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# $h$-STABILITY IN VOLTERRA DIFFERENCE SYSTEMS

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Abstract. We investigate $h-$stability of solutions of nonlinear Volterra difference systems and linear Volterra difference systems.

## 1. Introduction

Discrete Volterra systems arise mainly in the process of modeling of some real phenomena or by applying a numerical method to a Volterra integral equation. Medina and Pinto [13, 14] introduced the notion of $h-$stability which is an important extension of the notion of exponential asymptotic stability. In the study of the stability properties of difference systems, the notion of $h-$stability is very useful because, when we study the asymptotic stability it is not easy to work with non-exponential types of stability. For the study of the $h-$stability for difference systems, we refer to Choi et al. [2], Medina and Pinto [13]. Also, Choi et al. [6], Medina and Pinto [14] studied the $h-$stability for Volterra difference systems. In this paper, we investigate $h-$stability of solutions of nonlinear Volterra difference systems and linear Volterra difference systems.

## 2. Preliminaries

We consider the nonlinear Volterra difference system

$$x(n+1) = f(n, x(n)) + \sum_{s=n_0}^{n} g(n, s, x(s)), \tag{2.1}$$

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where $f : N(n_0) \times \mathbb{R}^m \to \mathbb{R}^m, g : N(n_0) \times N(n_0) \times \mathbb{R}^m \to \mathbb{R}^m, N(n_0) = \{n_0, n_0 + 1, \cdots, n_0 + k, \cdots\}$ ($n_0$ a nonnegative integer), $\mathbb{R}^m$ is the $m$-dimensional real Euclidean space. We assume that $f_x = \frac{\partial f}{\partial x}$ and $g_x = \frac{\partial f}{\partial x}$ exist and are continuously invertible on $N(n_0)\times\mathbb{R}^m$, $N(n_0)\times N(n_0)\times\mathbb{R}^m$, respectively. Also, we assume that $f(n, 0) = 0$ and $g(n, s, 0) = 0$. Let $x(n, n_0, x_0) = x_0$ denote the solution of (2.1) with $x(n_0, n_0, x_0) = x_0$. Also, we consider the associated variational systems

$$v(n+1) = f_x(n, 0)v(n)+\sum_{s=n_0}^{n} g_x(n, s, 0)v(s) \tag{2.2}$$

and

$$z(n+1) = f_x(n, x(n, n_0, x_0))z(n)+\sum_{s=n_0}^{n} g_x(n, s, x(s))z(s) \tag{2.3}$$

of (2.1). The fundamental matrix $\Phi(n, n_0, 0)$ of (2.2) is given by

$$\Phi(n, n_0, 0) = \frac{\partial}{\partial x_0} x(n, n_0, 0)$$

and the fundamental matrix $\Phi(n, n_0, x_0)$ of (2.3) is given by

$$\Phi(n, n_0, x_0) = \frac{\partial}{\partial x_0} x(n, n_0, x_0)$$

(See [12]). We now give the main definitions [3, 14] that we need in the sequel. Let $\mathbb{R}^m$ denote the Euclidean $m$-space. For $x \in \mathbb{R}^m$, let $| x |= (\sum_{j=1}^{m} x_j^2)^{\frac{1}{2}}$. For an $m \times m$ matrix $A$, define the norm $| A |$ of $A$ by $| A |= sup_{|x|\le 1} | Ax |$. Let $\mathbb{R}^+$ be the half line $[0, \infty)$.

Definition 2.1. The zero solution of (2.1), or more briefly system, is called *h-stable*($hS$) if there exist $c \ge 1, \delta > 0$ and a positive bounded function $h : N(n_0) \to \mathbb{R}$ such that

$$| x(n, n_0, x_0) |\le c | x_0 | h(n)h^{-1}(n_0)$$

for $n \ge n_0$ and $| x_0 |< \delta$ (here $h^{-1}(n) = \frac{1}{h(n)}$), $h-$*stable in variation*($hSV$) if the solution of system (2.3) is $hS$.

Remark 2.2. If $h(t) = e^{-t}$, then $h-$stability coincides with exponential stability, and if $h(t)$ is constant, then we have uniform Lipschitz stability.

For the various definitions of stability, we refer to [15] and we obtain the following possible implications for system (2.2) among the various types of stability as in [15]:

$$\begin{aligned} \text{h-stability} &\Rightarrow \text{uniform exponential stability} \\ &\Rightarrow \text{uniform Lipschitz stability} \\ &\Rightarrow \text{uniform stability} \end{aligned}$$

The following lemma says that the zero solution of (2.3) is $hS$ if and only if there exist two constants and a positive bounded function satisfying some conditions.

LEMMA 2.3. [15]. *The zero solution of (2.2) is $hS$ if and only if there exist $c \geq 1$ and a positive bounded function $h : N(n_0) \to \mathbb{R}$ such that for every $x_0 \in \mathbb{R}^m$*

$$|\ \Phi(n, n_0)\ | \leq ch(n)h^{-1}(n_0)$$

*for $n \geq n_0$ where $\Phi$ is a fundamental matrix solution of*

$$\Phi(n+1, n_0, x_0) = A(n)\Phi(n, n_0, x_0) + \sum_{s=n_0}^{n} B(n, s)\Phi(s, n_0, x_0), n \geq n_0,$$

*with $\Phi(n_0, n_0, x_0) = I$ (the identity matrix) and $A(n) = f_x(n, 0), B(n, s) = g_x(n, s, 0)$.*

In section 3, we use the following equivalent system by M. Zouyousefain and S. Leela [16] to show the $h-$stability of (2.1). This is a modification of Theorem 2.1 in [16].

LEMMA 2.4. [6]. *Assume that $(H_1)$ there exists an $m \times m$ matrix function $L(n, s)$ defined on $N(n_0) \times N(n_0)$ satisfying*

$$B(n, s) + L(n, s+1)A(s) - L(n, s) + \sum_{\sigma=s}^{n-1} L(n, \sigma+1)B(\sigma, s) = 0.$$

*Consider the linear system*

$$x(n+1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n, s)x(s) + f(n), x(n_0) = x_0, \tag{2.4}$$

*where $A(n)$ and $B(n, s)$ are $m \times m$ matrices for each $n, s \in N$ and $f : N(n_0) \to \mathbb{R}^m, n_0 \in N$. Then Equation (2.4) is equivalent to the linear difference equation*

$$y(n+1) = C(n)y(n) + L(n, n_0)x_0 + H(n), y(n_0) = x_0,$$

*where*

$$C(n) = A(n) - L(n, n) + B(n, n)$$

*and*

$$H(n) = f(n) + \sum_{s=n_0}^{n-1} L(n, s+1)f(s).$$

The following two theorems are given by S. K. Choi and N. J. Koo [3] and are concerned with $hS$.

THEOREM 2.5. [3]. *If the zero solution of (2.1) is $hS$, then the zero solution of (2.2) is also $hS$.*

THEOREM 2.6. [3]. *If the zero solution of (2.3) is $hS$, then the zero solution of (2.1) is also $hS$.*

By Theorems 2.5 and 2.6, we obtain the following corollary.

COROLLARY 2.7. *If the zero solution of (2.3) is $hS$, then the zero solution of (2.2) is also $hS$.*

THEOREM 2.8. [3]. *Assume that the zero solution of (2.2) is hS. Then the zero solution of (2.3) is also hS under the condition that for $| x |\le \rho$ with some $\rho > 0$,*

$$(i) \quad | f_x(n, x) - f_x(n, 0) |\le a(n),$$

*where $a : N(n_0) \to \mathbb{R}^+$,*

$$(ii) \quad | g_x(n, s, x) - g_x(n, s, 0) |\le b(n, s),$$

*where $b : N(n_0) \times N(n_0) \to \mathbb{R}^+$,*

$$(iii)\ \lambda(n) = h(n)[h^{-1}(n+1)a(n) + K] \in l_1(N(n_0)),$$

*and*

$$sup_{s\le\sigma\le n-1} \sum_{\sigma=n_0}^{n-1} h^{-1}(\sigma+1)b(\sigma, s) \le K,$$

*where $K > 0$.*

By Theorems 2.5 and 2.8, we obtain the following result.

COROLLARY 2.9. *Assume that the zero solution of (2.1) is $hS$. If the assumptions $(i), (ii)$ and $(iii)$ of Theorem 2.8 hold for $| x |\le \rho$ with some $\rho > 0$, then the zero solution of (2.1) is also $hSV$.*

THEOREM 2.10. [16]. *The solution $x(n, n_0, x_0)$ of (2.4) satisfies the relation*

$$x(n, n_0, x_0) = \Phi(n, n_0)x_0 + \sum_{s=n_0}^{n-1} \Phi(n, s+1)f(n),$$

*where $\Phi(n, n_0)$ is the fundamental matrix of the difference equation*

$$x(n+1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n,s)x(s)$$

*such that $\Phi(n_0, n_0)$ is the identity matrix.*

We need the following difference inequality which comes from the well-known Bihari-type inequality, to obtain the $hS$ of (3.4).

LEMMA 2.11. [4]. *Let $a(n), b(n)$ and $c(n)$ be non-negative functions defined on $N(n_0)$ and $d$ be a positive number. If, for $n \geq n_0$, the following inequality holds*

$$u(n) \leq d + \sum_{s=n_0}^{n-1} a(s)u(s) + \sum_{s=n_0}^{n-1} b(s) \sum_{l=n_0}^{s-1} c(l)u(l),$$

*then*

$$u(n) \leq dexp[\sum_{s=n_0}^{n-1} (a(s) + b(s) \sum_{l=n_0}^{s-1} c(l))], n \geq n_0.$$

LEMMA 2.12. [13]. *The unique solution $y(n, n_0, y_0)$ of (2.4) satisfying $y(n_0) = y_0$ is given by*

$$y(n, n_0, y_0) = R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s+1)f(s),$$

*where $R(n, m)$ is the unique solution of the matrix difference equation*

$$R(n,m) = R(n,m+1)A(m) + \sum_{r=m}^{n-1} R(n,r+1)B(r,m), n-1 \geq m \geq n_0, \tag{2.5}$$

*with $R(m, m) = I$.*

REMARK 2.13. In the special case when $f(n, x) = A(n)x(n)$ and $g(n, s, x) = B(n, s)x(s)$ in the nonlinear system (2.1), we note that the resolvent matrix $R(n, m)$ for equation (2.5) is closely related to the fundamental matrix $\Phi(n, n_0)$. By the uniqueness of solution, it is easy to see that $R(n, n_0) = \Phi(n, n_0)$.

## 3. Main results

In this section, we examine the property of hS for the nonlinear Volterra difference system (2.1) and the perturbation Volterra difference system (3.4) of (3.3) using the variation of parameters formula and Bihari-type inequality.

THEOREM 3.1. *Suppose that the assumption* $(H_1)$ *of Lemma 2.4 and all the assumptions* $(i), (ii)$ *and* $(iii)$ *of Theorem 2.8 hold. Then the hS property of the equation*

$$(3.1)\quad p(n, n_0, \mid x_0 \mid) = \mid \Psi(n, n_0) \mid\mid x_0 \mid + \sum_{s=n_0}^{n-1} \mid \Psi(n, s+1) \mid\mid L(s, n_0) \mid\mid x_0 \mid,$$

*where* $\Psi(n, s)$ *is the fundamental matrix solution of* $v(n+1) = C(n)v(n)$, *implies that the zero solution of (2.1) is* $hS$.

*Proof.* Set $A(n) = f_x(n, 0), B(n, s) = g_x(n, s, 0)$ and $f(n) \equiv 0$. We consider the equation (2.2). We see, in view of Lemma 2.4, that it is enough to investigate the equivalent equation with $H(n) = 0$

$$(3.2)\qquad v(n+1) = C(n)v(n) + L(n, n_0)x_0, v(n_0) = x_0.$$

By the variation of parameters formula, we have

$$v(n, n_0, x_0) = \Psi(n, n_0)x_0 + \sum_{s=n_0}^{n-1} \Psi(n, s+1)L(s, n_0)x_0.$$

It then follows from (3.1) that

$$\mid v(n) \mid \leq p(n, n_0, \mid x_0 \mid), n \geq n_0,$$

Therefore, from the assumption of (3.1), we obtain

$$\mid v(n, n_0, x_0) \mid \leq c \mid x_0 \mid h(n)h^{-1}(n_0), \quad n \geq n_0,$$

for some $c \geq 1$ and $\mid x_0 \mid < \delta$. Hence the zero solution of (3.2) is $hS$ and so is that of (2.2). This, with the assertion of Theorem 2.8, implies that the zero solution of (2.3) is $hS$. Hence, by Theorem 2.6, the zero solution of (2.1) is $hS$. This completes the proof.

COROLLARY 3.2. *Let the assumptions of Theorem 3.1 hold. Then the zero solution of (2.1) is also* $hSV$.

Consider the linear intergo-differential equation of Volterra type

$$x' = A(t)x + \int_{t_0}^{t} B(t,s)x(s)ds, x(t_0) = x_0,$$

and its perturbation

$$y' = A(t)y + \int_{t_0}^{t} B(t,s)y(s)ds + g(t,y), y(t_0) = y_0.$$

Corresponding to these Volterra integro-differential equations, we can consider

$$x(n+1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n,s)x(s), x(n_0) = x_0 \tag{3.3}$$

and its perturbation

$$y(n+1) = A(n)y(n) + \sum_{s=n_0}^{n} B(n,s)y(s) + g(n, y(n), Ty(n)), y(n_0) = y_0, \tag{3.4}$$

where $A(n)$ and $B(n,s)$ are $m \times m$ matrices for each $n, s \in N, g : N(n_0) \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ and $T : F(N(n_0), \mathbb{R}^m) \to \mathbb{R}^m$ is an operator on $F(N(n_0), \mathbb{R}^m) = \{y | y : N(n_0) \to \mathbb{R}^m$ is a sequence$\}$, $g(n, 0, 0) = 0$.

Using the discrete Bihari-type inequality, we obtain the property of $hS$ for (3.4).

THEOREM 3.3. *Suppose that the zero solution $x = 0$ of (3.3) is $hS$ with the positive function $h(n)$ and for any $n \geq n_0$*

$$| g(n, y, Ty) | \leq a(n)(| y(n) | + | Ty(n) |),$$

*where $a \in F(N(n_0), \mathbb{R}^+)$. Further suppose that the operator $T$ satisfies the inequality*

$$| Ty(n) | \leq \sum_{j=n_0}^{n-1} b(j) | y(j) |,$$

*where $b \in F(N(n_0), \mathbb{R}^+)$, and*

$$M(n) = exp \left[c_1 \sum_{j=n_0}^{n-1} [\frac{h(j)}{h(j+1)} a(j) \left(1 + \frac{1}{h(j)} \sum_{k=n_0}^{j-1} h(k)b(k)\right)]\right] < \infty.$$

*Then the zero solution $y = 0$ of (3.4) is $hS$.*

*Proof.* By Lemma 2.12, the solution $y(n)$ of (3.4) is given by

$$y(n) = R(n, n_0)y_0 + \sum_{j=n_0}^{n-1} R(n, j+1)g(j, y(j), Ty(j))$$

where $R(n, m)$ is the resolvent solution of the matrix difference equation (2.5). Then, by assumption, we have

$$\begin{aligned} | y(n, n_0, y_0) | &\leq | R(n, n_0) || y_0 | + \sum_{j=n_0}^{n-1} | R(n, j+1) || g(j, y(j), Ty(j)) | \\ &\leq c_1 h(n)h^{-1}(n_0) | y_0 | + \sum_{j=n_0}^{n-1} c_1 h(n)h^{-1}(j+1) | g(j, y(j), Ty(j)) | \\ &\leq c_1 h(n)h^{-1}(n_0) | y_0 | + c_1 \sum_{j=n_0}^{n-1} h(n)h^{-1}(j+1)[a(j)(| y(j) | \\ &\quad + \sum_{k=n_0}^{j-1} b(k) | y(k) |)]. \end{aligned}$$

Putting $u(n) = | y(n) | h^{-1}(n)$, we obtain the following inequality from Lemma 2.11

$$\begin{aligned} u(n) &\leq c_1 u(n_0) + c_1 \sum_{j=n_0}^{n-1} [\frac{h(j)}{h(j+1)} a(j)\big(u(j) + \frac{1}{h(j)} \sum_{k=n_0}^{j-1} h(k)b(k)u(k)\big)] \\ &\leq c_1 u(n_0) exp[c_1 \sum_{j=n_0}^{n-1} [\frac{h(j)}{h(j+1)} a(j)\big(1 + \frac{1}{h(j)} \sum_{k=n_0}^{j-1} h(k)b(k)\big)]] \\ &\leq c_1 u(n_0)M(n). \end{aligned}$$

Hence we obtain $| y(n) | \leq M | y_0 | h(n)h^{-1}(n_0)$, where $M = c_1 M(n) \geq 1$, for all $n \geq n_0$, and the proof is complete.

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