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THE INVERSION FORMULA OF THE STIELTJES TRANSFORM OF SPECTRAL DISTRIBUTION

Sang-Il Choi*

Abstract. In multivariate analysis, the inversion formula of the Stieltjes transform is used to find the density of a spectral distribution of random matrices of sample covariance type. Let $B_n = \frac{1}{n} Y_m^T T_m Y_m$ where $Y_m = [Y_{ij}]_{m\times n}$ is with independent, identically distributed entries and T_m is an $m \times m$ symmetric nonnegative definite random matrix independent of the Y_{ij}'s. In the present paper, using the inversion formula of the Stieltjes transform, we will find the density function of the limiting distribution of B_n away from zero.

1. Introduction

Let M be an $m \times m$ random matrix with real eigenvalues $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_m\}$. Then the spectral distribution function of M is the distribution function $F^M(x)$ with a jump of $\frac{1}{m}$ at each eigenvalue defined by

$$\forall x \in \mathbb{R}, \qquad F^M(x) = \frac{1}{m} \sum_{i=1}^{m} 1_{(-\infty,x]}(\Lambda_i)$$

where 1_S is the indicator function of the set S.

Let $\{Y_{ij}\}_{i,j\geq 1}$ be independent, identically distributed. real-valued random variables with $E|Y_{11} - EY_{11}|^2 = 1$. For each m in $\mathbb{N}$, the set of positive integers, let $Y_m = [Y_{ij}]_{m\times n}$, where $n = n(m)$ and $m/n \to c > 0$ as $m \to +\infty$, and let T_m be an $m \times m$ symmetric nonnegative definite random matrix independent of the Y_{ij}'s for which there exists a sequence of positive numbers $\{\mu_k\}_{k\geq 1}$ such that for each $k \in \mathbb{N}$,

$$\int_0^{+\infty} x^k dF^{T_m}(x) = \frac{1}{m} \operatorname{tr} T_m^k \to \mu_k, \text{almost surely, as } m \to +\infty$$

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where tr A means the trace of the matrix A and the μ_k's satisfy Carleman's sufficiency condition,

$$\sum_{k\geq 1} \mu_{2k}^{-\frac{1}{2k}} = +\infty,$$

for the existence and the uniqueness of the distribution function H having moments $\{\mu_k\}_{k\geq 1}$.([4])

Let $B_n = \frac{1}{n} Y_m^T T_m Y_m$. We have a limit theorem found in [3,4].

THEOREM 1.1. ([3, 4]) *The limiting spectral distribution function F of B_n is of the form*

$$\forall x \in \mathbb{R}, \qquad F(x) = (1-c)1_{[0,\infty)}(x) + cF_0(x),$$

where F_0 is the limiting spectral distribution function of $\frac{1}{n} Y_m Y_m^T T_m$ with moments

$$\forall k \in \mathbb{N}, \qquad \nu_k = \sum_{w=1}^{k} c^{k-w} \sum \frac{k!}{m_1! m_2! \ldots m_w! w!} \mu_1^{m_1} \ldots \mu_w^{m_w}$$

where the inner sum extends over all w-tuples of nonnegative integers $(m_1, \ldots, m_w)$ such that $\sum_{j=1}^{w} m_j = k - w + 1$ and $\sum_{j=1}^{w} j m_j = k$.

Moreover, F_0 is continuous on $\mathbb{R}_+$, the set of positive real numbers, and F is continuous away from 0.

Let $\mathbb{C}_+ = \{z \in \mathbb{C} : c(z) > 0\}$. For $z \in \mathbb{C}_+$, the Stieltjes transform of F, $m(z)$, is defined by,

$$m(z) = \int \frac{dF(\lambda)}{\lambda - z}.$$

In [1], it is shown that, for each $z \in \mathbb{C}_+$, $m = m(z)$ is the unique solution for $m \in \mathbb{C}_+$ to the equation

$$m = -\left(z - c \int \frac{\lambda dH(\lambda)}{1 + \lambda m} \right)^{-1}.$$

Therefore, on $\mathbb{C}_+$, $m(z)$ has an inverse, $z(m)$, given by

$$z(m) = -\frac{1}{m} + c \int \frac{\lambda dH(\lambda)}{1 + \lambda m} \qquad \text{for } m \in m(\mathbb{C}_+).$$

We can find the density function of F using Stieltjes transform of F. Here is the main theorem. The proof is deferred to the end of Section 2.

THEOREM 1.2. *For all $x \neq 0$, the density function of F is*

$$f(x) = \lim_{y \downarrow 0} \frac{1}{\pi} Im(m(x+iy)).$$

2. Stieltjes transform and inversion formula

Let $G(\cdot)$ be a distribution function and let $S(\cdot)$ be the Stieltjes transform of G. Let S_G be the support of G.

THEOREM 2.1. *For $z \in \mathbb{C} - S_G$, $S(z)$ is a well-defined analytic function. Moreover,*

$$S'(z) = \int \frac{dG(\lambda)}{(\lambda - z)^2}.$$

Proof. Fix $z_0 \in \mathbb{C} - S_G$. Since S_G is closed, we have $a \equiv \inf\{|\lambda - z_0| : \lambda \in S_G\} > 0$.

Therefore, $\forall \lambda \in \mathbb{R}$,

$$\frac{1_{S_G}(\lambda)}{|\lambda - z_0|} \leq \frac{1_{S_G}(\lambda)}{a}$$

and, since $\frac{1_{S_G}(\lambda)}{\lambda - z_0}$ is bounded and measurable, its integral with respect to G is well-defined.

For any $z \in \mathbb{C} - S_G$ s.t. $|z - z_0| < \frac{a}{2}$ and for $\lambda \in S_G$, we have $|\lambda - z| > \frac{a}{2}$, and

$$\frac{1_{S_G}(\lambda)}{|(\lambda - z)(\lambda - z_0)|} \leq \frac{1_{S_G}(\lambda)}{\frac{a^2}{2}}$$

and, by the Dominated Convergence Theorem,

$$\begin{aligned} \frac{S(z) - S(z_0)}{z - z_0} &= \frac{1}{z - z_0} \left(\int \frac{1}{\lambda - z} dG(\lambda) - \int \frac{1}{\lambda - z_0} dG(\lambda) \right) \\ &= \frac{1}{z - z_0} \int \frac{z - z_0}{(\lambda - z)(\lambda - z_0)} dG(\lambda) \\ &\to \int \frac{1}{(\lambda - z_0)^2} dG(\lambda) \text{ as } z \to z_0. \end{aligned}$$

Thus $S'(z_0)$ exists and is equal to $\int \frac{1}{(\lambda - z_0)^2} dG(\lambda)$.

Therefore, $S(z)$ is analytic on $\mathbb{C} - S_G$. □

Theorem 2.2 (Inversion Formula). *For any* $\lambda_1 < \lambda_2$,

$$\frac{1}{2}(G(\lambda_2) + G(\lambda_2^-)) - \frac{1}{2}(G(\lambda_1) + G(\lambda_1^-)) = \lim_{y \downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} Im(S(x+iy))dx,$$

where $G(\lambda^-) = \lim_{t \uparrow \lambda} G(t)$.

Proof. For $z = x + iy$, $y > 0$,

$$S(z) = \int \frac{\lambda - x}{(\lambda - x)^2 + y^2} dG(\lambda) + i \int \frac{y}{(\lambda - x)^2 + y^2} dG(\lambda).$$

Thus, from Fubini's Theorem,

$$\begin{aligned}
&\lim_{y \downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \mathrm{Im}(S(x+iy))dx \\
&\qquad = \frac{1}{\pi} \lim_{y \downarrow 0} \int_{\lambda_1}^{\lambda_2} \int \frac{y}{(\lambda - x)^2 + y^2} dG(\lambda) dx \\
&\qquad = \frac{1}{\pi} \lim_{y \downarrow 0} \int y \int_{\lambda_1}^{\lambda_2} \frac{1}{(\lambda - x)^2 + y^2} dx dG(\lambda) \\
&\qquad = \frac{1}{\pi} \lim_{y \downarrow 0} y \int \frac{1}{y} \left(- \arctan \frac{\lambda - x}{y} \Big|_{\lambda_1}^{\lambda_2} \right) dG(\lambda) \\
&\qquad = \frac{1}{\pi} \lim_{y \downarrow 0} \int \left(\arctan \frac{\lambda - \lambda_1}{y} - \arctan \frac{\lambda - \lambda_2}{y} \right) dG(\lambda).
\end{aligned}$$

Let $s(\lambda) = \arctan \frac{\lambda - \lambda_1}{y} - \arctan \frac{\lambda - \lambda_2}{y}$. Then

$$\lim_{y \downarrow 0} s(\lambda) = \begin{cases} 0 & \text{if } \lambda < \lambda_1, \\ \frac{\pi}{2} & \text{if } \lambda = \lambda_1, \\ \pi & \text{if } \lambda_1 < \lambda < \lambda_2, \\ \frac{\pi}{2} & \text{if } \lambda = \lambda_2, \\ 0 & \text{if } \lambda_2 < \lambda. \end{cases}$$

Therefore, by the Dominated Convergence Theorem,

$$\begin{aligned}
&\frac{1}{\pi}\lim_{y\downarrow 0}\int_{\lambda_1}^{\lambda_2}\mathrm{Im}(S(x+iy))dx\\
&\qquad = \frac{1}{\pi}\int\Big(\frac{\pi}{2}1_{\{\lambda_1\}}(\lambda)+\frac{\pi}{2}1_{\{\lambda_2\}}(\lambda)+\pi 1_{(\lambda_1,\lambda_2)}(\lambda)\Big)\,dG(\lambda)\\
&\qquad = \frac{1}{2}(G(\lambda_1)-G(\lambda_1^-)+G(\lambda_2)-G(\lambda_2^-))+G(\lambda_2^-)-G(\lambda_1)\\
&\qquad = \frac{1}{2}(G(\lambda_2)+G(\lambda_2^-))-\frac{1}{2}(G(\lambda_1)+G(\lambda_1^-)).
\end{aligned}$$

□

We can calculate G from the Inversion Formula,

$$G(x_2)-G(x_1)=\lim_{y\downarrow 0}\frac{1}{\pi}\int_{x_1}^{x_2}\mathrm{Im}(S(x+iy))dx$$

where x_1 and x_2 are continuity points of G.

Now we can find the density function of F.

Proof of Theorem 1.2. From the Inversion Formula, since F is continuous away from 0, for all $x_1 \neq 0$, $x_2 \neq 0$,

$$F(x_2)-F(x_1)=\lim_{y\downarrow 0}\frac{1}{\pi}\int_{x_1}^{x_2}\mathrm{Im}(m(x+iy))dx.$$

Therefore, for all $x_0 \neq 0$, the density function of F is

$$\begin{aligned}
f(x_0) &= \lim_{\bar{x}\to x_0}\frac{F(\bar{x})-F(x_0)}{\bar{x}-x_0}\\
&= \lim_{\bar{x}\to x_0}\frac{1}{\bar{x}-x_0}\lim_{y\downarrow 0}\frac{1}{\pi}\int_{x_0}^{\bar{x}}\mathrm{Im}(m(x+iy))dx\\
&= \lim_{y\downarrow 0}\frac{1}{\pi}\lim_{\bar{x}\to x_0}\frac{1}{\bar{x}-x_0}\int_{x_0}^{\bar{x}}\mathrm{Im}(m(x+iy))dx\\
&= \lim_{y\downarrow 0}\frac{1}{\pi}\mathrm{Im}(m(x_0+iy))
\end{aligned}$$

□

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Department of Mathematics
Hanseo University
Seosan Chungnam 356-820, Republic of Korea
E-mail: schoi@hanseo.ac.kr