{z + b}$$

with $a, b \in \mathbb{C} - \{0\}$. Then

$$f'_1(z) = 1 - \frac{a}{(z - b)^2} - \frac{a}{(z + b)^2} = \frac{(z^2 - b^2)^2 - 2az^2 - 2ab^2}{(z^2 - b^2)^2}.$$

Hence $f_1'(z) = 0$ has 4 roots

$$\left(b^2 + a + (4ab^2 + a^2)^{1/2}\right)^{1/2}.$$

The solutions of the equation $z^4 - 2(b^2 + a)z^2 + b^4 - 2ab^2 = 0$ are critical points of f_1. If it holds for c, then it is also satisfied for $-c$. So, if c is a critical point of f_1, then $-c$ is also a critical point of f_1. Hence we get the following theorem.

THEOREM 2.4. *The element $(a, a, b, -b) \in \mathbf{B}_3$ if and only if a, b satisfy the inequality*

$$|b^2 + a + (4ab^2 + a^2)^{1/2}| \cdot |b^2 - \frac{1}{2}a^2 + \frac{a}{2}(4ab^2 + a^2)^{1/2}|^2 < |b^4|$$

where the same value of $(4ab^2 + a^2)^{1/2}$ is taken on each side.

Proof. Let

$$f_1(z) = z + \frac{a}{z-b} + \frac{a}{z+b}.$$

Then $(a, a, b, -b) \in \mathbf{B}_3$ if and only if $|f_1| < 1$ at each critical points of f_1.

Note that $f_1'(z)$ has 4 roots

$$\left(b^2 + a + (4ab^2 + a^2)^{1/2}\right)^{1/2}$$

and

$$|f_1(z)|^2 = |z(1 + \frac{2a}{z^2 - b^2})|^2.$$

Hence $(a, a, b, -b) \in \mathbf{B}_3$ if and only if

$$\begin{aligned}
&|b^2 + a + (4ab^2 + a^2)^{1/2}|\,|\frac{3a + (4ab^2 + a^2)^{1/2}}{a + (4ab^2 + a^2)^{1/2}}|^2 \\
=&|b^2 + a + (4ab^2 + a^2)^{1/2}|\,|\frac{(3a + (4ab^2 + a^2)^{1/2})(a - (4ab^2 + a^2)^{1/2})}{-4ab^2}|^2 \\
=&|b^2 + a + (4ab^2 + a^2)^{1/2}|\,|\frac{b^2 - \frac{1}{2}a^2 + \frac{a}{2}(4ab^2 + a^2)^{1/2}}{b^2}|^2 < 1.
\end{aligned}$$

So we get desired conclusion. □

Now, we find the condition for a point in $\mathbf{B}_3$ with multiplicity 2.

THEOREM 2.5. *Let*

$$f_1 = f_{a,a,b,-b}(z) = z + \frac{a}{z-b} + \frac{a}{z+b}$$

with $a, b \in \mathbb{C} - \{0\}$.

All the critical points of f_1 *are of multiplicity 2 if* $a = -4b^2$. *The point* 0 *is a critical point of* f_1 *with multiplicity 2 if* $b^2 = 2a$.

Proof. We represent

$$f_1'(z) = \frac{g(z^2)}{(z^2 - b^2)^2}$$

where $g(z^2) = (z^2 - b^2)^2 - 2az^2 - 2ab^2$. Hence $g(z^2) = z^4 - 2(b^2 + a)z^2 + b^4 - 2ab^2$.

Since the discriminant of $g(z)$ is

$$(b^2 + a)^2 - (b^4 - 2ab^2) = 4ab^2 + a^2 = a(4b^2 + a),$$

all the critical points of f_1 are of multiplicity 2 if $a = -4b^2$.

In order to find a condition for 0 to be a critical point of f_1 with multiplicity 2, we use the quadratic formula for $g(z)$. The equation

$$(b^2 + a) + (4ab^2 + a^2)^{1/2} = 0$$

holds if and only if

$$(b^2 + a)^2 = 4ab^2 + a^2.$$

Hence 0 is a critical point of f_1 with multiplicity 2 if and only if $b^2 = 2a$. □

Now we find some elements $(a, a, b, -b)$ of $\mathbf{B}_3$.

EXAMPLE 2.6. 1) Let $a = 1/200$ and $b = 1/10$. Then $(a, a, b, -b)$ satisfies the inequality in Theorem 2.4 and so it belongs to $\mathbf{B}_3$. Note that 0 is a critical point of $f_{a,a,b,-b}$ with multiplicity 2 since $b^2 = 2a$.

In fact, the critical points of $f_{a,a,b,-b}$ are

$$\{\pm\frac{\sqrt{3}}{10}, 0, 0\}.$$

2) Let $a = -1/25$ and $b = 1/10$. Then $(a, a, b, -b)$ belongs to $\mathbf{B}_3$. We notice that all the critical points of $f_{a,a,b,-b}$ are of multiplicity 2 since $a = -4b^2$.

In fact, the critical points of $f_{a,a,b,-b}$ are

$$\{\pm\frac{\sqrt{3}}{10}i, \pm\frac{\sqrt{3}}{10}i\}.$$

3) Let $a = 9/400$ and $b = 1/10$. Then $(a, a, b, -b)$ belongs to $\mathbf{B}_3$. The critical points of $f_{a,a,b,-b}$ are

$$\{\pm\frac{\sqrt{7}}{10}, \pm\frac{\sqrt{2}}{20}i\}.$$

Note that all the critical points are simple.

3. Projection mapping

We study the mapping from the coefficient body onto the set of critical points of the functions $f_{\mathbf{a},\mathbf{b}}$ or that of the critical values, i.e. the images of critical points.

DEFINITION 3.1. Let Γ be the set of all points $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n$ such that the corresponding rational map $f_{\mathbf{a},\mathbf{b}}$ has a non-simple critical point or has a pair of critical points whose images are the same. Γ is called the *collision locus*.

It implies that the rational map $f_{\mathbf{a},\mathbf{b}}$ has $2n-2$ simple critical values if $(\mathbf{a}, \mathbf{b})$ in $\mathbf{B}_n - \Gamma$. We denote the set of simple critical values of $f_{\mathbf{a},\mathbf{b}}$ by

$$V_{\mathbf{a},\mathbf{b}} = \{\alpha_1, \cdots, \alpha_{2n-2}\},$$

where $\alpha_j = f_{\mathbf{a},\mathbf{b}}(c_j)$ for every j if we let $\{c_j\}_{j=1}^{2n-2}$ be the set of the simple critical points of $f_{\mathbf{a},\mathbf{b}}$. This set can be considered as a point in $B_{0,2n-2}U$ where $B_{0,2n-2}\mathbb{C}$ is the quotient space of $F_{0,2n-2}\mathbb{C} = \{(z_1, \cdots, z_{2n-2}) \in \mathbb{C}^{2n-2} : z_i \neq z_j \text{ if } i \neq j\}$ by the symmetric group S_{2n-2}. In fact $V_{\mathbf{a},\mathbf{b}}$ is a point in $B_{0,2n-2}U$ where $U \subset \mathbb{C}^{2n-2}$ is the unit disc.

Consider the projection

$$\pi_V : \mathbf{B}_n - \Gamma \to B_{0,2n-2}U$$

defined by

$$\pi_V(\mathbf{a}, \mathbf{b}) = V_{\mathbf{a},\mathbf{b}}.$$

Since

$$f_{\mathbf{a},\mathbf{b}}(z) = z + \sum_{k=1}^{n-1} \frac{a_k}{(z-b_k)^2},$$

$$\begin{aligned} f'_{\mathbf{a},\mathbf{b}}(z) &= 1 - \sum_{k=1}^{n-1} \frac{a_k}{(z-b_k)^2} \\ &= \prod_{j=1}^{n-1}(z-b_j)^2 \left(1 - \sum_{k=1}^{n-1} \frac{a_k}{(z-b_k)^2}\right). \end{aligned}$$

Hence, for every point $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n - \Gamma$, the critical points $c_1, \cdots, c_{2n-2}$ of $f_{\mathbf{a},\mathbf{b}}$ are the simple solutions of the algebraic equation $f'_{\mathbf{a},\mathbf{b}}(z) = 0$. c_j moves holomorphically with respect to $(\mathbf{a}, \mathbf{b})$ and so does the image α_j of c_j for each $j = 1, \cdots, 2n-2$. Therefore the map π_S is holomorphic.

For the projection π_V the following theorem is known (see [8]).

Theorem 3.2. *The projection π_S is a*

$$(2n-2)!\, n^{n-3}$$

-sheeted proper holomorphic covering of $B_{0,2n-2}U$ for every $n \geq 2$.

It means the number of points in $\pi_V^{-1}(\mathrm{V})$ of V by π_V is always $(2n-2)!\, n^{n-3}$. The number

$$\frac{(2n-2)!\, n^{n-3}}{n!}$$

is known as a Hurwitz number (see [5]).

Now we define another projection.

Definition 3.3. Let $\Delta \subset \Gamma$ be the set of all points $(\mathbf{a}, \mathbf{b})$ in $(\mathbb{C}^*)^{n-1} \times F_{0,n-1}\mathbb{C}$ where $(\mathbb{C}^*) = \mathbb{C} - \{0\}$ such that the corresponding rational map $f_{\mathbf{a},\mathbf{b}}$ has a non-simple critical point. It is called the *non-simple locus*.

Then for every point $(\mathbf{a}, \mathbf{b})$ in $(\mathbb{C}^*)^{n-1} \times F_{0,n-1}\mathbb{C} - \Delta$, the rational function $f_{\mathbf{a},\mathbf{b}}$ has $2n-2$ simple critical points. We denote the set of simple critical points of $f_{\mathbf{a},\mathbf{b}}$ by

$$C_{\mathbf{a},\mathbf{b}} = \{c_1, \cdots, c_{2n-2}\}.$$

We see that $C_{\mathbf{a},\mathbf{b}}$ can be considered as a point in $B_{0,2n-2}\mathbb{C}$.

Thus the projection

$$\pi_C : (\mathbb{C}^*)^{n-1} \times F_{0,n-1}\mathbb{C} - \Delta \to B_{0,2n-2}\mathbb{C}$$

defined by

$$\pi_C(\mathbf{a}, \mathbf{b}) = C_{\mathbf{a},\mathbf{b}}$$

is a well defined holomorphic map.

The following theorem can be checked in [8].

THEOREM 3.4. *For every point* C *in* $B_{0,2n-2}\mathbb{C}$, *there are at most*

$$\frac{(2n-2)!}{n!}$$

preimages of C *by* π_C.

The number

$$\frac{(2n-2)!}{n!\,(n-1)!}$$

is called the *n-th Catalan number*. For every fixed C in $B_{0,2n-2}\mathbb{C}$, there are

$$\frac{(2n-2)!}{n!\,(n-1)!}$$

classes of rational functions of degree n which have C as the set of critical points ([4]).

EXAMPLE 3.5. In Example 2.6 we find that the set of critical points of $f_{a,a,b,-b}$ is

$$C = \{\pm\frac{\sqrt{7}}{10}, \pm\frac{\sqrt{2}}{20}i\}$$

where $a = 9/400$ and $b = 1/10$.

On the other hand, for

$$C = \{\pm\frac{\sqrt{7}}{10}, \pm\frac{\sqrt{2}}{20}i\} \in B_{0,2n-2}\mathbb{C},$$

there are at most $4!/3!$ preimages of C by π_C by Theorem 3.4. Two of them are known as

$$(\frac{9}{400}, \frac{9}{400}, \frac{1}{10}, -\frac{1}{10}), (\frac{9}{400}, \frac{9}{400}, -\frac{1}{10}, \frac{1}{10})$$

By calculation we find another 2 preimages

$$(\frac{1}{48}, \frac{1}{48}, \frac{\sqrt{7}}{10\sqrt{6}}, -\frac{\sqrt{7}}{10\sqrt{6}}), (\frac{1}{48}, \frac{1}{48}, -\frac{\sqrt{7}}{10\sqrt{6}}, \frac{\sqrt{7}}{10\sqrt{6}}).$$

So we find 4 preimages of C by π_C and there are $4!/(3!2!) = 2$ classes of rational functions of degree 3 which have C as a set of critical points. They are

$$f_1 = z + \frac{9/400}{z-1/10} + \frac{9/400}{z+1/10}$$

and

$$f_2 = z + \frac{1/48}{z - \frac{\sqrt{7}}{10\sqrt{6}}} + \frac{1/48}{z + \frac{\sqrt{7}}{10\sqrt{6}}}.$$

We know that $|f_1(c_i) = \alpha_i| < 1$ at each critical point c_i. The set of critical values of f_1 is

$$\mathrm{V} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{\pm\frac{21\sqrt{7}}{120}, \pm\frac{\sqrt{2}}{10}i\} \in B_{0,2n-2}U.$$

So we have 4! preimages of V by π_V by Theorem 3.2.

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Department of Mathematics
The University of Suwon
Kyungkido, 445-743, Korea
E-mail: mjeong@suwon.ac.kr