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RELATIVE REGULAR RELATIONS

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Abstract. In this paper the relative regular relations with respect to a homomorphism are defined and it will be given the necessary and sufficient conditions for the relations to be transitive.

1. Introduction

Given a transformation group (X, T), we may regard T as a set of self-homomorphisms of X. The enveloping semigroup $E(X)$ of (X, T) is defined to be the closure of T in X^X, taken with the product topology. $E(X)$ has both semigroup and transformation group structures, so it plays an important role to investigate the properties of transformation group (X, T).

The notion of proximal relation in transformation group and regular minimal sets have been strengthened and extended by consideration of homomorphisms [6]. For a given homomorphism $\pi : X \to Y$, with Y minimal and for $y \in Y$, $E(\pi, y)$ and $P_\pi(y)$ have been constructed, which in some ways generalize the transformation group structure of $E(X)$ and the proximal relation of X, respectively. The informations about $E(\pi, y)$ and $P_\pi(y)$ were studied intensively in [6]. The relative regular relations, as generalized notions of relative proximal relations, were introduced in [7].

In this paper, we will define relative regular relations with respect to a homomorphism $\pi : X \to Y$, which are analogous, but not identical to those of [7]. The relations are reflexive and symmetric, but are neither transitive nor closed. The necessary and sufficient conditions for the relations to be transitive will be given.

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2. Preliminaries

An arbitrary, but fixed, topological group will be denoted by T throughout this paper and we will consider the transformation group (X, T) with a compact Hausdorff space X. The compact Hausdorff space X carries a natural uniformity $U[X]$ whose indices are the neighborhoods of the diagonal in $X \times X$.

A closed nonempty subset A of (X, T) is called a *minimal set* if for every $x \in A$ the orbit xT is a dense subset of A. A point whose orbit closure is a minimal set is called an *almost periodic point.* If X is itself minimal, we say that it is a *minimal transformation group* or a *minimal set.*

Let (X, T) and (Y, T) be transformation groups. A function $\pi : X \to Y$ is called a *homomorphism* if π is continuous and $\pi(xt) = \pi(x)t$ $(x \in X, t \in T)$. Onto homomorphism is called an *epimorphism.* A homomorphism of X into itself is called an *endomorphism* and bijective endomorphism is called an *automorphism* of X. We denote the group of automorphisms of X by $A(X)$.

Definition 2.1. [4] Let (X, T) be a transformation group. Two points x and $x^{'}$ of X are called *proximal* provided that for each index $\alpha \in U[X]$, there exists a $t \in T$ such that $(xt, x^{'}t) \in \alpha$. The set of all proximal pairs of points is called the *proximal relation* and is denoted by $P(X, T)$ or $P(X)$.

Definition 2.2. [8] Let (X, T) be a transformation group. Two points x and $x^{'}$ are said to be *regular* if $h(x)$ and $x^{'}$ are proximal for some automorphism h of X. This is, $(h(x), x^{'}) \in P(X, T)$ for some $h \in A(X)$. The set of all regular pairs of points in X is called the *regular relation* and is denoted by $R(X, T)$ or $R(X)$.

A Minimal transformation group (X, T) is called *regular minimal* if x, y in X, then there is an automorphism h of (X, T) such that $h(x)$ and y are proximal.

A Minimal transformation group (M, T) is said to be *universal* if every minimal transformation group with phase group T is a homomorphic image of (M, T). The group of automorphisms of M is denoted by $A(M)$.

The enveloping semigroup $E(X)$ of X is defined to be the closure of T in X^X, providing X^X with its product topology. The *minimal right* *ideal* I is the nonempty subset of $E(X)$ with $IE(X) \subset I$, which contains no proper nonempty subset with the same property. Two idempotents u and v in $E(X)$ are said to be *equivalent*, writing $u \sim v$ if $uv = u$ and $vu = v$.

THEOREM 2.3. [4] *Let $E(X)$ be the enveloping semigroup of (X, T). Then*
(1) The maps $\theta_x : E(X) \to X$ defined by $\theta_x(p) = xp$ are homomorphisms with range $\overline{xT}$ for $x \in X$.
(2) Given an epimorphism $\pi : X \to Y$, there exists a unique epimorphism $\theta : (E(X), T) \to (E(Y), T)$ such that $\pi\theta_x = \theta_{\pi(x)}\theta$ for $x \in X$.
(3) If (X, T) coincides with (Y, T), then θ is the identity map.

Let $\pi : X \to Y$ be a fixed homomorphism, with Y minimal. Suppose $y \in Y$. Then $X^{\pi^{-1}(y)}$ is a transformation group whose elements are functions from $\pi^{-1}(y)$ to X.

DEFINITION 2.4. [6] Define $z_y : \pi^{-1}(y) \to X$ by $z_y(x) = x$ for all $x \in \pi^{-1}(y)$. Let $E(\pi, y)$ be the orbit closure of z_y. That is, $E(\pi, y) = \overline{z_y T} \subset X^{\pi^{-1}(y)}$.

Shoenfeld [6] showed that the minimal sets of $E(\pi, y)$ are isomorphic and independent of the choice of y.

DEFINITION 2.5. [6] Let $\pi : X \to Y$ be a homomorphism with Y minimal. Two points x and x' are called *relatively proximal* (to π) if $\pi(x) = \pi(x')$ and $(x, x') \in P(X)$. The set of all relatively proximal pairs is called the *relative proximal relation with respect to* π and is denoted by $P_\pi(Y)$.
In particular, for a fixed $y \in Y$, we define

$$P_\pi(y) = \{(x, x') \in X \times X \mid \pi(x) = \pi(x') = y \textit{ and } (x, x') \in P(X)\}$$

$P_\pi(Y)$ and $P_\pi(y)$ are reflexive and symmetric relations, but are neither transitive nor closed.

DEFINITION 2.6. [7] A homomorphism $\pi : X \to Y$ is called *regular* if for each x, x' with $\pi(x) = \pi(x')$, there exists an automorphism h of X such that $(h(x), x') \in P(X)$ and $\pi h = \pi$.

THEOREM 2.7. [6] *Let $\pi : X \to Y$ be a homomorphism with Y minimal*
(1) $P_\pi(y)$ is an equivalence relation for $y \in Y$ if and only if $E(\pi, y)$ contains just one minimal set.

(2) The relative proximal relation $P_\pi(Y)$ is an equivalence relation if and only if $E(\pi, y)$ contains just one minimal set, for all $y \in Y$.

3. Relative regular relations

Throughout this section, we will consider the homomorphism $\pi : X \to Y$, where Y is a minimal transformation group.

In [7], the relative regular relations $R_\pi(y)$ and $R_\pi(Y)$ of a given $\pi : X \to Y$ were defined as follows. For $y \in Y$,

$$R_\pi(y) = \{(x, x^{'}) | \pi(x) = \pi(x^{'}) = y, (h(x), x^{'}) \in P(X)\ for\ some\ h \in A(X)\}$$

$$R_\pi(Y) = \bigcup\{R_\pi(y) \mid y \in Y\}$$

Several properties of these relations were studied.

Given a homomorphism $\pi : X \to Y$, we define

$$Aut(\pi) = \{h \mid h\ is\ an\ automorphism\ of\ X\ satisfying\ \pi\ h = \pi\}$$

DEFINITION 3.1. Let $\pi : X \to Y$ be a homomorphism. π is called a *group extension* if whenever $x, x^{'} \in X$ and $\pi(x) = \pi(x^{'})$, there exists an $h \in Aut(\pi)$ such that $h(x) = x^{'}$.

Now, we will define relative regular relations analogously, but not all the same to [7].

DEFINITION 3.2. Let $\pi : X \to Y$ be a homomorphism. Two points x and $x^{'}$ in X are *relatively regular* if $\pi(x) = \pi(x^{'})$ and $(h(x), x^{'}) \in P(X)$ for some $h \in Aut(\pi)$. The set of all relatively regular pairs is called the *relative regular relation with respect to* π and is denoted by $R_\pi(Y)$.
In particular, for a fixed $y \in Y$, we define $R_\pi(y)$ to be the set of all $(x, x^{'}) \in \pi^{-1}(y) \times \pi^{-1}(y)$ such that $(h(x), x^{'}) \in P(X)$ for some $h \in Aut(\pi)$.

REMARK 3.3. 1. The followings are obvious from the definitions.
(1) $P_\pi(y) \subset P_\pi(Y) \subset P(X) \subset R(X)$
(2) $P_\pi(y) \subset R_\pi(y) \subset R_\pi(Y) \subset R(X)$
(3) $P_\pi(Y) \subset R_\pi(Y)$
2. If X is a minimal set and Y is a singleton $\{y\}$, then $R_\pi(Y)$ coincides with $R(X)$.
3. Note that the only automorphism of a proximal transformation group is the identity. Therefore, $P_\pi(Y)$ coincides with $R_\pi(Y)$ in a proximal transformation group.

Note that a homomorphism of minimal sets is a group extension if and only if it is distal and regular. Therefore, if $\pi : X \to Y$ is a group extension, then the following are verified easily.
(1) $P_\pi(y) = \{(x,x) \mid x \in \pi^{-1}(y)\}$ for each $y \in Y$.
(2) $R_\pi(y) = \{(x,x') \mid \pi(x) = \pi(x') = y\}$ for each $y \in Y$.
(3) $P_\pi(Y) = \{(x,x) \mid x \in X\}$.
(4) $R_\pi(Y) = \{(x,x') \mid \pi(x) = \pi(x')\}$.

Lemma 3.4. *Let $\pi : X \to Y$ be a homomorphism and let $h \in Aut(\pi)$. Then*
(1) $h^{-1} \in Aut(\pi)$
(2) If $(x,x') \in P_\pi(y)$, then $(h(x),h(x')) \in P_\pi(y)$.

Proof. (1) Obvious.
(2) Let $(x,x') \in P_\pi(y)$ and $h \in Aut(\pi)$. $(x,x') \in P_\pi(y)$ implies $\pi(x) = \pi(x') = y$ and $(x,x') \in P(X)$. Then $\pi h(x) = \pi(x) = y = \pi(x') = \pi h(x')$ and $(h(x),h(x')) \in P(X)$, because $(h(x),h(x'))$ is the homomorphic image of proximal pair (x,x'). Therefore, $(h(x),h(x')) \in P_\pi(y)$. □

Theorem 3.5. *Let $\pi : X \to Y$ be a homomorphism, and let $y \in Y$.*
(1) $R_\pi(Y)$ and $R_\pi(y)$ are reflexive and symmetric relations.
(2) If $E(\pi,y)$ contains just one minimal set, then $R_\pi(y)$ is an equivalence relation.
(3) If $P_\pi(y)$ (resp. $P_\pi(Y)$) is an equivalence relation, then so is $R_\pi(y)$ (resp. $R_\pi(Y)$).
(4) If $R_\pi(y)$ (resp. $R_\pi(Y)$) is closed, then $R_\pi(y)$ (resp. $R_\pi(Y)$) is transitive for $y \in Y$.

Proof. The proofs are similar to those of Lemma 4.7, Theorem 4.8, Corollary 4.10, Theorem 4.15 in [7], and therefore, we omit the proofs. □

Theorem 3.6. *Let $\pi : X \to Y$ be a homomorphism with X regular minimal and Y minimal, and let x, x' in X such that $\pi(x) = \pi(x')$. Then $(h(x),x') \in P(X)$ for some $h \in Aut(\pi)$.*

Proof. Let x, x' in X such that $\pi(x) = \pi(x')$. Since X is regular minimal, we can find an $h \in A(X)$ such that $h(x)$ and x' are proximal in X. Therefore, $h(x)u = x'$ for some minimal idempotent u of $E(X)$. This implies that

$$h(xu) = h(x)u = h(x)uu = x'u$$

and hence

$$\pi h(xu) = \pi(x^{'}u) = \pi(x^{'})\theta(u) = \pi(x)\theta(u) = \pi(xu)$$

Since Y is minimal, we have $\pi h = \pi$. That is, $h \in Aut(\pi)$. Consequently, $(h(x), x^{'}) \in P(X)$ for some $h \in Aut(\pi)$. □

COROLLARY 3.7. *Let $\pi : X \to Y$ be a homomorphism with X regular minimal and Y minimal. Then $R_\pi(y)$ and $R_\pi(Y)$ are equivalence relations.*

Proof. It suffices to show that $R_\pi(Y)$ and $R_\pi(y)$ are transitive. Consider points x, $x^{'}$, $x^{''}$ in X such that $(x, x^{'}) \in R_\pi(Y)$ and $(x^{'}, x^{''}) \in R_\pi(Y)$. Then $\pi(x) = \pi(x^{'}) = \pi(x^{''})$. By Theorem 3.6, we can find an $h \in Aut(\pi)$ such that $(h(x), x^{''}) \in P(X)$, which implies that $(x, x^{''}) \in R_\pi(Y)$. Therefore, $R_\pi(Y)$ is transitive. Similarly $R_\pi(y)$ is also transitive. □

LEMMA 3.8. [2] *Let (X, T) be a minimal transformation group and let $\gamma : M \to X$ and $\delta : M \to X$ be homomorphisms. Then there exists $\alpha \in A(M)$ such that $\delta = \gamma\alpha$.*

THEOREM 3.9. *Let $\pi : X \to Y$ and $\gamma : M \to X$ be homomorphisms and let u and v be the equivalent idempotents of $E(X)$ such that $yu = yv = y$, and $x \in \pi^{-1}(y)$. Then $(xu, xv) \in R_\pi(y)$ if and only if there exist $\alpha \in A(M)$ and $h \in Aut(\pi)$ such that $h\gamma\alpha = \gamma$, $\pi\gamma\alpha = \pi\gamma$ and $(\gamma\alpha(m), \gamma(m)) = (xu, xv)$ for some $m \in M$.*

Proof. Suppose first that $(xu, xv) \in R_\pi(y)$. Then there exists an $h \in Aut(\pi)$ such that $(h(xu), xv) \in P(X)$. By Lemma 3.8, for given $\gamma : M \to X$ and $h\gamma : M \to X$, we can find an $\alpha \in A(M)$ such that $h\gamma\alpha = \gamma$. Therefore, we also have $\pi\gamma\alpha = \pi\gamma$, because $h \in Aut(\pi)$. Since γ is an epimorphism, $\gamma(m) = xv$ for some $m \in M$. Since $(h(xu), xv)$ is both proximal and almost periodic pair of points of X, $h(xu) = xv$. Since $h\gamma\alpha(m) = \gamma(m) = xv = h(xu)$ and h is an automorphism, it follows that $\gamma\alpha(m) = xu$.
Conversely, suppose that there exist an $\alpha \in A(M)$ and an $h \in Aut(\pi)$ such that $h\gamma\alpha = \gamma$, $\pi\gamma\alpha = \pi\gamma$ and $(\gamma\alpha(m), \gamma(m)) = (xu, xv)$ for some $m \in M$. Then $h\gamma\alpha(m) = \gamma(m)$ implies that $h(xu) = xv$ and $\pi\gamma\alpha(m) = \pi\gamma(m)$ implies that $\pi(xu) = \pi(xv) = y$. Therefore, $(xu, xv) \in R_\pi(y)$. □

If we take $h = 1_X$, the identity automorphism of X, in the proof of Theorem 3.9, then the following corollary is immediate.

COROLLARY 3.10. *If the conditions of Theorem 3.9 are fulfilled, then* $(xu, xv) \in P_\pi(y)$ *if and only if there exists an* $\alpha \in A(M)$ *such that* $\gamma\alpha = \gamma$, $\pi\gamma\alpha = \pi\gamma$ *and* $(\gamma\alpha(m), \gamma(m)) = (xu, xv)$ *for some* $m \in M$.

Let $\pi : X \to Y$ be a homomorphism and let $h \in Aut(\pi)$. Define

$$R_\pi^h(y) = \{(x, x^{'}) \mid \pi(x) = \pi(x^{'}) = y \text{ and } (h(x), x^{'}) \in P(X)\}$$

Note that if $h = 1_X$, then $R_\pi^h(y) = P_\pi(y)$.

THEOREM 3.11. *Let* $\pi : X \to Y$ *be a homomorphism. Then* $P_\pi(y)$ *is closed if and only if* $R_\pi^h(y)$ *is closed for all* $h \in Aut(\pi)$.

Proof. Suppose that $P_\pi(y)$ is closed. Let $h \in Aut(\pi)$, and let $((x_n, x_n^{'}))$ be a net in $R_\pi^h(y)$ such that $(x_n, x_n^{'})$ converges to $(x, x^{'})$. Then $(h(x_n), x_n^{'}) \in P_\pi(y)$ for all n and $(h(x_n), x_n^{'})$ converges to $(h(x), x^{'})$. Since $P_\pi(y)$ is closed, $(h(x), x^{'}) \in P_\pi(y)$ and therefore $(x, x^{'}) \in R_\pi^h(y)$, which shows that $R_\pi^h(y)$ is closed.
Conversely, let $R_\pi^h(y)$ be closed for all $h \in Aut(\pi)$, and let $((x_n, x_n^{'}))$ be a net in $P_\pi(y)$ such that $(x_n, x_n^{'})$ converges to $(x, x^{'})$. Then $h^{-1} \in Aut(\pi)$ and $(h^{-1}(x_n), x_n^{'}) \in R_\pi^h(y)$ for each n and $(h^{-1}(x_n), x_n^{'})$ converges to $(h^{-1}(x), x^{'})$. Since $R_\pi^h(y)$ is closed, $(h^{-1}(x), x^{'}) \in R_\pi^h(y)$. This shows that $(x, x^{'}) \in P_\pi(y)$, and therefore $P_\pi(y)$ is closed. □

THEOREM 3.12. *Let* $\pi : X \to Y$ *be a homomorphism and let* $y \in Y$. *The following are equivalent.*
(1) $R_\pi(y)$ *is an equivalence relation.*
(2) Let u *be an idempotent of* $E(X)$ *such that* $yu = y$. *Then* $(xu, x^{'}u) \in R_\pi(y)$ *for* $(x, x^{'}) \in R_\pi(y)$.
(3) For h, k *in* $Aut(\pi)$, *there exists* l *in* $Aut(\pi)$ *such that* $R_\pi^h(y) \circ R_\pi^k(y) \subset R_\pi^l(y)$.
(4) Let u *and* v *be the equivalent idempotents of* $E(X)$ *such that* $yu = yv = y$. *Then* $(xu, xv) \in R_\pi(y)$ *for* $x \in \pi^{-1}(y)$.
(5) Let u *and* v *be the equivalent idempotents of* $E(X)$ *such that* $yu = yv = y$, *and* $x \in \pi^{-1}(y)$. *There exist* $\alpha \in A(M)$ *and* $h \in Aut(\pi)$ *such that* $h\gamma\alpha = \gamma$, $\pi\gamma\alpha = \pi\gamma$ *and* $(\gamma\alpha(m), \gamma(m)) = (xu, xv)$ *for some* $m \in M$.

Proof. (1) ⇒ (2) Let $u^2 = u$ such that $yu = y$. Since (xu, x), $(x, x^{'})$ and $(x^{'}, x^{'}u)$ are in $R_\pi(y)$ for $(x, x^{'}) \in R_\pi(y)$ and $R_\pi(y)$ is transitive, it follows that $(xu, x^{'}u) \in R_\pi(y)$ for $(x, x^{'}) \in R_\pi(y)$.

(2) $\Rightarrow$ (3) Let h, k in $Aut(\pi)$ and let $(x, x^{'}) \in R_\pi^h(y) \circ R_\pi^k(y)$. Then $(x, x^{''}) \in R_\pi^k(y) \subset R_\pi(y)$ and $(x^{''}, x^{'}) \in R_\pi^h(y) \subset R_\pi(y)$ for some $x^{''} \in \pi^{-1}(y)$. By (2), $(xu, x^{''}u) \in R_\pi(y)$ and $(x^{''}u, x^{'}u) \in R_\pi(y)$, for idempotent u of $E(X)$ such that $yu = y$. Therefore,

$$(\phi_1(xu), x^{''}u) \in P(X) \textit{ and } (\phi_2(x^{''}u), x^{'}u) \in P(X)$$

for some ϕ_1 and ϕ_2 in $Aut(\pi)$. We also have

$$(\phi_2\phi_1(xu), \phi_2(x^{''}u)) \in P(X)$$

by Lemma 3.4.(2). Since $(\phi_2\phi_1(xu), \phi_2(x^{''}u))$ and $(\phi_2(x^{''}u), x^{'}u)$ are both proximal and almost periodic pairs of points, $\phi_2\phi_1(xu) = \phi_2(x^{''}u) = x^{'}u$.

Since $\pi\phi_2\phi_1(x) = \pi(x) = \pi(x^{'}) = y$ and $(l(x), x^{'}) \in P(X)$ for $l = \phi_2\phi_1 \in Aut(\pi)$, it follows that $(x, x^{'}) \in R_\pi^l(y)$.

(3) $\Rightarrow$ (4) Let $yu = yv = y$ and let $x \in \pi^{-1}(y)$. Observe that $(xu, x) \in R_\pi(y)$, $(x, xv) \in R_\pi(y)$ and $\pi(xu) = \pi(xv) = \pi(x) = y$. That is, $(xu, x) \in R_\pi^k(y)$ and $(x, xv) \in R_\pi^h(y)$ for some h, k in $Aut(\pi)$. Therefore, $(xu, xv) \in R_\pi^h(y) \circ R_\pi^k(y)$. By (3), $(xu, xv) \in R_\pi^l(y)$ for some $l \in Aut(\pi)$. Consequently, $(xu, xv) \in R_\pi(y)$.

(4) $\Leftrightarrow$ (5) By Theorem 3.9.

(4) $\Rightarrow$ (1) It suffices to show that $R_\pi(y)$ is transitive. Let $(x, x^{'}) \in R_\pi(y)$ and $(x^{'}, x^{''}) \in R_\pi(y)$. Then $(h(x), x^{'}) \in P_\pi(y)$ and $(k(x^{'}), x^{''}) \in P_\pi(y)$ for $h, k \in Aut(\pi)$. There exist minimal right ideals I, K of $E(X)$ and automorphisms h, k in $A(X)$ such that

$$h(x)p = x^{'}p, \ k(x^{'})q = x^{''}q$$

for all $p \in I$ and $q \in K$. Let u and v be the equivalent idempotents in I and K such that $yu = y = yv$. Then

$$h(x)u = x^{'}u, \ k(x^{'})v = x^{''}v$$

By hypothesis, we get $(x^{'}u, x^{'}v) \in R_\pi(y)$, $(x^{''}u, x^{''}v) \in R_\pi(y)$. Then $(\phi_1(x^{'}u), x^{'}v) \in P(X)$, $(\phi_2(x^{''}u), x^{''}v) \in P(X)$ for some ϕ_1, ϕ_2 in $Aut(\pi)$. Since $(\phi_1(x^{'}u), x^{'}v)$ and $(\phi_2(x^{''}u), x^{''}v)$ are both proximal and almost periodic pair of points, we have

$$\phi_1(x^{'}u) = x^{'}v, \ \phi_2(x^{''}u) = x^{''}v$$

Therefore,

$$k\phi_1(x^{'}u) = k(x^{'}v) = x^{''}v = \phi_2(x^{''}u)$$

Since $k(\phi_1(x^{'}))u = k\phi_1(x^{'}u) = k\phi_1(h(x)u) = k\phi_1 h(x)u$ and $k\phi_1 h(x)u = k\phi_1(x^{'}u) = k(x^{'}v) = x^{''}v = \phi_2(x^{''}u)$, we obtain

$$\phi_2^{-1}k\phi_1 h(x)u = x^{''}u$$

which shows that $(\phi_2^{-1}k\phi_1 h(x), x^{''}) \in P(X)$ and $\phi_2^{-1}k\phi_1 h \in Aut(\pi)$. That is, $(x, x^{''}) \in R_\pi(y)$. □

THEOREM 3.13. *Let $\pi : X \to Y$ be a regular homomorphism. Then $R_\pi(Y)$ and $R_\pi(y)$ are closed equivalence relations.*

Proof. Let (p, q) be any element of the closure of $R_\pi(Y)$. There exists a net $((x_n, x_n^{'}))$ in $R_\pi(X)$ such that $(x_n, x_n^{'})$ converges to (p, q). Then $(\pi(x_n), \pi(x_n^{'}))$ converges to $(\pi(p), \pi(q))$. Since $(x_n, x_n^{'}) \in R_\pi(Y)$ for all n, it follows that $\pi(x_n) = \pi(x_n^{'})$ and $\pi(p) = \pi(q)$. Since $\pi : X \to Y$ is a regular homomorphism, there exists $h \in Aut(\pi)$ such that $h(p)$ and q are proximal, and hence $(p, q) \in R_\pi(Y)$. Therefore, $R_\pi(Y)$ is closed. Similarly $R_\pi(y)$ is also closed. □

Since a group extension $\pi : X \to Y$ is always regular, we have the following corollary.

COROLLARY 3.14. *Let $\pi : X \to Y$ be a group extension. Then $R_\pi(Y)$ and $R_\pi(y)$ are closed equivalence relations.*

THEOREM 3.15. *Let $\pi : X \to Y$ be a homomorphism with X, Y minimal, and suppose that the subspace $Aut(\pi)$ of X^X admits a compact Hausdorff topology making it a topological group and its action on X jointly continuous. Then the relative regular relation $R_\pi(Y)$ is closed if and only if π is represented as a composition of an regular homomorphism π_1 and a distal homomorphism π_2.*

Proof. Suppose that the relative regular relation $R_\pi(Y)$ is closed. Then $R_\pi(Y)$ is an equivalence relation by Theorem 3.5(4). Let $\pi_1 : X \to X/R_\pi(Y)$ be the projection $\pi_1(x) = [x]$, the equivalence class of x, and $\pi_2 : X/R_\pi(Y) \to Y$ the natural correspondence $\pi_2([x]) = \pi(x)$. Then $\pi_2\pi_1(x) = \pi_2([x]) = \pi(x)$ for $x \in X$. Therefore, it follows that $\pi = \pi_2\pi_1$. To show that π_1 is a regular homomorphism, let $x, x^{'} \in X$ such that $\pi_1(x) = \pi_1(x^{'})$. Then $(x, x^{'})$ is a regular pair and $\pi(x) = \pi(x^{'})$. There exists $h \in Aut(\pi)$ such that $(h(x), x^{'}) \in P(X)$. Therefore π_1 is a regular homomorphism.
Now, we show that π_2 is a distal homomorphism. Suppose that $\pi_2(z) =$

$\pi_2(z')$ and (z, z') is a proximal pair. Since π_1 is an epimorphism, there exists a proximal pair (x, x') such that $(\pi_1(x), \pi_1(x')) = (z, z')$. Then

$$\pi(x) = \pi_2\pi_1(x) = \pi_2(z) = \pi_2(z') = \pi_2\pi_1(x') = \pi(x').$$

Therefore, $\pi_1(x) = \pi_1(x')$ and hence $z = z'$. This shows that π is a distal homomorphism.

For the converse, suppose that $((x_n, x'_n))$ is a net in $R_\pi(Y)$ such that $((x_n, x'_n))$ converges to (x, x'). Then for each n, $\pi(x_n) = \pi(x'_n)$ and $(h_n(x_n), x'_n) \in P(X)$ for some $h_n \in Aut(\pi)$. Therefore, $\pi_1 h_n(x_n)$ and $\pi_1(x'_n)$ are proximal, and hence regular for each n. Therefore, $\pi_2(\pi_1 h_n(x_n)) = \pi_2(\pi_1(x'_n))$. Since π_2 is distal, $\pi_1(h_n(x_n)) = \pi_1(x'_n)$. By the hypothesis of $Aut(\pi)$, we can assume h_n converges to $h \in Aut(\pi)$, and therefore $\pi_1(h_n(x_n))$ converges to $\pi_1 h(x) = \pi_1(x')$. Since $\pi h_n = \pi$, $\pi h = \pi$. Therefore, $h(x)$ and x' are proximal and hence x and x' are regular. That is, $(x, x') \in R_\pi(Y)$. □

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