JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 3, September 2009

A FUNCTIONAL CENTRAL LIMIT THEOREM FOR LINEAR RANDOM FIELD GENERATED BY NEGATIVELY ASSOCIATED RANDOM FIELD

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Abstract. We prove a functional central limit theorem for a linear random field generated by negatively associated multi-dimensional random variables. Under a finite second moment condition we extend the result in Kim, Ko and Choi[Kim,T.S, Ko,M.H and Choi, Y.K.,2008. The invariance principle for linear multi-parameter stochastic processes generated by associated fields. Statist. Probab. Lett. 78, 3298-3303] to the negatively associated case.

1. Introduction

Let p be a positive integer and let $\{\xi(t_1,\cdots,t_p),(t_1,\cdots,t_p)\in\mathbb{Z}^p\}$ a field of random variables defined on a probability space $(\Omega,\mathcal{F},\mathcal{P})$. The field $\{\xi(t_1,\cdots,t_p),(t_1,\cdots,t_p)\in\mathbb{Z}^p\}$ is called negatively associated (NA) if for every pair of disjoint subsets S,T of $\mathbb{Z}^p$ and any pair of coordinatewise increasing functions $f(\xi(t_1,\cdots,t_p),(t_1,\cdots,t_p)\in S), g(\xi(t_1,\cdots,t_p),(t_1,\cdots,t_p)\in T)$ with $Ef^2(\xi(t_1,\cdots,t_p),(t_1,\cdots,t_p)\in S)<\infty$ and $Eg^2(\xi(t_1,\cdots,t_p),(t_1,\cdots,t_p)\in T)<\infty$, it holds that $Cov(f(\xi(t_1,\cdots,t_p),(t_1,\cdots,t_p)\in S), g(\xi(t_1,\cdots,t_p),(t_1,\cdots,t_p)\in T))\le 0$. The concept of NA was introduced by Joag-Dev and Proschan(1983). As pointed out and proved by Joag-Dev and Proschan(1983), a number of well-known multivariate distributions possess the NA property, such as multinomial distribution, multivariate hypergeometric distribution, Dirichlet distribution, negatively correlated normal distribution, permutation distribution, and joint distribution of ranks. Because of their wide applications in multivariate statistical analysis and reliability theory, the concept of negatively associated random variables has received extensive attention

Received June 22, 2009; Revised August 19, 2009; Accepted August 21, 2009.
2000 Mathematics Subject Classification: Primary 60F05, 60G10.
Key words and phrases: functional central limit theorem, linear process, negative association, random field, maximal inequality.

recently. We refer to Joag-Dev and Proschan(1983) for fundamental properties. In the case of $p = 1$, we refer to Newman(1984) for the central limit theorem, Matula(1992) for the three series theorem, and Su et al.(1997) for a moment inequality and weak convergence. In the case of $p \geq 2$, Roussas(1994) studied the central limit theorems for weak stationary NA random fields, Zhang(2000) obtained the weak convergence for NA random fields with finite $2+\delta$th moment and Zhang and Wen(2001) proved the weak convergence for identically distributed NA fields with only finite second moment.

Let $W(\cdot, \cdots, \cdot)$ denote multi-parameter standard Brownian motion, *i.e.*, a zero-mean Gaussian process with covariance function satisfying $E[W(t_1, \cdots, t_p)W(s_1, \cdots, s_p)] = \Pi_{j=1}^{p} min(t_j, s_j)$, and let D_p be the space of "cadlag" functions from $[0,1]^p$ to $\mathbb{R}$; it is possible to introduce on D_p a metric topology which makes it complete and separable, and indeed D_p is the multi-dimensional analogue of the Skorohod space $D[0,1]$, see Straf(1970) or Poghosyan and Roelly(1998) for details. Without loss of generality, we can assume that $E\xi(t_1, \cdots, t_p) = 0$ for all $(t_1, \cdots, t_p) \in \mathbb{Z}^p$. Define, in the Skorohod space D_p, the partial sum process,

$$W_n(r_1, \cdots, r_p) = \frac{1}{\sigma n^{p/2}} \sum_{t_1=1}^{[nr_1]} \cdots \sum_{t_p=1}^{[nr_p]} \xi(t_1, \cdots, t_p), \tag{1.1}$$

where $0 \leq r_1, \cdots, r_p \leq 1$, $(t_1, \cdots, t_p) \in \mathbb{N}^p$, and $[.]$ the integer-part function and

$$\sigma^2 = \sum_{(t_1, \cdots, t_p) \in \mathbb{Z}^p} Cov(\xi(0, \cdots, 0), \xi(t_1, \cdots, t_p)) < \infty. \tag{1.2}$$

Proposition 1.1. *(Zhang and Wen, 2001) Let $\{\xi(t_1, \cdots, t_p), (t_1, \cdots, t_p) \in \mathbb{Z}^p\}$ be a field of stationary NA random variables with $E\xi(0, 0, \cdots, 0) = 0$ and $E\xi(0, \cdots, 0)^2 < \infty$. Then there exists a positive constant K, depending only on p, such that*

$$\limsup_{n\to\infty} n^{-p} E(\max_{k_1, \cdots, k_p \leq n} \sum_{t_1=1}^{k_1} \cdots \sum_{t_p=1}^{k_p} \xi(t_1, \cdots, t_p))^2 \leq KE\xi(0, 0, \cdots, 0)^2.$$

Theorem 1.2. *(Zhang and Wen, 2001) Let $\{\xi(t_1, \cdots, t_p), (t_1, \cdots, t_p) \in \mathbb{Z}^p\}$ be a field of stationary NA random variables with $E\xi(0, 0, \cdots, 0) = 0$ and $0 < E\xi(0, 0, \cdots, 0)^2 < \infty$. Then $W_n(r_1, \cdots, r_p) \Rightarrow W$ as $n \to \infty$, in the space D_p endowed with th Skorohod topology, where $\{W(r_1, \cdots, r_p), (r_1, \cdots, r_p) \in [0,1]^p\}$ is a p-dimensional standard Wiener process and $\Rightarrow$ means weak convergence in D_p.*

Marinucci and Poghosyan (2001) generalized a result known for the case of $p = 1$ as the Beveridge-Nelson decomposition(cf. Phillips and Solo, 1992) to the case of $p \geq 2$ and proved the following functional central limit theorem for linear random fields generated by a field of independent and identically distributed random variables $\{\xi(t_1, \cdots, t_p)\}$.

THEOREM 1.3. *(Kim, Ko and Choi, 2008) Let* $\{\xi(t_1, \cdots, t_p), (t_1, \cdots, t_p) \in \mathbb{Z}^p\}$ *be a field of identically distributed and associated random variables with* $E\xi(t_1, \cdots, t_p) = 0$, $0 < E\xi(t_1, \cdots, t_p)^2 < \infty$ *and* $E|\xi(t_1, \cdots, t_p)|^q < \infty,\ q > 2$ *and* $\{a(k_1, \cdots, k_p)\}$ *be a sequence of positive constants. Assume* $\sum_{i_1=0}^{\infty} \cdots \sum_{i_p=0}^{\infty} \sum_{k_1=i_1+1}^{\infty} \cdots \sum_{k_p=i_p+1}^{\infty} a(k_1, \cdots, k_p) < \infty$. *Then, for* $0 \leq r_1, \cdots, r_p \leq 1$

$$\frac{1}{\sigma n^{\frac{p}{2}}} \sum_{t_1=1}^{[nr_1]} \cdots \sum_{t_p=1}^{[nr_p]} u(t_1, \cdots, t_p) \Rightarrow A(1, \cdots, 1) W(r_1, \cdots, r_p) \text{ as } n \to \infty,$$

where σ^2 *is defined in (1.2),* $u(t_1, \cdots, t_p)$ *and* $a(i_1, \cdots, i_p)$ *are defined in (2.1) and (2.2) below.*

In this note we will extend Theorem 1.2 to the linear random field generated by a field of negatively associated random variables(see Theorem 3.2) as well as Theorem 1.3 to the negatively associated case with the finite second moment condition $E\xi(t_1, \cdots, t_p)^2 < \infty$.

2. Preliminaries

Define a linear random field by

(2.1)
$$u(t_1, \cdots, t_p) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_p=0}^{\infty} a(i_1, \cdots, i_p) \xi(t_1 - i_1, \cdots, t_p - i_p), (t_1, \cdots, t_p) \in \mathbb{Z}^p,$$

where $\{\xi(t_1, \cdots, t_p)\}$ is a field of identically distributed NA random variables with $E\xi(t_1, \cdots, t_p) = 0$ and $0 < E(\xi(t_1, \cdots, t_p))^2 < \infty$.

Put

(2.2)
$$A(x_1, \cdots, x_p) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_p=0}^{\infty} a(i_1, \cdots, i_p) x_1^{i_1} \cdots x_p^{i_p},\ (x_1, \cdots, x_p) \in \mathbb{R}^p,$$

and assume that $|x_i| \leq 1,\ i = 1, \cdots, p$,

$$a(i_1, \cdots, i_p) > 0, \tag{2.3}$$

$$(2.4)\qquad \sum_{i_1=0}^{\infty}\cdots\sum_{i_p=0}^{\infty}\sum_{k_1=i_1+1}^{\infty}\cdots\sum_{k_p=i_p+1}^{\infty} a(k_1,\cdots,k_p)<\infty,$$

which implies $A(1,\cdots,1)=\sum_{i_1=0}^{\infty}\cdots\sum_{i_p=0}^{\infty} a(i_1,\cdots,i_p)<\infty$.

Marinucci and Poghosyan(2001) generalized a result known for the case of $p=1$ as the Beveridge-Nelson decomposition as follow:

LEMMA 2.1. *(Marinucci and Poghosyan, 2001) Let Γ_p be the class of all 2^p subsets γ of $\{1,2,\cdots,p\}$. Let $y_i=x_i$ if $j\in\gamma$ and $y_i=1$ if $j\notin\gamma$. Then we have*

$$A(x_1,\cdots,x_p)=\sum_{\gamma\in\Gamma_p}\{\Pi_{j\in\gamma}(x_j-1)\}A_\gamma(y_1,\cdots,y_p),$$

where it is assumed that $\Pi_{j\in\phi}=1$, and

$$(2.5)\qquad A_\gamma(y_1,\cdots,y_p)=\sum_{i_1=0}^{\infty}\cdots\sum_{i_p=0}^{\infty} a_\gamma(i_1,\cdots,i_p)y_1^{i_1}\cdots y_p^{i_p},$$

$$(2.6)\qquad a_\gamma(i_1,\cdots,i_p)=\sum_{s_1=i_1+1}^{\infty}\cdots\sum_{s_p=i_p+1}^{\infty} a(s_1,\cdots,s_p),$$

where the sums go over indices $s_j, j\in\gamma$, where as $s_j=i_j$ if $j\notin\gamma$.

Marinucci and Poghosyan(2001) also considered the partial backshift operator satisfying

$$(2.7)\quad B_i\xi(t_1,\cdots,t_i,\cdots,t_p)=\xi(t_1,\cdots,t_i-1,\cdots,t_p),\ i=1,2,\cdots,p,$$

which enables us to write (2.1) more compactly as

$$(2.8)\qquad \begin{aligned} u(t_1,\cdots,t_p)&=\sum_{i_1=0}^{\infty}\cdots\sum_{i_p=0}^{\infty} a(i_1,\cdots,i_p)B_1^{i_1}\cdots B_p^{i_p}\xi(t_1,\cdots,t_p)\\ &=A(B_1,\cdots,B_p)\xi(t_1,\cdots,t_p),\end{aligned}$$

where

$$A(B_1,\cdots,B_p)=\sum_{i_1=0}^{\infty}\cdots\sum_{i_p=0}^{\infty} a(i_1,\cdots,i_p)B_1^{i_1}\cdots B_p^{i_p}.$$

The above ideas shall be exploited here to establish the functional central limit theorem for the linear random fields. To this aim, write

$$(2.9)\qquad \xi_\gamma(t_1,\cdots,t_p)=A_\gamma(L_1,\cdots,L_p)\xi(t_1,\cdots,t_p),$$

where the operator L_i is defined as $L_i = B_i$ for $i \in \gamma$, $L_i = 1$ otherwise(see Marinucci and Poghosyan(2001) for more details).

REMARK 2.2. 2.1 Note that $0 \leq \sum_{i_1=0}^{\infty} \cdots \sum_{i_p=0}^{\infty} a_\gamma(i_1, \cdots, i_p) < \infty$. 2.2 Note that $\xi_\gamma(t_1, \cdots, t_p) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_p=0}^{\infty} a_\gamma(i_1, \cdots, i_p)\xi(t_1 - i_1, \cdots, t_p - i_p)$ and that $\xi_\gamma(t_1, \cdots, t_p)^{'}$s are negatively associated by the properties of negative association(see Joag-Dev and Proschan, 1983).

3. A functional central limit theorem

To prove Theorem 3.2 we need the following lemma.

LEMMA 3.1. *Let $\{\xi(t_1, \cdots, t_p)\}$ be a field of identically distributed NA random variables $E\xi(t_1, \cdots, t_p) = 0$, and $0 < E(\xi(t_1, \cdots, t_p))^2 < \infty$. Assume that (2.3) and (2.4) hold. Then*

$$E(\xi_\gamma(t_1, \cdots, t_p))^2 < \infty \ for\ \gamma \in \Gamma_P \tag{3.1}$$

Proof. From Remarks in Section 2 we have

$$\begin{aligned} \xi_\gamma(0, \cdots, 0) &= \sum_{i_1=0}^{\infty} \cdots \sum_{i_p=0}^{\infty} a_\gamma(i_1, \cdots, i_p)\xi(-i_1, \cdots, -i_p) \\ &= \sum_{i=0}^{\infty} a_\gamma(\phi(i))\xi(-\phi(i)), \end{aligned}$$

where $\phi : \mathbb{Z} \to \mathbb{Z}^p$ and $\{\xi(-\phi(i))\}$ is a sequence of identically distributed NA random variables. Hence,

$$\begin{aligned} E(\xi_\gamma(t_1, \cdots, t_p))^2 &= E(\xi_\gamma(0, \cdots, 0))^2 \\ &= E(\sum_{i=0}^{\infty} a_\gamma(\phi(i))\xi(-\phi(i)))^2 \\ &= [\{E(\sum_{i=0}^{\infty} a_\gamma(\phi(i))\xi(-\phi(i)))^2\}^{\frac{1}{2}}]^2 \\ &\leq [\sum_{i=0}^{\infty} a_\gamma(\phi(i))(E(\xi(-\phi(i)))^2)^{\frac{1}{2}}]^2 \\ &\leq C[\sum_{i=0}^{\infty} a_\gamma(\phi(i))]^2 \\ &< \infty \text{ by Remark 2.1.} \end{aligned}$$

Note that the above first inequality follows from Minkowski's inequality. □

THEOREM 3.2. *Let* $u(t_1, \cdots, t_p)$ *be defined as in (2.1) and* $\{\xi(t_1, \cdots, t_p), (t_1, \cdots, t_p) \in \mathbb{Z}^p\}$ *a field of the identically distributed NA random variables with* $E\xi(t_1, \cdots, t_p) = 0$ *and* $0 < E(\xi(t_1, \cdots, t_p))^2 < \infty$. *Assume that (2.3) and (2.4) hold. Then, for* $0 \le r_1, \cdots, r_p \le 1$

$$\frac{1}{\sigma n^{p/2}} \sum_{t_1=1}^{[nr_1]} \cdots \sum_{t_p=1}^{[nr_p]} u(t_1, \cdots, t_p) \Rightarrow A(1, \cdots, 1) W(r_1, \cdots, r_p) \tag{3.2}$$

where $\sigma^2 = \sum_{(t_1, \cdots, t_p) \in \mathbb{Z}^p} Cov(\xi(0, \cdots, 0), \xi(t_1, \cdots, t_p)) < \infty$ *and* $\Rightarrow$ *means weak convergence.*

Proof. See Appendix. □

COROLLARY 3.3. *Let* $u(t_1, \cdots, t_p)$ *satisfy model (2.1) and* $\{\xi(t_1, \cdots, t_p)\}$ *a field of identically distributed NA random variables with* $E\xi(t_1, \cdots, t_p) = 0$ *and* $E(\xi(t_1, \cdots, t_p))^2 < \infty$. *If* $a(i_1, \cdots, i_p) = 1$ *for* $i_1 = \cdots = i_p = 0$, $a(i_1, \cdots, i_p) = 0$ *otherwise, then (3.2) holds.*

REMARK 3.4. Corollary 3.3 is a special case of Theorem 3.2. Hence Theorem 3.2 is an extension of Theorem 1.2.

EXAMPLE 3.5. *Let* $\{\xi(t_1, t_2), (t_1, t_2) \in \mathbb{Z}^2\}$ *be a field of identically distributed NA random variables with* $E\xi(0, 0) = 0$ and $0 < E\xi(0, 0)^2 < \infty$. *Let*

$$\begin{aligned} u(t_1, t_2) &= \xi(t_1, t_2) + \xi(t_1 - 1, t_2) + \xi(t_1, t_2 - 1) + \xi(t_1 - 1, t_2 - 1) \\ &= A(B_1, B_2)\xi(t_1, t_2) \text{ for } A(B_1, B_2) = 1 + B_1 + B_1 B_2 + B_2^2. \end{aligned}$$

Then, by Theorem 3.2

$$(\sigma n)^{-1} \sum_{t_1=1}^{[nr_1]} \sum_{t_2=1}^{[nr_2]} u(t_1, t_2) \Rightarrow 4W(r_1, r_2),$$

where $\sigma^2 = \sum_{(t_1, t_2) \in \mathbb{Z}^2} Cov(\xi(0, 0), \xi(t_1, t_2)) < \infty$.

COROLLARY 3.6. *Let* $\{\xi(t_1, \cdots, t_p), (t_1, \cdots, t_p) \in \mathbb{Z}^p\}$ *be a field of i.i.d. random variables with* $E\xi(t_1, \cdots, t_p) = 0$ *and* $0 < E\xi(t_1, \cdots, t_p)^2 = \sigma^2 < \infty$. *If (2.3) and (2.4) hold, then, for* $0 \le r_1, \cdots, r_p \le 1$

$$\frac{1}{\sigma n^{\frac{p}{2}}} \sum_{t_1=1}^{[nr_1]} \cdots \sum_{t_p=1}^{[nr_p]} u(t_1, \cdots, t_p) \Rightarrow A(1, \cdots, 1) W(r_1, \cdots, r_p) \text{ as } n \to \infty,$$

where $\Rightarrow$ *denotes weak convergence in* D_p.

From Theorem 3.2 we obtain sufficient conditions so that $\sum_{i_1=0}^{[nr_1]} \cdots \sum_{i_p=1}^{[nr_p]} u(t_1, \cdots, t_p)$*(properly normalized) converges weakly to Wiener measure if the corresponding result for* $\sum_{t_1=1}^{[nr_1]} \cdots \sum_{t_p=1}^{[nr_p]} \xi(t_1, \cdots, t_p)$ *is true.*

Corollary 3.7. *Let* $(t_1, \cdots, t_p)$ *be defined as in (2.1) and* $\{\xi(t_1, \cdots, t_p), (t_1, \cdots, t_p) \in \mathbb{Z}^p\}$ *a field of the identically distributed NA random variables with* $E\xi(t_1, \cdots, t_p) = 0$ *and* $0 < E\xi(t_1, \cdots, t_p)^2 < \infty$*. Assume that (2.3) and (2.4) hold. If, for* $0 \le r_1, \cdots, r_p \le 1$ $\frac{1}{\sigma n^{\frac{p}{2}}} \sum_{t_1=1}^{[nr_1]} \cdots \sum_{t_p=1}^{[nr_p]} \xi(t_1, \cdots, t_p) \Rightarrow W(r_1, \cdots, r_p)$ *holds, then (3.2) holds.*

Appendix

Proof of Theorem 3.2 The proof of this theorem is obtained by the similar method to the proof of Theorem 3.1 in Kim, Ko and Choi(2008) only taking $q = 2$. For completeness we repeat it here. From Theorem 1.2 we have

$$\frac{1}{\sigma n^{p/2}} \sum_{t_1=1}^{[nr_1]} \cdots \sum_{t_p=1}^{[nr_p]} \xi(t_1, \cdots, t_p) \Rightarrow W(r_1, \cdots, r_p). \tag{A.1}$$

From Proposition 1.1 and Lemma 3.1, we have some constant $C > 0$ such that

$$\limsup_{n\to\infty} n^{-p} E \max_{t_1, \cdots, t_p \le n} \left(\sum \xi_\gamma(t_1, \cdots, t_p)\right)^2 \le C. \tag{A.2}$$

We start from the case of $p = 2$, where we provide full details; the extension to the case of $p > 2$ is discussed afterwards. If we apply Lemma 2.1 to the backshift polynomial $A(B_1, \cdots, B_p)$, we find that the following a.s. equality holds:

$$u(t_1, t_2) = A(1, 1)\xi(t_1, t_2) + (B_1 - 1)A_1(B_1, 1)\xi(t_1, t_2)$$
$$+(B_2 - 1)A_2(1, B_2)\xi(t_1, t_2) + (B_1 - 1)(B_2 - 1)A_{12}(B_1, B_2)\xi(t_1, t_2)$$

which implies that, for $0 \le r_1, r_2 \le 1$

$$\sum_{t_1=1}^{[nr_1]} \sum_{t_2=1}^{[nr_2]} u(t_1, t_2) = \sum_{t_1=1}^{[nr_1]} \sum_{t_2=1}^{[nr_2]} A(1, 1)\xi(t_1, t_2) - \sum_{t_2=1}^{[nr_2]} \xi_1([nr_1], t_2) + \sum_{t_2=1}^{[nr_2]} \xi_1(0, t_2)$$

$$- \sum_{t_1=1}^{[nr_1]} \xi_2(t_1, [nr_2]) + \sum_{t_1=1}^{[nr_1]} \xi_2(t_1, 0) - \xi_{12}(0, [nr_2]) + \xi_{12}(0, 0) \tag{A.3}$$

$$-\xi_{12}([nr_1],0)+\xi_{12}([nr_1],[nr_2])$$

$$=\sum_{t_1=1}^{[nr_1]}\sum_{t_2=1}^{[nr_2]}A(1,1)\xi(t_1,t_2)+R_n(r_1,r_2).$$

Note that $\xi_1(\cdot,\cdot)$, $\xi_2(\cdot,\cdot)$ and $\xi_{12}(\cdot,\cdot)$ are negatively associated(see Remarks in Section 2).

From Markov's inequality, Proposition 1.1 and Lemma 3.1, for $0 \le r_1, r_2 \le 1$,

$$P\{\max_{0\le r_1,r_2\le 1} n^{-1}\sum_{t_2=1}^{[nr_2]}\xi_1([nr_1],t_2)>\delta\}$$

$$\le \frac{E\max_{0\le r_1,r_2\le 1}(\sum_{t_2=1}^{[nr_2]}\xi_1([nr_1],t_2))^2}{n^2\delta^2}$$

$$\le Cn^{-1}=o(1) \tag{A.4}$$

as $n\to\infty$. We can apply exactly the same argument to establish also

$$P\{\max_{0\le r_1,r_2\le 1} n^{-1}\sum_{t_1=1}^{[nr_1]}\xi_2(t_1,[nr_2])>\delta\}=o(1)\ as\ n\to\infty. \tag{A.5}$$

By Lemma 3.1 we have for $0\le r_1,r_2\le 1$ $E(\xi_{12}([nr_1],[nr_2]))^2<\infty$ and hence by the same argument as above we also have

$$P\{\max_{0\le r_1,r_2\le 1} n^{-1}\xi_{12}([nr_1],[nr_2])>\delta\}=o(1)\ as\ n\to\infty. \tag{A.6}$$

Thus,

$$n^{-1}\sum_{t_1=1}^{[nr_1]}\sum_{t_2=1}^{[nr_2]}u(t_1,t_2)=n^{-1}\sum_{t_1=1}^{[nr_1]}\sum_{t_2=1}^{[nr_2]}A(1,1)\xi(t_1,t_2)+n^{-1}R_n(r_1,r_2),$$

$$\sup_{0\le r_1,r_2\le 1}|n^{-1}R_n(r_1,r_2)|=o_p(1),$$

which implies

$$(\sigma n)^{-1}\sum_{t_1=1}^{[nr_1]}\sum_{t_2=1}^{[nr_2]}u(t_1,t_2)\Rightarrow A(1,1)W(r_1,r_2)\ as\ n\to\infty.$$

In the case of $p > 2$, the argument is analogous; we have

$$(A.7) \quad \sum_{t_1=1}^{[nr_1]} \cdots \sum_{t_p=1}^{[nr_p]} u(t_1, \cdots, t_p) = A(1, \cdots, 1) \sum_{t_1=1}^{[nr_1]} \cdots \sum_{t_p=1}^{[nr_p]} \xi(t_1, \cdots, t_p) + R_n(r_1, \cdots, r_p)$$

where

$$(A.8) \quad R_n(r_1, \cdots, r_p) = \sum_{\gamma \in \Gamma_p, \gamma \neq \phi} \{\Pi_{j \in \gamma}(B_j - 1)\} \sum_{t_1=1}^{[nr_1]} \cdots \sum_{t_p=1}^{[nr_p]} A_\gamma(L_1, \cdots, L_p) \xi(t_1, \cdots, t_p)$$

with L_i defined as in (2.9); note that for $j \in \gamma$

$$(A.9) \quad \begin{aligned} \textstyle\sum_{t_j=1}^{[nr_j]} & (B_j - 1) A_\gamma(L_1, \cdots, L_p) \xi(t_1, \cdots, t_p) \\ = & \sum_{t_j=1}^{[nr_j]} A_\gamma(L_1, \cdots, L_p) \xi(t_1, \cdots, t_j - 1, \cdots, t_p) \\ & - \sum_{t_j=1}^{[nr_j]} A_\gamma(L_1, \cdots, L_p) \xi(t_1, \cdots, t_p) \\ = & A_\gamma(L_1, \cdots, L_p) \xi(t_1, \cdots, 0, \cdots, t_p) \\ & - A_\gamma(L_1, \cdots, L_p) \xi(t_1, \cdots, [nr_j], \cdots, t_p). \end{aligned}$$

Thus the right-hand side of (A.8) can be written more explicitly as

$$(A.10) \quad \begin{aligned} & \sum_{t_2=1}^{[nr_2]} \sum_{t_3=1}^{[nr_3]} \cdots \sum_{t_p=1}^{[nr_p]} A_1(B_1, \cdots, 1) \xi(0, \cdots, t_p) \\ & - \sum_{t_2=1}^{[nr_2]} \sum_{t_3=1}^{[nr_3]} \cdots \sum_{t_p=1}^{[nr_p]} A_1(B_1, \cdots, 1) \xi(n_1, \cdots, t_p) \\ & + \sum_{t_1=1}^{[nr_1]} \sum_{t_3=1}^{[nr_3]} \cdots \sum_{t_p=1}^{[nr_p]} A_2(1, B_2, \cdots, 1) \xi(0, \cdots, t_p) \\ & - \sum_{t_1=1}^{[nr_1]} \sum_{t_3=1}^{[nr_3]} \cdots \sum_{t_p=1}^{[nr_p]} A_2(1, B_2, \cdots, 1) \xi(n_1, \cdots, t_p) + \cdots \end{aligned}$$

$$+A_{12\cdots p}(B_1,\cdots,B_p)\xi(0,\cdots,0)$$
$$-A_{12\cdots p}(B_1,\cdots,B_p)\xi(0,\cdots,n_p)$$
$$-A_{12\cdots p}(B_1,\cdots,B_p)\xi(n_1,\cdots,0)+\cdots$$
$$+A_{12\cdots p}(B_1,\cdots,B_p)\xi(n_1,\cdots,n_p)$$

where in view of (A.9) the sums corresponding to each $A_\gamma(\cdot,\cdots,\cdot)$ run over t_i such that $i \notin \gamma$. Now

$$\frac{1}{\sigma n^{p/2}}A(1,\cdots,1)\sum_{t_1=1}^{[nr_1]}\cdots\sum_{t_p=1}^{[nr_p]}\xi(t_1,\cdots,t_p) \Rightarrow A(1,1,\cdots,1)W(r_1,\cdots,r_p)$$

by Theorem 1.2, so it is sufficient to prove that

$$(A.11) \qquad \sup_{0\le r_1,\cdots,r_p\le 1}|n^{-p/2}R_n(r_1,\cdots,r_p)| = o_p(1)$$

by Theorem 4.1 in Billingsley(1968). Considering for instance the first term on the right-hand side of (A.10), note that $\xi_1(t_1,\cdots,t_p)$ are negatively associated for different values of $t_1,\cdots,t_p$. Thus, for $0\le r_1,\cdots,r_p\le 1$, we have, from the same argument as for the case of $p=2$ and from Proposition 1.1 and Lemma 3.1

$$(A.12) \qquad P\{\max_{0\le r_1,\cdots,r_p\le 1} n^{-p/2}\sum_{t_1=1}^{[nr_1]}\cdots\sum_{t_p=1}^{[nr_p]}\xi_1([nr_1],\cdots,t_p) > \delta\}$$

$$\le Cn^{-p}\, n^{(p-1)} = Cn^{-1} = o(1) \ as\ n\to\infty.$$

More generally, let $\sharp(\gamma)$ denote the cardinality of γ; each other term in (A.11) is $n^{-p/2}$ times a partial sum of $n^{p-\sharp(\gamma)}$ elements, and we can apply iteratively the same argument to complete the proof.

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