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THE INTEGRATION BY PARTS FOR THE C-INTEGRAL

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Abstract. In this paper, we define the C-integral and prove the integration by parts formula for the C-integral.

1. Introduction and preliminaries

It is well-known [8] that the integration by parts formula is valid for the Lebesgue, Denjoy, Perron, and Henstock integrals. In this paper, we prove the integration by parts formula for the C-integral. Throughout this paper, $I_0 = [a, b]$ is a compact interval in R. Let $D = \{(I_i, \xi_i)\}_{i=1}^n$ be a finite collection of non-overlapping tagged intervals of I_0 and let δ be a positive function on I_0. We say that D is

(a) a δ - fine McShane partition of I_0 if $\cup_{i=1}^n I_i = I_0$, $I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in I_o$ for all $i = 1, 2, ..., n$,

(b) a δ - fine $C_\epsilon-$partition of I_0 if it is a δ - fine McShane partition of I_0 and satisfying

$$\sum_{i=1}^{n} dist(\xi_i, I_i) < \frac{1}{\epsilon},$$

where $\text{dist}(\xi_i, I_i) = inf\{|t - \xi_i| : t \in I_i\}$.

Given a $\delta-$fine partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ we write

$$S(f, D) = \sum_{i=1}^{n} f(\xi_i)|I_i|,$$

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whenever $f : I_0 \to R$.

2. Properties of the C-integral

We present the definition of the C-integral.

DEFINITION 2.1. [2] A function $f : I_0 \to R$ is C-integrable if there exists a real number A such that for each $\epsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that

$$|S(f, D) - A| < \epsilon$$

for each $\delta-$fine C_ϵ-partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0. The real number A is called the C-integral of f on I_0. and we write $A = \int_{I_0} f$ or $A = (C)\int_{I_0} f$.

The function f is C-integrable on the set $E \subset I_0$ if the function $f\chi_E$ is C-integrable on I_0, we write $\int_E f = \int_{I_0} f\chi_E$.

We can easily get the following two theorems.

THEOREM 2.2. *A function $f : I_0 \to R$ is C-integrable if and only if for each $\epsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that*

$$|S(f, D_1) - S(f, D_2)| < \epsilon$$

for arbitrary δ-fine C_ϵ -partitions D_1 and D_2 of I_0.

THEOREM 2.3. *Let $f : I_0 \to R$. Then*

(1) If f is C-integrable on I_0, then f is C-integrable on every subinterval of I_0.

(2) If f is C-integrable on each of the intervals I_1 and I_2, where I_1 and I_2 are non-overlapping and $I_1 \cup I_2 = I_0$, then f is C-integrable on I_0 and $\int_{I_1} f + \int_{I_2} f = \int_{I_0} f$.

The following theorem shows that the C-integral is linear.

THEOREM 2.4. *Let f and g be C-integrable functions on I_0. Then*
(1) αf is C-integrable on I_0 and $\int_{I_0} \alpha f = \alpha \int_{I_0} f$ for each $\alpha \in R$,
(2)$f + g$ is C-integrable on I_0 and $\int_{I_0}(f + g) = \int_{I_0} f + \int_{I_0} g$.

DEFINITION 2.5. Let $F : I_0 \to R$ and let E be a subset of I_0.

(a) F is said to be AC_c on E if for each $\epsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta : I_0 \to R^+$ such that $\sum_i |F(I_i)| < \epsilon$ for each δ-fine partial C_ϵ-partition $D = \{(I_i, \xi_i)\}$ of I_0 satisfying $\xi_i \in E$ and $\sum_i |I_i| < \eta$.

(b) F is said to be ACG_c on E if E can be expressed as a countable union of sets on each of which F is AC_c.

THEOREM 2.6. [13] *If a function $f : I_0 \to R$ C-integrable on I_0 if and only if there is an ACG_c function F on I_0 such that $F' = f$ almost everywhere on I_0.*

3. The integration by parts for the C-integral

To prove the integration by parts for the C-integral, we need the following two theorems.

THEOREM 3.1. *If F is ACG_c on $[a, b]$, then F is continuous on $[a, b]$.*

Proof. Let $[a, b] = \cup_{n=1}^{\infty} E_n$ where F is AC_c on each E_n. Let $c \in [a, b]$ and choose an index n such that $c \in E_n$. Let $\epsilon > 0$. Since F is AC_c on E_n, there exist a positive number $\eta > 0$ and a positive function $\delta : [a, b] \to \mathbb{R}^+$ such that $\sum_i |F(I_i)| < \epsilon$ for each $\delta-$fine partial $C_\epsilon-$partition $D = \{(\xi_i, I_i)\}_{i=1}^{n}$ of $[a, b]$ satisfying $\sum_i |I_i| < \eta$ and $\xi_i \in E_n$. Let $r = \min\{\delta(c), \eta\}$. Suppose that $x \in (c - r, c + r) \cap [a, b]$. Then $([c, x], c)$ $(or([x, c], c))$ is a $\delta-$fine partial $C_\epsilon-$partition with $|x - c| < \eta$. Hence, $|F(x) - F(c)| < \epsilon$. It follows that F is continuous at c. □

THEOREM 3.2. *If F and G are ACG_c on $[a, b]$, then FG is ACG_c on $[a, b]$.*

Proof. Since F and G are continuous on $[a, b]$ by Theorem 3.1, there exist real numbers M_1 and M_2 with $M_1, M_2 \geq 1$ such that $|F(t)| \leq M_1$ and $|G(t)| \leq M_2$ for each $t \in [a, b]$. Since F is ACG_c on $[a, b]$, we have $[a, b] = \cup_{n=1}^{\infty} E_n$ and F is AC_c on each E_n. Since G is ACG_c on $[a, b]$, we have $[a, b] = \cup_{k=1}^{\infty} A_k$ and G is AC_c on each A_k. then $[a, b] = \cup_{n=1}^{\infty} \cup_{k=1}^{\infty} (E_n \cap A_k)$.

To show that FG is AC_c on each $E_n \cap A_k$, fix n and k. Let $\epsilon > 0$. Since F is AC_c on E_n, there exist a constant $\eta_1 > 0$ and a positive function $\delta_1 : [a, b] \to \mathbb{R}^+$ such that

$$\sum_{i=1}^{n} |F(I_i)| < \frac{\epsilon}{2M_2}$$

for each δ_1-fine partial $McShane$ partition $\{(x_i, I_i)\}_{i=1}^{p}$ of $[a, b]$ satisfying $\sum_{i=1}^{p} dist(x_i, I_i) < \frac{2M_2}{\epsilon}$ and $\sum_i |I_i| < \eta_1$ and $x_i \in E_n$. Since G

is AC_c on A_k, there exist a constant $\eta_2 > 0$ and a positive function $\delta_2 : [a, b] \to \mathbb{R}^+$ such that

$$\sum_{j=1}^{n} |G(J_j)| < \frac{\epsilon}{2M_1}$$

for each δ_2-fine partial *McShane* partition $\{(y_j, J_j)\}_{j=1}^{q}$ of $[a, b]$ satisfying $\sum_{j=1}^{q} dist(y_j, J_j) < \frac{2M_1}{\epsilon}$ and $\sum_j |J_j| < \eta_2$ and $y_j \in A_k$.

Let $\delta = \min\{\delta_1, \delta_2\}$ and let $\eta = \min\{\eta_1, \eta_2\}$. Let $D = \{(\xi_i, [c_i, d_i])\}_{i=1}^{m}$ be a $\delta-$fine partial *McShane* partition satisfying $\sum_{i=1}^{m} dist(\xi_i, [c_i, d_i]) < \frac{1}{\epsilon}$ and $\sum_i (d_i - c_i) < \eta$ and $\xi_i \in E_n \cap A_k$. Then, since $\sum_{i=1}^{m} dist(\xi_i, [c_i, d_i]) < \frac{2M_1}{\epsilon}$ and $\sum_{i=1}^{m} dist(\xi_i, [c_i, d_i]) < \frac{2M_2}{\epsilon}$, we have

$$\begin{aligned}
&\sum_{i=1}^{m} |F(d_i)G(d_i) - F(c_i)G(c_i)| \\
\le\ &\sum_{i=1}^{m} |F(d_i)G(d_i) - F(c_i)G(d_i)| + \sum_{i=1}^{m} |F(c_i)G(d_i) - F(c_i)G(c_i)| \\
=\ &\sum_{i=1}^{m} |G(d_i)||F(d_i) - F(c_i)| + \sum_{i=1}^{m} |F(c_i)||G(d_i) - G(c_i)| \\
\le\ &M_2 \sum_{i=1}^{m} |F(d_i) - F(c_i)| + M_1 \sum_{i=1}^{m} |G(d_i) - G(c_i)| \\
<\ &M_2 \frac{\epsilon}{2M_2} + M_1 \frac{\epsilon}{2M_1} = \epsilon.
\end{aligned}$$

Hence, FG is AC_c on $E_n \cap A_k$. □

THEOREM 3.3. *Let $f : [a, b] \to \mathbb{R}$ be $C-$integrable on $[a, b]$ and let $F(x) = (C)\int_a^x f$ for each $x \in [a, b]$. If $G : [a, b] \to \mathbb{R}$ is AC on $[a, b]$, then fG is $C-$integrable on $[a, b]$ and*

$$(C)\int_a^b fG = F(b)G(b) - (L)\int_a^b FG'.$$

Proof. Since F is ACG_c on $[a, b]$ and the AC function G is AC_c on $[a, b]$, FG is ACG_c on $[a, b]$ by Theorem 3.1. Hence, $(FG)'$ is $C-$integrable on $[a, b]$. Since F is bounded and measurable, FG' is Lebesgue integrable on $[a, b]$. Since $fG = (FG)' - FG'$ almost everywhere on $[a, b]$, fG is $C-$integrable on $[a, b]$ and

$$\begin{aligned}(C)\int_a^b fG &= (C)\int_a^b (FG)' - (L)\int_a^b FG' \\ &= F(b)G(b) - (L)\int_a^b FG'.\end{aligned}$$

□

COROLLARY 3.4. *Let $f : [a, b] \to \mathbb{R}$ be $C-$integrable on $[a, b]$ and let $F(x) = (C)\int_a^x f$ for each $x \in [a, b]$. If $G : [a, b] \to \mathbb{R}$ is AC on $[a, b]$, then fG is $C-$integrable on $[a, b]$ and*

$$(C)\int_a^b fG = F(b)G(b) - \int_a^b FdG,$$

where the second integral is the $Riemann - Stieltjes$ integral of F with respect to G.

Proof. By Theorem 3.3, the function fG is $C-$integrable on $[a, b]$. Since F is continuous and G is AC on $[a, b]$,

$$(L)\int_a^b FG' = \int_a^b FdG.$$

Hence,

$$(C)\int_a^b fG = F(b)G(b) - \int_a^b FdG.$$

□

THEOREM 3.5. *Let $f : [a, b] \to \mathbb{R}$ be $C-$integrable on $[a, b]$ and let $F(x) = (C)\int_a^x f$ for each $x \in [a, b]$. If $G : [a, b] \to \mathbb{R}$ is an ACG_c function of bounded variation on $[a, b]$, then fG is $C-$integrable on $[a, b]$ and*

$$(C)\int_a^b fG = F(b)G(b) - \int_a^b FdG.$$

Proof. Since F is ACG_c on $[a, b]$, FG is ACG_c on $[a, b]$ by Theorem 3.2. Hence, $(FG)'$ is $C-$integrable on $[a, b]$. Since F is bounded and measurable, FG' is Lebesgue integrable on $[a, b]$. Since $fG = (FG)' - FG'$ almost everywhere on $[a, b]$, fG is $C-$integrable on $[a, b]$ and hence

fG is *Henstock* integrable on $[a, b]$. By [8, Theorem 12.21],

$$\begin{aligned}(C)\int_a^b fG &= (H)\int_a^b fG \\ &= F(b)G(b) - \int_a^b FdG.\end{aligned}$$

□

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