JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 3, September 2009

PROBABILITIES OF ANALOGUE OF WIENER PATHS CROSSING CONTINUOUSLY DIFFERENTIABLE CURVES

Kun Sik Ryu*

Abstract. Let φ be a complete probability measure on $\mathbb{R}$, let m_φ be the analogue of Wiener measure over paths on $[0,T]$ and let $f(t)$ be continuously differentiable on $[0,T]$. In this note, we give the analogue of Wiener measure m_φ of $\{x \text{ in } C[0,T] | x(0) < f(0)$ and $x(s_0) \geq f(s_0)$ for some s_0 in $[0,T]\}$ by use of integral equation techniques. This result is a generalization of Park and Paranjape's 1974 result[1].

1. Introduction

Let $T > 0$ be given and let m_w be the standard Wiener measure on $C_0[0,T]$, the space of all continuous functions x with $x(0) = 0$. From [4] and [5], we can found the following equations ; for $b \geq 0$,

$$\begin{aligned} (1.1) \qquad & m_w(\{x \text{ in } C_0[0,T] | \sup_{0\leq t\leq T} x(t) \geq b\}) \\ &= 2m_w(\{x \text{ in } C_0[0,T] | x(T) \geq b\}) \\ &= 2\int_{b/\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \end{aligned}$$

and

$$\begin{aligned} (1.2) \qquad & m_w(\{x \text{ in } C_0[0,T] | \sup_{0\leq t\leq T} (x(t) - at) \geq b\}) \\ &= \int_{(aT+b)/\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + e^{-2ab} \int_{-\infty}^{(aT-b)/\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du. \end{aligned}$$

Received July 03, 2009; Accepted August 14, 2009.
2000 Mathematics Subject Classification: Primary 28C20.
Key words and phrases: analogue of Wiener measure, Volterra integral equation, conditional expectation.
This paper was supported by the Research Fund in Hannam University in 2009.

In 1974, Park and Paranjape proved the following theorem [1].

THEOREM 1.1. *Let $f(t)$ be continuous on $[0,T]$, differentiable in $(0,T)$, and satisfy $|f'(t)| \le \frac{C}{t^p}$ $(0 < p < \frac{1}{2})$ for some constant C. Then for $b \ge -f(0)$,*

$$
\begin{aligned}
(1.3) \qquad & m_w(\{x \text{ in } C_0[0,T] |\ \sup_{0\le t\le T}(x(t)-f(t)) \ge b\}) \\
&= 2\int_{(f(T)+b)/\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}du \\
&\quad -4\int_0^T M(T,t)\big(\int_{(f(T)+b)/\sqrt{T}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du\big)dt \\
&\quad +\sum_{n=1}^{\infty}4^n\int_0^T K_n(T,t)\big[2\int_{(f(t)+b)/\sqrt{t}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du \\
&\quad -4\int_0^t M(t,s)\int_{(f(s)+b)/\sqrt{s}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}duds\big]dt
\end{aligned}
$$

where

$$
M(t,s) = \begin{cases} \frac{\partial}{\partial s}\int_{-\infty}^{(f(t)-f(s))/\sqrt{t-s}} \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du & (0 \le s < t \le T) \\ 0 & (0 \le t < s \le T) \end{cases},
$$

$$
K_1(T,t) = \int_t^T M(T,s)M(s,t)ds
$$

and

$$
K_{n+1}(T,t) = \int_t^T K_n(T,s)K_1(s,t)ds.
$$

In 2002, the author and Dr. Im presented the definition and the theories of analogue of Wiener measure m_φ on $C[0,T]$, the space of all continuous functions on $[0,T]$. This measure is a kind of generalization of standard Wiener measure. Indeed, if φ is the Dirac measure δ_0 at the origin in $\mathbb{R}$ then m_φ is the standard Wiener measure m_w.

The main purpose of this note is to find the analogue of Wiener measure m_φ of $\{x \text{ in } C[0,T] |\ \sup_{0\le t\le T}(x(t)-f(t)) \ge 0\}$ for continuously differentiable function f on $[0,T]$, which is a generalization of Theorem 1.1.

Throughout in this note, $\int_a^b f(u)du$ means the Henstock integral of f.

2. Statement of the result and proof

Let φ be a complete probability measure on $\mathbb{R}$ and let m_φ be the analogue of Wiener measure on $C[0,T]$ for giving a measure φ.

From [2], we can find the following theorem.

THEOREM 2.1. *(The Wiener integration formula for analogue of Wiener measure) If $g : \mathbb{R}^{n+1} \to \mathbb{C}$ is a Borel measurable function then the following equality holds.*

$$\begin{aligned}&\int_{C[a,b]} g(x(t_0), x(t_1), \cdots, x(t_n))\, d\omega_\varphi(x)\\ &\overset{*}{=} \int_{\mathbb{R}^{n+1}} g(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, u_1, \cdots, u_n)\\ &\quad d(\prod_{j=1}^{n} m_L \times \varphi)((u_1, u_2, \cdots, u_n), u_0)\end{aligned}$$

where $\overset{*}{=}$ means that if one side exists then both sides exist and the two values are equal.

Let $f : [0,T] \to \mathbb{R}$ be continuously differentiable and we let $f(s) = 0$ if $s \le 0$. For t in $[0,T]$, the limit $\lim_{s\to t^-} \frac{f(t)-f(s)}{\sqrt{t-s}}$ exists and equals to 0.

For x in $C[0,T]$, let $\tau(x)$ be the first hitting time of the curve f from below by x, that is, $x(\tau(x)) = f(\tau(x))$. If x never reaches the curve f, we let $\tau(x) = +\infty$.

For t in $[0,T]$, let

$$\begin{aligned}A_t &\\ &= \{x \text{ in } C[0,T] | x(0) < f(0) \text{ and for some } s_0 \text{ in } [0,t],\\ &\quad x(s_0) \ge f(s_0)\}.\end{aligned} \tag{2.1}$$

Let $G : \mathbb{R} \to \mathbb{R}$ be a function with

$$G(t) = \begin{cases} 0 & (t < 0) \\ m_\varphi(A_t) & (0 \le t \le T) \\ m_\varphi(A_T) & (T < t) \end{cases}.$$

LEMMA 2.2. *G is increasing continuous with $G(0) = 0$.*

Proof. It is clear that G is increasing and $G(0) = 0$. Let t be in $[0,T)$ and let $\langle t_n \rangle$ be a decreasing sequence in $[0,T]$ with $\lim_{n\to\infty} t_n = t$. Then $A_t = \cap_{n=1}^\infty A_{t_n}$, so $G(t) = m_\varphi(\cap_{n=1}^\infty A_{t_n}) = \lim_{n\to\infty} m_\varphi(A_{t_n})$

$= \lim_{n\to\infty} G(t_n)$ which implies that G is right-continuous.
Let t be in $(0,T]$ and let $\langle t_n\rangle$ be an increasing sequence in $[0,T]$ with $\lim_{n\to\infty} t_n = t$. Putting $N = \{x \text{ in } C[0,T] | x(0) < f(0) \text{ and } \tau(x) = t\}$, $m_\varphi(N) = 0$ and $A_t = N \cup (\cup_{n=1}^\infty A_{t_n})$, so $G(t) = m_\varphi(N \cup (\cup_{n=1}^\infty A_{t_n}))$ $= m_\varphi(\cup_{n=1}^\infty A_{t_n}) = \lim_{n\to\infty} m_\varphi(A_{t_n}) = \lim_{n\to\infty} G(t_n)$ which implies that G is left-continuous. Hence, G is continuous. $\square$

LEMMA 2.3. *If $0 \le s < t \le T$ then $\tau(x) = s$ and $x(t) - x(s)$ are independent.*

Proof. Let A and B be two Borel subsets of $\mathbb{R}$. By the integration formula for analogue of Wiener measure [2],

$$\begin{aligned}
& m_\varphi(\{x \text{ in } C[0,T] | x(s) \text{ is in } A \text{ and } x(t)-x(s) \text{ is in } B\}) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{\frac{1}{\sqrt{s}}(A-u_0)}(v_1)\chi_{\frac{1}{\sqrt{t-s}}B}(v_2)\frac{1}{2\pi}\exp\{-\frac{1}{2}(v_1^2+v_2^2)\} \\
&\quad dv_1 dv_2 d\varphi(u_0) \\
&= m_\varphi(\{x \text{ in } C[0,T] | x(s) \text{ is in } A\}) \\
&\quad \cdot m_\varphi(\{x \text{ in } C[0,T] | x(t)-x(s) \text{ is in } B\}),
\end{aligned}$$

as desired. $\square$

The following theorem is one of main theorems in this note.

THEOREM 2.4. *For $0 < t \le T$, $G(t)$ satisfies the following Volterra's integral equation of the second kind*

$$\begin{aligned}
&G(t) \\
&= 2\int_{-\infty}^{f(0)} \Big(\int_{f(t)}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(u_1-u_0)^2)}{2t}\} du_1\Big) d\varphi(u_0) \\
&\quad -2\int_0^t G(s)M(t,s)ds
\end{aligned} \tag{2.2}$$

where

$$M(t,s) = \begin{cases} \frac{\partial}{\partial s}\int_{-\infty}^{(f(t)-f(s))/\sqrt{t-s}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du & (0 \le s < t \le T) \\ 0 & (0 \le t \le s \le T) \end{cases}.$$

Proof. For $0 < t \le T$,

$$\begin{aligned} & G(t) \\ &= m_{\varphi}(A_t \cap \{x \text{ in } C[0,T] | x(t) \ge f(t)\}) \\ &\quad + m_{\varphi}(A_t \cap \{x \text{ in } C[0,T] | x(t) < f(t)\}) \\ &= m_{\varphi}(\{x \text{ in } C[0,T] | x(0) < f(0) \text{ and } x(t) \ge f(t)\}) \\ &\quad + m_{\varphi}(\{x \text{ in } C[0,T] | x(0) < f(0), x(t) < f(t) \text{ and} \\ &\quad \text{for some } s_0 \text{ in } [0,t], x(s_0) = f(s_0)\}), \end{aligned}$$

$$\begin{aligned} & m_{\varphi}(\{x \text{ in } C[0,T] | x(0) < f(0) \text{ and } x(t) \ge f(t)\}) \\ &= \int_{-\infty}^{f(0)} \Big(\int_{f(t)}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(u_1 - u_0)^2)}{2t}\} du_1\Big) d\varphi(u_0) \end{aligned}$$

and

$$\begin{aligned} & m_{\varphi}(\{x \text{ in } C[0,T] | x(0) < f(0), x(t) < f(t) \text{ and} \\ & \text{ for some } s_0 \text{ in } [0,t], x(s_0) = f(s_0)\}) \\ &\overset{(1)}{=} \int_0^t E^{\varphi}(x(t) < f(t) | \tau(x) = s) dG(s) \\ &\overset{(2)}{=} \int_0^t E^{\varphi}(x(t) - x(s) < f(t) - x(s) | \tau(x) = s) dG(s) \\ &\overset{(3)}{=} \int_0^t E^{\varphi}(x(t) - x(s) < f(t) - x(s)) dG(s) \\ &\overset{(4)}{=} \int_0^t \Big(\int_{C[0,T]} \chi_{\{x \text{ in } C[0,T] | x(t) - x(s) < f(t) - x(s)\}}(x) dm_{\varphi}(x)\Big) dG(s) \\ &\overset{(5)}{=} \int_0^t \Big(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(f(t) - f(s))/\sqrt{t-s}} e^{-\frac{v^2}{2}} dv\Big) dG(s). \end{aligned}$$

Step (1) follows from the basic properties of conditional expectation. From $x(s) = f(s)$, we have Step (2). By Lemma 2.3, we obtain Step (3). Using the Wiener integration formula for analogue of Wiener measure, we can check Step (4). Step (5) come from the change of variables theorem.

Putting

$$U(t,s) = \begin{cases} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(f(t)-f(s))/\sqrt{t-s}} e^{-\frac{v^2}{2}} dv & \text{for } 0 \le s < t \le T \\ \frac{1}{2} & \text{for } t \le s \le T \end{cases},$$

$$\int_0^t \Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(f(t)-f(s))/\sqrt{t-s}} e^{-\frac{v^2}{2}}dv\Big)dG(s)$$
$$= \int_0^t U(t,s)dG(s)$$
$$\overset{(1)}{=} \lim_{s\to t^+} U(t,s)\lim_{s\to t^+} G(s) - \lim_{s\to 0^-} U(t,s)\lim_{s\to 0^-} G(s) - \int_0^t G(s)dU(t,s)$$
$$\overset{(2)}{=} \frac{1}{2}G(t) - \int_0^t G(s)\frac{d}{ds}U(t,s)ds.$$

By the integration by part, we have Step (1). Since $\lim_{s\to 0^-} G(s) = G(0) = 0$, we obtain Step (2). Hence, we have the equality (2.2). □

The equality (2.2) and the change of order of integration gives

(2.3) $G(t)$
$$= 2\int_{-\infty}^{f(0)} \Big(\int_{f(t)}^{+\infty} \frac{1}{\sqrt{2\pi t}}\exp\{-\frac{(u_1-u_0)^2)}{2t}\}du_1\Big)d\varphi(u_0)$$
$$-4\int_0^t \Big[\int_{-\infty}^{f(0)} \Big(\int_{f(s)}^{+\infty} \frac{1}{\sqrt{2\pi s}}\exp\{-\frac{(u_1-u_0)^2)}{2s}\}du_1\Big)d\varphi(u_0)\Big]$$
$$M(t,s)ds + 4\int_0^t \Big(\int_z^t M(s,z)M(t,s)ds\Big)G(z)dz$$

if $M(s,z)M(t,s)G(z)$ is integrable on $\{(s,z)|0 \le z < s \le t\}$.

By [6], we obtain the main theorem in this note.

THEOREM 2.5. *If $\int_z^t M(s,z)M(t,s)ds$ is square integrable on $\{(z,t)| 0 \le z < t \le T\}$ then the equation (2.2) has one and essentially only one solution in the class L_2. This solution is given by the formula*

(2.4) $G(t)$
$$= 2\int_{-\infty}^{f(0)} \Big(\int_{f(t)}^{+\infty} \frac{1}{\sqrt{2\pi t}}\exp\{-\frac{(u_1-u_0)^2)}{2t}\}du_1\Big)d\varphi(u_0)$$
$$+\sum_{n=1}^{\infty}(-1)^n 2^{n+1}\int_0^t \Big[\int_{-\infty}^{f(0)} \Big(\int_{f(s)}^{+\infty} \frac{1}{\sqrt{2\pi s}}\exp\{-\frac{(u_1-u_0)^2)}{2s}\}$$
$$du_1\Big)d\varphi(u_0)\Big]H_n(t,s)ds$$

where $H_1(t,s) = M(t,s)$ and $H_{n+1}(t,s) = \int_s^t H_n(t,z)H_1(z,s)dz$.

Proof. We know that

$$M(t,s) = \frac{1}{\sqrt{2\pi(t-s)}}\big(f'(s) + \frac{f(t)-f(s)}{2(t-s)}\exp\{-\frac{(f(t)-f(s))^2}{2(t-s)}\}\big)$$

for $0 \le s < t \le T$.
Since f is continuous differentiable, $f'(s) + \frac{f(t)-f(s)}{2(t-s)}\exp\{-\frac{(f(t)-f(s))^2}{2(t-s)}\}$ is bounded on $\{(s,t)|0 \le s < t \le T\}$. So, for some K, $|M(t,s)| \le \frac{1}{\sqrt{2\pi(t-s)}}K$ and $|M(s,z)| \le \frac{1}{\sqrt{2\pi(s-z)}}K$. Hence

$$\begin{aligned}
&|\int_z^t M(s,z)M(t,s)ds| \\
&\le \frac{K^2}{2\pi}|\int_z^t \frac{1}{\sqrt{(s-z)(t-s)}}ds| \\
&= \frac{K^2}{2},
\end{aligned}$$

that is, $|\int_z^t M(s,z)M(t,s)ds|^2$ is a bounded function. Hence $\int_z^t M(s,z)$ $M(t,s)ds$ is square integrable on $\{(z,t)|0 \le z < t \le T\}$, as desired. $\Box$

REMARK 2.6. *If $\varphi = \delta_0$ then the equation* (1.3) *and the equation* (2.4) *are exactly same.*

REMARK 2.7. *Let $\varphi = \delta_0$ and let $f(t) = b$ be a constant function with $b \ge 0$. The $M(t,s) = 0$ for $0 \le s < t \le T$, so we have the equation* (1.1), *that is,*

$$G(t) = 2\int_b^{+\infty} \frac{1}{\sqrt{2\pi}}\exp\{-\frac{u^2}{2t}\}du.$$

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Department of Mathematics Education
Hannam University
Daejeon 306-791, Republic of Korea
E-mail: ksr@hannam.ac.kr