JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 3, September 2009

ON THE STATISTICALLY COMPLETE FUZZY NORMED LINEAR SPACE.

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Abstract. In this paper, we introduce the notion of the statistically complete fuzzy norm on a linear space. And we consider some relations between the fuzzy statistical completeness and ordinary completeness on a linear space.

1. Introduction

The notions of fuzzy vector spaces and fuzzy topological vector spaces were introduced in Katsaras and Liu [6]. These ideas were modified by Katsaras [4], and in [5] Katsaras defined the fuzzy norm on a vector space. In [7] Krishna and Sarma discussed the generation of a fuzzy vector topology from an ordinary vector topology on a vector space. Also Krishna and Sarma [8] observed the convergence of sequence of fuzzy points. Rhie et al.[12] introduced the notion of fuzzy α-Cauchy sequence of fuzzy points and fuzzy completeness.

In this paper, we firstly observe some properties relative to the statistically convergent sequences in a normed linear space. Secondly, we introduce the notion of the statistically complete fuzzy norm using the statistical convergence of sequences on a linear space. And we consider some relations between the fuzzy statistical completeness and the ordinary completeness on a linear space.

Received July 23, 2009; Accepted August 20, 2009.
2000 Mathematics Subject Classification: Primary 54A40.
Key words and phrases: fuzzy norm, equivalent fuzzy norm, statistical convergence, statistical Cauchy sequence, fuzzy statistically complete.
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*This paper has been supported by the 2009 Hannam University Research Fund.

2. Preliminaries

Throughout this paper, X is a vector space over the field $K(R$ or $C)$. Fuzzy subsets of X are denoted by Greek letters in general. χ_A denotes the characteristic function of the set A.

Definition 2.1. [6] For two fuzzy subset μ_1 and μ_2 of X, the fuzzy subset $\mu_1 + \mu_2$ is defined by

$$(\mu_1 + \mu_2)(x) = \sup_{x_1 + x_2 = x} min\{\mu_1(x_1), \mu_2(x_2)\}$$

And for a scalar t of K and a fuzzy subset μ of X, the fuzzy subset $t\mu$ is defined by

$$(t\mu)(x) = \begin{cases} \mu(x/t) & \text{if} \quad t \neq 0 \\ 0 & \text{if} \quad t = 0 \ \text{ and } \ x \neq 0 \\ \sup_{y \in X} \mu(y) & \text{if} \quad t = 0 \ \text{ and } \ x = 0. \end{cases}$$

Definition 2.2. [4] $\mu \in I^X$ is said to be

1. *convex* if $t\mu + (1-t)\mu \leq \mu$ for each $t \in [0, 1]$
2. *balanced* if $t\mu \leq \mu$ for each $t \in K$ with $|\, t \,| \leq 1$
3. *absorbing* if $sup_{t>0}\, t\mu(x) = 1$ for all $x \in X$.

Definition 2.3. [4] Let (X, τ) be a topological space and $\omega(\tau) = \{f : (X, \tau) \to [0, 1] \mid f \ \ is \ \ lower \ \ semicontinuous\}$. Then $\omega(\tau)$ is a fuzzy topology on X. This topology is called the fuzzy topology generated by τ on X. The fuzzy usual topology on K means the fuzzy topology generated by the usual topology of K.

Definition 2.4. [4] A *fuzzy linear topology* on a vector space X over K is a fuzzy topology on X such that the two mappings

$$\begin{aligned} + \ &: \ X \times X \to X, \qquad (x, y) \to x + y \\ \cdot \ &: \ K \times X \to X, \qquad (t, x) \to tx \end{aligned}$$

are continuous when K has the fuzzy usual topology. A linear space with a fuzzy linear topology is called a *fuzzy topological linear space* or a *fuzzy topological vector space*.

Definition 2.5. [4] Let x be a point in a fuzzy topological space X. A family F of neighborhoods of x is called a base for the system of all neighborhoods of x if for each neighborhood μ of x and each $0 < \theta < \mu(x)$, there exists $\mu_1 \in F$ with $\mu_1 \leq \mu$ and $\mu_1(x) > \theta$.

DEFINITION 2.6. [5] A *fuzzy seminorm* on X is a fuzzy set ρ in X which is convex, balanced and absorbing. If in addition $inf_{t>0}\, t\rho(x) = 0$ for every nonzero x, then ρ is called a *fuzzy norm*.

THEOREM 2.7. [5] *If ρ is a fuzzy seminorm on X, then the family $B_\rho = \{\theta \wedge (t\rho) \mid 0 < \theta \leq 1,\ t > 0\}$ is a base of zero for a fuzzy linear topology τ_ρ. The fuzzy topology τ_ρ is called the fuzzy topology induced by the fuzzy seminorm ρ . And a linear space equipped with a fuzzy seminorm (resp. fuzzy norm) is called a fuzzy seminormed (resp. fuzzy normed) linear space.*

DEFINITION 2.8. [7] Let ρ be a fuzzy seminorm on X. $P_\epsilon : X \to R_+$ is defined by $P_\epsilon(x) = \wedge\{t > 0 \mid t\rho(x) > \epsilon\}$ for each $0 < \epsilon < 1$.

THEOREM 2.9. [7] *The seminorm P_ϵ is a seminorm for each $\epsilon \in (0, 1)$. Further P_ϵ is a norm on X for each $0 < \epsilon < 1$ if and only if ρ is a fuzzy norm on X.*

THEOREM 2.10. [11] *A metric space (X, d) is complete if and only if for any nested sequence $A_1 \supset A_2 \supset \cdots$ of nonempty closed sets of X such that diameter $A_n \to 0$, $\cap_{n\in Z} A_n \neq \emptyset$.*

3. Statistical convergence on a normed linear space.

In [2], H. Fast introduced an extension of the usual concept of sequential limits which he called statistical convergence. In [14] I. J. Schoenberg gave some basic properties of statistical convergence and also studied the concept as a summability method. In [15], one may find a resent trend for this topics. In this section, we prove that every statistical Cauchy sequence on a Banach space is statistically convergent ; as an extension of Fridy's [3] result to a normed linear space.

DEFINITION 3.1. [15] The natural density of a positive integer set K is defined by $\delta(K) = \lim_{n\to\infty} \frac{1}{n} \mid \{k \in K : k \leq n\} \mid$, where $\mid \{k \in K : k \leq n\} \mid$ is the number of elements of K not exceeding n.

It is clear that for a finite set K, we have $\delta(K) = 0$. The natural density may not exist for each set K and is different from zero which means $\delta(K) > 0$. Besides that, $\delta(K^c) = 1 - \delta(K)$ where K^c means the complement of K.

NOTATION. For facilitation, we introduce the following notation: if $< x_k >$ is a sequence such that x_k satisfies property P for all k except a set of natural density zero (equivalently for all k in a positive integer set with natural density one), then we say that x_k satisfies P for "almost all k", and we abbreviate this by "$a.a.\ \ k$."

DEFINITION 3.2. The sequence $< x_k >$ on a normed linear space is statistically convergent to the vector x provided that for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \in n : \parallel x_k - x \parallel \geq \epsilon\} \mid = 0,$$

i.e.,

$$(*) \qquad \parallel x_k - x \parallel < \epsilon \quad a.a. \ \ k.$$

In this case we write $st - \lim x_k = x$.

Example [3] Define $x_k = x$ if k is a square and $x_k = 0$ otherwise. Then $\mid \{k \leq n : x_k \neq 0\} \mid \leq \sqrt{n}$, so $st - \lim x_k = 0$. Note that we could have assigned any values whatsoever to x_k when k is a square, and we would still have $st - \lim x_k = 0$. It is clear that if the inequality in $(*)$ holds for all but finitely many k, then $\lim x_k = x$. It follows that $\lim x_k = x$ implies $st - \lim x_k = x$. As most convergence theories, we introduce the statistical analogue of the Cauchy convergence criterion.

DEFINITION 3.3. The sequence $< x_k >$ on a normed linear space is statistical Cauchy sequence if for every $\epsilon > 0$ there exists a number $N (= N(\epsilon))$ such that

$$\parallel x_k - x_N \parallel < \epsilon \quad a.a. \ \ k,$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n : \parallel x_k - x_N \parallel \geq \epsilon\} \mid = 0,$$

THEOREM 3.4. *Every statistically convergent sequence is a statistical Cauchy sequence on a normed linear space.*

Proof. Suppose $st - \lim x_k = x$ and $\epsilon > 0$. Then $\parallel x_k - x \parallel < \epsilon/2$ $a.a.\ \ k$ and if N is chosen so that $\parallel x_N - x \parallel < \epsilon/2$ then we have

$$\begin{aligned} \parallel x_k - x_N \parallel &< \parallel x_k - x \parallel + \parallel x_N - x \parallel \\ &< \ \epsilon/2 + \epsilon/2 \quad a.a. \ \ k. \end{aligned}$$

Hence, $< x_k >$ is a statistically Cauchy sequence. □

THEOREM 3.5. *For every statistical Cauchy sequence $< x_k >$ on a complete normed linear space, there is a convergent sequence $< y_k >$ such that $x_k = y_k$ $a.a.\ \ k$.*

Proof. Let $< x_k >$ be a statistical Cauchy sequence. Then we can choose N so that the closed ball $C = \{x \in X :\| x_N - x \| \leq 1\}$ contains x_k *a.a. k.* Also we can choose M so that $C' = \{x \in X :\| x_M - x \| \leq 1/2\}$ contains x_k *a.a. k.* We assert that $C_1 = C \cap C'$ contains x_k *a.a. k.*

For,

$$\{k \leq n : x_k \notin C \cap C'\}$$
$$= \{k \leq n : x_k \notin C\} \cup \{k \leq n : x_k \notin C'\},$$

so

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n : x_k \notin C \cap C'\} \mid$$
$$\leq \lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n : x_k \notin C\} \mid + \lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n : x_k \notin C'\} \mid = 0.$$

Therefore C_1 is a closed set of diameter less than or equal to 1 that contains x_k *a.a. k.* Now we proceed by choosing $N(2)$ so that $C'' = \{x \in X :\| x_{N(2)} - x \| \leq 1/4\}$ contains x_k *a.a. k*, and by the preceding argument $C_2 = C_1 \cap C''$ contains x_k *a.a. k*, and C_2 has the diameter less than or equal to $1/2$. Continuing inductively we construct a sequence $\{C_m\}_{m=1}^{\infty}$ of closed sets such that for each m, $C_m \supseteq C_{m+1}$, the diameter of C_m is not greater than 2^{1-m}, and $x_k \in C_m$ *a.a. k.* By Theorem 2.10, there is a vector y such that $\cap_{m=1}^{\infty} C_m = \{y\}$. Using the fact that $x_k \in C_m$ *a.a. k.* we choose an increasing positive integer sequence $\{T_m\}_{m=1}^{\infty}$ such that

$$(*) \qquad \frac{1}{n} \mid \{k \leq n : x_k \notin C_m\} \mid < \frac{1}{m} \quad if \quad n > T_m.$$

Now define a subsequence z of $< x_k >$ consisting of terms x_k such that $k > T_1$ and

$$if \quad T_m < k < T_{m+1} \quad then \;\; x_k \notin C_m.$$

Next define a sequence $< y_k >$ by

$$y_k = \begin{cases} y & \text{if} \quad x_k \; is \; a \; term \; of \; z, \\ x_k & otherwise \end{cases}$$

Then $\lim y_k = y$; for, if $\epsilon > 1/m > 0$ and $k > T_m$ then either x_k is a term of z, which means $y_k = y$, or $y_k = x_k \in C_m$ and $\| y_k - y \| \leq$ diameter of $C_m \leq 2^{1-m}$. We also assert that $x_k = y_k$ a.a. k. To verify this we observe that if $T_m < n < T_{m+1}$ then

$$\{k \leq n : y_k \neq x_k\} \subseteq \{k \leq n : x_k \notin C_m\},$$

so by $(*)$

$$\frac{1}{n} \mid \{k \leq n : y_k \neq x_k\} \mid \leq \frac{1}{n} \mid \{k \leq n : x_k \notin C_m\} \mid < \frac{1}{m} \, .$$

Hence, the limit as $n \to \infty$ is 0 and $x_k = y_k$ *a.a.* k. This completes the proof. □

THEOREM 3.6. *If $< x_k >$ is a sequence on a Banach space for which there is a convergent sequence $< y_k >$ such that $x_k = y_k$ a.a. k, then it is a statistically convergent sequence.*

Proof. Let $x_k = y_k$ *a.a.* k and $\lim y_k = L$. Suppose $\epsilon > 0$. Then for each n,

$$\{k \leq n :\| x_k - L \|\geq \epsilon\}$$

$$\subseteq \{k \leq n : x_k \neq y_k\} \cup \{k \leq n :\| y_k - L \|> \epsilon\};$$

since $\lim y_k = L$, the latter set contains a fixed number of integers, say $l = l(\epsilon)$. Therefore

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n :\| x_k - L \|\geq \epsilon\} \mid$$

$\leq \lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n : x_k \neq y_k\} \mid + \lim \frac{l}{n} = 0$ because $x_k = y_k$ *a.a.* k. Hence, $\| x_k - L \|< \epsilon$ *a.a.* k, so the proof is complete.

□

4. Fuzzy statistical convergence and fuzzy statistical completeness.

In this section, we introduce the notion of a statistically complete fuzzy norm on a linear space. And we consider some relations between the fuzzy statistical completeness and the ordinary completeness. Now, we define the statistical convergence of sequences in a fuzzy normed linear space.

DEFINITION 4.1. Let (X, ρ) be a fuzzy normed linear space. A sequence $< x_k > \subset X$ is said to statistically converge to $x \in X$ if for every $t > 0$ and $0 < \epsilon < 1$, there exists a positive integer set K with natural density one such that $k \in K$ implies $t\rho(x_k - x) > 1 - \epsilon$,
i.e., $t\rho(x_k - x) > 1 - \epsilon$ *a.a.* k.

THEOREM 4.2. *Let (X, ρ) be a fuzzy normed linear space. A sequence $< x_k > \subset X$ statistically converges to $x \in X$ if and only if for every $t > 0$ and $0 < \epsilon < 1$,* $P_{1-\epsilon}(x_k - x) < t$ *a.a.* k.

Proof. Let $t > 0$ and $\epsilon \in (0, 1)$ be given. Since $< x_k >$ statistically converges to x, there exists a positive integer set K with natural density one such that

$$\begin{aligned} & x \in K \quad \text{implies} \quad \tfrac{t}{2}\rho(x_k - x) > 1 - \epsilon \\ \Longrightarrow \quad & P_{1-\epsilon}(x_n - x) \le \tfrac{t}{2} < t \quad a.a. \quad k. \end{aligned}$$

For the converse, let $t > 0$ and $\epsilon > 0$ be given. Then

$$\begin{aligned} & P_{1-\epsilon}(x_n - x) < t \quad a.a. \quad k \\ \Longrightarrow \quad & t'\rho(x_n - x) > 1 - \epsilon \quad a.a. \quad k \\ & \qquad \text{for some} \quad t' \in (P_{1-\epsilon}(x_n - x), t) \\ \Longrightarrow \quad & t\rho(x_n - x) \ge t'\rho(x_n - x) > 1 - \epsilon \quad a.a. \quad k. \end{aligned}$$

This completes the proof.

□

The next definition of the statistical Cauchy sequence in a fuzzy normed linear space is an extension of the fuzzy Cauchy sequence.

DEFINITION 4.3. Let (X, ρ) be a fuzzy normed linear space. A sequence $< x_k > \subset X$ is a statistical Cauchy sequence if and only if for every $t > 0$ and $0 < \epsilon < 1$, there exists a positive integer set K with natural density one such that $k, l \in K$ implies $t\rho(x_k - x_l) > 1 - \epsilon$,
i.e., $t\rho(x_k - x_l) > 1 - \epsilon \quad a.a. \quad k,\ l.$

THEOREM 4.4. *Let (X, ρ) be a fuzzy normed linear space. A sequence $< x_k > \subset X$ is a statistical Cauchy sequence if and only if for every $t > 0$ and $0 < \epsilon < 1$, $P_{1-\epsilon}(x_k - x_l) < t \quad a.a. \quad k,\ l.$*

Proof. The proof is similar to that of Theorem 4.2. We omit it.

□

The following theorem is easily verified with elementary skill from Theorem 4.2. and Theorem 4.4.

THEOREM 4.5. *Every statistically convergent sequence in a fuzzy normed linear space is a statistical Cauchy sequence.*

Now, we introduce the statistically complete fuzzy norm using the statistical Cauchy sequence defined above.

DEFINITION 4.6. A fuzzy normed linear space (X, ρ) is said to be fuzzy statistically complete if every statistical Cauchy sequence in X statistically converges to a point in X.

Lemma 4.7. *Let $(X, \| \cdot \|)$ be a normed linear space and B the closed unit ball of X. Then every statistical Cauchy sequence in the fuzzy normed linear space (X, χ_B) is a statistical Cauchy sequence with respect to the ordinary norm.*

Proof. Let $< x_k > \subset X$ be a statistical Cauchy sequence on (X, χ_B) and $\delta > 0$. Since $< x_k >$ is a statistical Cauchy sequence, for this δ and for every $0 < \epsilon < 1$, there exists a positive integer set K with natural density one such that $k, l \in K$ implies

$$\begin{array}{ll} & \frac{\delta}{2}\chi_B(x_k - x_l) > 1 - \epsilon \\ \Longrightarrow & \chi_B(\frac{2}{\delta}(x_k - x_l)) > 1 - \epsilon \quad a.a.\ \ k,\ l \\ \Longrightarrow & \chi_B(\frac{2}{\delta}(x_k - x_l)) = 1 \quad a.a.\ \ k,\ l \\ \Longrightarrow & \| x_k - x_l \| \le \frac{\delta}{2} < \delta \quad a.a.\ \ k,\ l. \end{array}$$

Therefore $< x_k >$ is a statistical Cauchy sequence in $(X, \| \cdot \|)$.
This prove the lemma.

□

Theorem 4.8. *Let $(X, \| \cdot \|)$ be a Banach space. Then the fuzzy normed linear space (X, χ_B) is fuzzy statistically complete where B is the closed unit ball of X.*

Proof. Let $< x_k >$ be a statistically Cauchy sequence in (X, χ_B). Then it is a statistical Cauchy sequence with respect to the ordinary norm $\| \cdot \|$ by the above lemma. Since $(X, \| \cdot \|)$ is a Banach space, there exists an $x \in X$ such that x_k statistically converges to x by Theorem 3.5 and 3.6. Now, we show that $< x_k >$ statistically converges to this x in (X, χ_B). Let $t > 0$ and $0 < \epsilon < 1$. Then there exists a positive integer set K with natural density one such that

$$\begin{array}{ll} & k \in K \ \ \text{implies} \quad \| x_k - x \| < t \\ \Longrightarrow & \| \frac{1}{t}(x_k - x) \| < 1 \quad a.a.\ \ k \\ \Longrightarrow & \chi_B(\frac{1}{t}(x_k - x)) = 1 \quad a.a.\ \ k \\ \Longrightarrow & t\,\chi_B(x_k - x) > 1 - \epsilon \quad a.a.\ \ k \end{array}$$

That is $< x_k >$ statistically converges to x, therefore (X, χ_B) is fuzzy statistically complete. This completes the proof. □

Corollary 4.9. *The field K(R or C) with the fuzzy topology generated by the usual topology on K is a fuzzy statistically complete fuzzy normed linear space.*

Definition 4.10. [5] Two fuzzy seminorms ρ_1, ρ_2 on X are said to be *equivalent* iff $\tau_{\rho_1} = \tau_{\rho_2}$.

THEOREM 4.11. [12] *Let* $(X, \|\cdot\|)$ *be a normed linear space. If* ρ *is a lower semi-continuous fuzzy norm on* X*, and has the bounded support:* $\{x \in X | \ \rho(x) > 0\}$ *is bounded, then* ρ *is equivalent to the fuzzy norm* χ_B *where* B *is the closed unit ball of* X*.*

By Theorem 4.8. and 4.11 we get the following theorem.

THEOREM 4.12. *If* X *is a Banach space and* ρ *is a lower semicontinuous fuzzy norm on* X *having the bounded support, then the fuzzy normed linear space* (X, ρ) *is fuzzy statistically complete.*

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