

ON ASYMPTOTIC PROPERTY IN VARIATION FOR NONLINEAR DIFFERENTIAL SYSTEMS

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ABSTRACT. We show that two notions of asymptotic equilibrium and asymptotic equilibrium in variation for nonlinear differential systems are equivalent via t_∞ -similarity of associated variational systems. Moreover, we study the asymptotic equivalence between nonlinear system and its variational system.

1. Introduction and basic notions

The aim of this paper is to study asymptotic properties - asymptotic equilibrium and asymptotic equivalence - of the nonlinear differential system and its variational system. To do this, we need the concepts of strong stability and t_∞ -similarity due to G. Ascoli [1] and R. Conti [8], respectively.

Consider the nonlinear differential system

$$(1.1) \quad x'(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$, and \mathbb{R}^n is the n -dimensional real Euclidean space. We assume that the Jacobian matrix $f_x = \frac{\partial f}{\partial x}$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. The symbol $|\cdot|$ denotes arbitrary vector norm on \mathbb{R}^n .

Let $x(t) = x(t, t_0, x_0)$ be the unique solution of (1.1) satisfying $x(t_0) = x_0$. Also, we consider the associated variational systems

$$(1.2) \quad v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0,$$

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and

$$(1.3) \quad z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0.$$

The fundamental matrix solutions $\Phi(t, t_0, 0)$ of (1.2) and $\Phi(t, t_0, x_0)$ of (1.3) are given by

$$\Phi(t, t_0, 0) = \frac{\partial}{\partial x_0} x(t, t_0, 0)$$

and

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

respectively [12].

System (1.1) is *strongly stable* [1, 9] if for each $\varepsilon > 0$ there exists a corresponding $\delta = \delta(\varepsilon) > 0$ such that any solution $x(t)$ of (1.1) which satisfies the inequality $|x(t_1, t_0, x_0)| < \delta$ for some $t_1 \geq t_0$ exists and satisfies the inequality $|x(t, t_0, x_0)| < \varepsilon$ for all $t \geq t_0$.

It is clear that strong stability implies uniform stability, which in turn implies ordinary stability. The linear differential system

$$(1.4) \quad x'(t) = A(t)x(t),$$

where $A(t)$ is an $n \times n$ continuous matrix on \mathbb{R}^+ , is strongly stable if and only if there exists a constant $M > 0$ such that

$$|X(t)| \leq M, \quad |X^{-1}(t)| \leq M \text{ for all } t \geq t_0,$$

where $X(t)$ is a fundamental matrix of (1.4) [9, Theorem 1, p. 54].

Let $A(t)$ and $B(t)$ be $n \times n$ continuous matrices on \mathbb{R}^+ . They are t_∞ -similar [8] if there exists an $n \times n$ continuous matrix $F(t)$ with

$$\int_0^\infty |F(t)| dt < \infty$$

and continuously differentiable matrix $S(t)$ such that

$$(1.5) \quad S'(t) + S(t)B(t) - A(t)S(t) = F(t),$$

where $S(t)$ is bounded and invertible with bounded $S^{-1}(t)$.

The notion of t_∞ -similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on \mathbb{R}^+ , and it preserves some stability concepts [8, 10]. Trench [14] introduced the concept of t_∞ -quasisimilarity which is not symmetric or transitive, but still preserves stability properties.

Hewer [10] and Choi et al. [4, 5] studied the stability in variation for nonlinear differential systems using the notion of t_∞ -similarity. For stability in variation of difference systems using the notion of n_∞ -similarity, which is the corresponding notion of t_∞ -similarity for the discrete case,

see [3]. Also, Choi and Koo [6] studied the asymptotic property for linear integro-differential systems.

This paper deals with the asymptotic equilibrium for nonlinear differential systems in section 2. We show that two notions of asymptotic equilibrium and asymptotic equilibrium in variation for nonlinear systems are equivalent via the notion of t_∞ -similarity of associated variational systems in section 3. Moreover, we study the asymptotic equivalence between nonlinear system and its variational system in section 3.

2. Asymptotic equilibria for nonlinear systems

System (1.1) has *asymptotic equilibrium* [11] if every solution of (1.1) tends to a finite limit vector $\xi \in \mathbb{R}^n$ as $t \rightarrow \infty$ and to every constant vector $\eta \in \mathbb{R}^n$ there is a solution $x(t)$ of (1.1) such that $\lim_{t \rightarrow \infty} x(t) = \eta$.

Also, System (1.1) has *asymptotic equilibrium in variation* if for every solution of (1.1) the corresponding variational system (1.3) has asymptotic equilibrium.

Firstly, we show that the associated variational differential system (1.2) inherits the property of asymptotic equilibrium from the original nonlinear differential system (1.1) in the following theorem. To show this we need the following lemma.

LEMMA 2.1. [4, Theorem 3.2] *Linear system (1.4) has asymptotic equilibrium if and only if $\lim_{t \rightarrow \infty} X(t)$ exists and is invertible, where $X(t)$ is a fundamental matrix of (1.4).*

THEOREM 2.2. *If (1.1) has asymptotic equilibrium, then (1.2) also has asymptotic equilibrium.*

Proof. We begin by showing that the fundamental matrix $\Phi(t, t_0, 0)$ of (1.2) given by $\frac{\partial}{\partial x_0} x(t, t_0, 0)$ is convergent as $t \rightarrow \infty$. Let $x_0(h) \equiv (0, \dots, h, \dots, 0)$ be a vector of small length $|h|$ in the j -th coordinate direction for each $j = 1, \dots, n$. Since (1.1) has asymptotic equilibrium, $\lim_{t \rightarrow \infty} x(t, t_0, x_0(h)) = x_\infty$ exists for fixed nonzero h . Let $\varepsilon > 0$ be given. For any given $0 < |h| < \varepsilon$, there exists a positive large number $N = N(t_0, x_0(h))$ such that $|x(t, t_0, x_0(h)) - x(s, t_0, x_0(h))| < |h|^2$ for any $t, s \geq N$ and $j = 1, \dots, n$, since $x(t, t_0, x_0(h))$ has the Cauchy property for given $(t_0, x_0(h))$ and for each $j = 1, \dots, n$. Then we obtain for each $j = 1, \dots, n$,

$$\left| \frac{\partial}{\partial x_{0j}} x(t, t_0, 0) - \frac{\partial}{\partial x_{0j}} x(s, t_0, 0) \right|$$

$$\begin{aligned}
&= \left| \lim_{h \rightarrow 0} \frac{x(t, t_0, x_0(h)) - x(t, t_0, 0)}{h} - \lim_{h \rightarrow 0} \frac{x(s, t_0, x_0(h)) - x(s, t_0, 0)}{h} \right| \\
&= \left| \lim_{h \rightarrow 0} \frac{x(t, t_0, x_0(h)) - x(s, t_0, x_0(h))}{h} \right| < \lim_{h \rightarrow 0} \frac{|h|^2}{|h|} < \varepsilon, \text{ for } t, s \geq N.
\end{aligned}$$

This implies that $\lim_{t \rightarrow \infty} \Phi(t, t_0, 0) = \Phi_\infty$ exists.

Note that the fact that f_x exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ assures the existence and continuity of $x(t, t_0, x_0)$ and $\Phi(t, t_0, x_0) = \frac{\partial x(t, t_0, x_0)}{\partial x_0}$, respectively [7, Theorem 7.2, p. 25]. Furthermore, the change of limits of $\Phi(t, t_0, x_0)$ holds:

$$\begin{aligned}
&\lim_{t \rightarrow \infty} \lim_{h \rightarrow 0} \frac{x_j(t, t_0, x_{0j}(h)) - x_j(t, t_0, 0)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\lim_{t \rightarrow \infty} x_j(t, t_0, x_{0j}(h))}{h}, \quad j = 1, \dots, n.
\end{aligned}$$

Now, by using Lemma 2.1, it suffices to prove that the limit Φ_∞ is invertible. Given linearly independent vectors $\hat{x}_{0j} \in \mathbb{R}^n$ in the j -coordinate direction for each $j = 1, \dots, n$, it follows from asymptotic equilibrium of (1.1) that there exist the solutions $x_j(t, t_0, x_{0j}(h))$ of (1.1) which are convergent to $h\hat{x}_{0j}$ for each $j = 1, \dots, n$ and fixed $h \neq 0$. Then we have

$$\begin{aligned}
&\lim_{t \rightarrow \infty} \Phi(t, t_0, 0) \\
&= \lim_{t \rightarrow \infty} \left[\frac{\partial}{\partial x_{01}} x_1(t, t_0, x_{01}(h)), \dots, \frac{\partial}{\partial x_{0n}} x_n(t, t_0, x_{0n}(h)) \right] \\
&= \lim_{t \rightarrow \infty} \left[\lim_{h \rightarrow 0} \frac{x_1(t, t_0, x_{01}(h))}{h}, \dots, \lim_{h \rightarrow 0} \frac{x_n(t, t_0, x_{0n}(h))}{h} \right] \\
&= \left[\lim_{h \rightarrow 0} \frac{\lim_{t \rightarrow \infty} x_1(t, t_0, x_{01}(h))}{h}, \dots, \lim_{h \rightarrow 0} \frac{\lim_{t \rightarrow \infty} x_n(t, t_0, x_{0n}(h))}{h} \right] \\
&= [\hat{x}_{01}, \dots, \hat{x}_{0n}] = \Phi_\infty.
\end{aligned}$$

Since the vectors $\hat{x}_{01}, \dots, \hat{x}_{0n}$ are linearly independent, Φ_∞ is invertible. This completes the proof. \square

Note that the converse of Theorem 2.2 does not hold in general.

EXAMPLE 2.3. If we consider the nonlinear scalar differential equation

$$(2.1) \quad x'(t) = f(t, x(t)) = \frac{e^t x^2(t)}{1 + x^2(t)}, \quad x(t_0) = x_0,$$

then its variational equation has a constant fundamental solution and equation (2.1) has an unbounded solution.

Now, for the converse of Theorem 2.2, we examine asymptotic equilibrium for the perturbed system of linear differential system (1.2) by using the comparison principle. To do this we need the following comparison principle in [11].

LEMMA 2.4. Suppose that $w(t, r) \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ is monotone nondecreasing in r for each fixed $t \geq t_0 \geq 0$ with the property that

$$u(t) - \int_{t_0}^t w(s, u(s))ds < v(t) - \int_{t_0}^t w(s, v(s))ds, \quad t \geq t_0 \geq 0$$

for $u, v \in C(\mathbb{R}^+, \mathbb{R}^+)$. If $u(t_0) < v(t_0)$, then $u(t) < v(t)$ for all $t \geq t_0 \geq 0$.

Setting $f_x(t, 0) = A(t)$ and using the mean value theorem, the nonlinear differential system (1.1) can be written as

$$(2.2) \quad x'(t) = A(t)x(t) + G(t, x(t)), \quad x(t_0) = x_0,$$

where $G(t, x) = f(t, x(t)) - f_x(t, 0)x(t) = \int_0^1 [f_x(t, \theta x) - f_x(t, 0)]d\theta x$.

LEMMA 2.5. Assume that

- (i) (1.2) is strongly stable,
- (ii) for each $t \geq t_0$ and $x \in \mathbb{R}^n$, $G(t, x)$ in (2.2) satisfies

$$|G(t, x)| \leq g(t, |x|),$$

where $g(t, u) \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ is nondecreasing in u for $t \geq t_0 \geq 0$.

Also, we consider the scalar differential equation

$$(2.3) \quad u'(t) = Mg(t, u(t)), \quad u(t_0) = u_0 > 0,$$

and suppose that

- (iii) all solutions of (2.3) are bounded on \mathbb{R}^+ .

Then all solutions of the scalar equation

$$(2.4) \quad \begin{aligned} r'(t) &= \omega(t, r(t)) \\ &= |f_x(t, 0)|r(t) + Mg(t, r(t)), \quad r(t_0) = r_0 > 0, \end{aligned}$$

are bounded provided $Mr_0 < u_0$ with $r_0 = |x_0|$.

Proof. By using the variation of constants formula, the solution $x(t)$ of (1.1) satisfies the following inequality

$$\begin{aligned} r(t, t_0, |x_0|) &= |x(t, t_0, x_0)| \\ &= |\Psi(t, t_0)x_0 + \int_{t_0}^t \Psi(t, s)G(s, x(s))ds| \\ &\leq Mr_0 + M \int_{t_0}^t g(s, r(s))ds, \end{aligned}$$

where $\Psi(t, t_0) = \Phi(t, t_0, 0)$ is a fundamental matrix of (1.2). This implies that

$$\begin{aligned} r(t, t_0, r_0) &= M \int_{t_0}^t g(s, r(s)) ds \\ &\leq Mr_0 < u_0 \\ &= u(t) - M \int_{t_0}^t g(s, u(s)) ds. \end{aligned}$$

Then, by using Lemma 2.4, we obtain

$$r(t) < u(t) \quad \text{for each } t \geq t_0,$$

provided $Mr_0 < u_0$. □

We obtain the following result by using Theorems 9.1 and 9.6 in [2].

THEOREM 2.6. *Assume that*

- (i) (1.2) has asymptotic equilibrium,
- (ii) for each $t \geq t_0$ and $x \in \mathbb{R}^n$, $G(t, x)$ in (2.2) satisfies

$$|G(t, x)| \leq g(t, |x|),$$

where $g(t, u) \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ is nondecreasing in u for $t \geq t_0 \geq 0$.

Also, we consider the scalar differential equation

$$(2.3) \quad u'(t) = Mg(t, u(t)), u(t_0) = u_0 > 0,$$

and suppose that

- (iii) all solutions of (2.3) are bounded on \mathbb{R}^+ .

Then (1.1) has asymptotic equilibrium provided $M|x_0| < u_0$.

Proof. Let $x(t, t_0, x_0)$ be any solution of (1.1). From the variation of constants formula in [12] and conditions (i) and (ii), we obtain

$$\begin{aligned} |x(t)| &= |\Psi(t, t_0)x_0 + \int_{t_0}^t \Psi(t, s)G(s, x(s))ds| \\ &\leq |\Psi(t, t_0)||x_0| + \int_{t_0}^t |\Psi(t, s)||G(s, x(s))|ds \\ &\leq M|x_0| + M \int_{t_0}^t g(s, |x(s)|)ds, \end{aligned}$$

where M is a constant from the boundedness of $\Psi(t, s)$ for each $t, s \geq t_0$. Then we have the following integral inequality:

$$\begin{aligned} |x(t)| &= M \int_{t_0}^t g(s, |x(s)|) ds \\ &\leq |x_0| < u_0 \\ &= u(t) - M \int_{t_0}^t g(s, u(s)) ds. \end{aligned}$$

By letting $w(t, r) = Mg(t, r)$ and using Lemma 2.4, we obtain

$$|x(t)| \leq u(t) \quad \text{for each } t \geq t_0,$$

provided $M|x_0| < u_0$.

Now, we prove that the solution $x(t)$ of (2.2) converges to a vector as $t \rightarrow \infty$. Consider the integral function

$$(2.5) \quad p(t, t_0, x_0) = \int_{t_0}^t \Psi^{-1}(s, t_0) G(s, x(s, t_0, x_0)) ds.$$

From the monotonicity of the function g and asymptotic equilibrium of (1.2), we obtain

$$\begin{aligned} |p(t) - p(s)| &\leq \int_s^t |\Psi^{-1}(\tau, t_0)| |G(\tau, x(\tau))| d\tau \\ &\leq M \int_s^t g(\tau, |x(\tau)|) d\tau \\ &\leq M \int_s^t g(\tau, u(\tau)) d\tau = u(t) - u(s), \end{aligned}$$

for any $t \geq s \geq t_0$. Since $u(t)$ has the Cauchy property, $p(t)$ converges to a vector p_∞ as $t \rightarrow \infty$. Hence there exists a vector $\xi \in \mathbb{R}^n$ such that any solution $x(t, t_0, x_0)$ of (2.2) satisfies the asymptotic relationship

$$x(t) = \xi + o(1) \quad \text{as } t \rightarrow \infty,$$

where $\xi = \Psi_\infty[x_0 + p_\infty]$.

Conversely, let $\eta \in \mathbb{R}^n$ be any vector. We note that for each $(t_0, \lambda) \in \mathbb{R}^+ \times \mathbb{R}^+$, we have

$$(2.6) \quad \int_{t_0}^\infty \omega(s, \lambda) ds < \infty,$$

by the boundedness of the solution $r(t)$ of (2.4) in Lemma 2.5. Let $r(t, t_0, |\eta|)$ be the maximal solution of (2.4) such that $R = \lim_{t \rightarrow \infty} r(t, t_0, |\eta|)$

exists. By (2.6) with $\lambda = 2R$ we can choose $t_0 \geq 0$ so large that

$$(2.7) \quad \int_{t_0}^{\infty} \omega(s, 2R) ds < R.$$

We define the operator S on Ω by

$$Sx(t) = \eta - \int_t^{\infty} f(s, x(s)) ds,$$

where Ω is the set of all bounded continuous functions $x : [t_0, \infty) \rightarrow \mathbb{R}^n$ with the norm $\|x\| = \sup_{t \geq t_0} |x(t)| \leq 2R$. We note that Ω is closed, bounded, and convex. In view of the continuity of f and (2.6), we easily see that S is continuous. Moreover $S(\Omega) \subset \Omega$, since

$$\begin{aligned} |Sx(t)| &\leq |\eta| + \int_t^{\infty} |f(s, x(s))| ds \\ &\leq R + \int_t^{\infty} [|f_x(s, 0)| |x(s)| + Mg(s, |x(s)|)] ds \\ &\leq R + \int_{t_0}^{\infty} \omega(s, 2R) ds < 2R. \end{aligned}$$

This inequality also shows that the set $\mathfrak{F} = \{Sx(t) : x \in \Omega\}$ is uniformly bounded and equicontinuous on any interval $[t_0, T]$, and that $\lim_{t \rightarrow \infty} Sx(t) = \eta$ uniformly in $x \in \Omega$, i.e., \mathfrak{F} is compact. Consequently Schauder's fixed point theorem [9] implies that S has a fixed point $x \in \Omega$, i.e., there is a solution $x(t)$ of (1.1) such that

$$(2.8) \quad x(t) = \eta - \int_t^{\infty} f(s, x(s)) ds = \eta + o(1) \text{ as } t \rightarrow \infty,$$

since $\int_t^{\infty} f(s, x(s)) ds \rightarrow 0$ as $t \rightarrow \infty$. This completes the proof. \square

As a consequence of Theorem 2.6 we obtain the following.

COROLLARY 2.7. *Instead of the condition (i) of Theorem 2.6 we assume that $\int_{t_0}^{\infty} |f_x(s, 0)| ds$ is finite. Then (1.1) has asymptotic equilibrium.*

3. Asymptotic equivalence in variation

In view of Theorem 2.2 and the following theorem, we obtain two corollaries about asymptotic equilibria in variation for the nonlinear system (1.1).

THEOREM 3.1. [4] *Asymptotic equilibrium for linear systems is preserved by t_∞ -similarity.*

COROLLARY 3.2. *Assume that (1.1) has asymptotic equilibrium and $f_x(t, 0)$ and $f_x(t, x(t, t_0, x_0))$ are t_∞ -similar with $\lim_{t \rightarrow \infty} S(t) = S_\infty$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some $\delta > 0$. Then (1.1) has asymptotic equilibrium in variation.*

COROLLARY 3.3. *Assume that (1.1) has asymptotic equilibrium and for some $\delta > 0$ we have*

$$\int_{t_0}^{\infty} |f_x(t, 0) - f_x(t, x(t, t_0, x_0))| dt < \infty,$$

for $|x_0| \leq \delta$. Then (1.1) has asymptotic equilibrium in variation.

Proof. It follows from Theorem 2.2 that (1.2) has asymptotic equilibrium. Letting $F(t) = |f_x(t, 0) - f_x(t, x(t, t_0, x_0))|$ with $S(t) = I$ for each $t \geq t_0 \geq 0$, we obtain that $F(t)$ is absolutely integrable. Thus $f_x(t, x(t, t_0, x_0))$ and $f_x(t, 0)$ are t_∞ -similar with $\lim_{t \rightarrow \infty} S(t) = S_\infty = I$. Here I is the identity matrix. This implies that (1.3) has also asymptotic equilibrium by Theorem 3.1. \square

Two differential systems $x'(t) = f(t, x(t))$ and $y'(t) = k(t, y(t))$ with $k \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ are said to be *asymptotically equivalent* if, for every solution $x(t)$, there exists a solution $y(t)$ such that

$$x(t) = y(t) + o(1) \text{ as } t \rightarrow \infty$$

and conversely, for every solution $y(t)$, there exists a solution $x(t)$ satisfying the asymptotic relationship.

Now, we obtain the result of asymptotic equivalence between the nonlinear system (1.1) and its variational system (1.2) under the hypotheses of Theorem 2.6.

THEOREM 3.4. *Let the assumptions be the same as in Theorem 2.6. Then (1.1) and (1.2) are asymptotically equivalent.*

Proof. Let $x(t)$ be any solution of (1.1). Then we have $\lim_{t \rightarrow \infty} x(t) = x_\infty$ since (1.1) has asymptotic equilibrium. Setting $y_0 = \Psi_\infty^{-1}x_\infty - p_\infty$ as in Theorem 2.6, there exists a solution $y(t, t_0, y_0)$ of (1.2) such that

$$\begin{aligned} \lim_{t \rightarrow \infty} [y(t) - x(t)] &= \Psi_\infty[y_0 + p_\infty] - x_\infty \\ &= \Psi_\infty[\Psi_\infty^{-1}x_\infty - p_\infty + p_\infty] - x_\infty = 0. \end{aligned}$$

For the converse asymptotic relationship, we easily see that the asymptotic relationship also holds by setting $x_0 = y_0 + p_\infty$. This completes the proof. \square

EXAMPLE 3.5. To illustrate Theorem 3.4, we consider the Ricatti scalar differential equation in [13]

$$(3.1) \quad \begin{aligned} x' &= f(t, x) = \lambda(t)(-x + x^2), \\ x(t_0) &= x_0, \quad t \geq t_0 \geq 0, \quad |x_0| \leq \frac{1}{2}, \end{aligned}$$

whose general solution is $x(t, t_0, x_0) = x_0[x_0 + (1 - x_0) \exp \int_{t_0}^t \lambda(s) ds]^{-1}$ with $\lambda \in C(\mathbb{R}^+, \mathbb{R}^+)$ and its associated variational differential equation

$$(3.2) \quad v' = f_x(t, 0)v(t) = -\lambda(t)v(t),$$

where $f(t, x) = \lambda(t)[-x + x^2]$ and $f_x(t, x) = \lambda(t)[-1 + 2x]$ with $\int_{t_0}^\infty \lambda(s) ds < \infty$. Then (3.1) has asymptotic equilibrium. Furthermore, (3.1) and (3.2) are asymptotically equivalent.

Proof. The fundamental solution $\Phi(t, t_0, 0)$ of (3.2) is given by $\Phi(t, t_0, 0) = \exp(-\int_{t_0}^t \lambda(s) ds)$. Then it is not hard to show that (3.2) has asymptotic equilibrium. Setting $f_x(t, 0) = A(t)$ and using the mean value theorem, (3.1) can be written as

$$(3.3) \quad x'(t) = A(t)x(t) + G(t, x(t)) = -\lambda(t)x(t) + \lambda(t)x^2(t),$$

where $G(t, x) = \int_0^1 [f_x(t, \theta x) - f_x(t, 0)] d\theta x$. Then we obtain

$$\begin{aligned} |G(t, x)| &\leq \left| \int_0^1 [f_x(t, \theta x) - f_x(t, 0)] d\theta x \right| \\ &= |\lambda(t)x(t)| \leq \lambda(t)|x(t)| = g(t, |x|), \end{aligned}$$

where $g(t, u) = \lambda(t)u$ is nondecreasing in $u > 0$. The solution of scalar differential equation

$$(3.4) \quad u'(t) = g(t, u(t)) = \lambda(t)u, \quad 0 < u(0) = u_0,$$

is given by $u(t, t_0, u_0) = u_0 \exp(\int_{t_0}^t \lambda(s) ds)$. Thus all solutions $u(n)$ of (3.4) are bounded on \mathbb{R}^+ . Note that $\lim_{t \rightarrow \infty} \exp(-\int_{t_0}^t \lambda(s) ds)|x_0|$ exists. Since all conditions of Theorem 3.4 are satisfied, (3.1) has asymptotic equilibrium. It follows that (3.1) and (3.2) are asymptotically equivalent by Theorem 3.4.

Furthermore, there exists $F(t)$ absolutely integrable over \mathbb{R}^+ , such that

$$S'(t) + S(t)f_x(t, 0) - f_x(t, x(t, t_0, x_0))S(t) = F(t)$$

for some $S(t) = \exp(-\int_{t_0}^t \lambda(s)ds)$, since

$$\begin{aligned} \int_{t_0}^{\infty} |F(s)|ds &\leq \int_{t_0}^{\infty} 5\lambda(s) \exp(-\int_{t_0}^s \lambda(\tau)d\tau)ds \\ &= \left[-5 \exp(-\int_{t_0}^t \lambda(s)ds) \right]_{t=t_0}^{t=\infty} < \infty, \quad |x_0| \leq \frac{1}{2}. \end{aligned}$$

Thus $f_x(t, 0)$ and $f_x(t, x(t, t_0, x_0))$ are t_∞ -similar with $\lim_{t \rightarrow \infty} S(t) = S_\infty < \infty$, and so (3.1) has asymptotic equilibrium in variation. \square

We conclude that two concepts of asymptotic equilibrium and asymptotic equilibrium in variation for nonlinear system (1.1) are equivalent via t_∞ -similarity of the associated variational systems.

THEOREM 3.6. *In addition to the assumptions of Theorem 2.6, we assume that $f_x(t, 0)$ and $f_x(t, x(t, t_0, x_0))$ are t_∞ -similar with $\lim_{t \rightarrow \infty} S(t) = S_\infty$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some $\delta > 0$. Then (1.1) has asymptotic equilibrium if and only if (1.1) has asymptotic equilibrium in variation.*

Proof. It follows from Theorems 3.1, 2.2 and 2.6. \square

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