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ANALYTIC FOURIER-FEYNMAN TRANSFORM AND CONVOLUTION OF FUNCTIONALS IN A GENERALIZED FRESNEL CLASS

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Abstract. Huffman, Park and Skoug introduced various results for the L_p analytic Fourier-Feynman transform and the convolution for functionals on classical Wiener space which belong to some Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick. Also Chang, Kim and Yoo extended the above results to an abstract Wiener space for functionals in the Fresnel class $\mathcal{F}(B)$ which corresponds to $\mathcal{S}$. Moreover they introduced the L_p analytic Fourier-Feynman transform for functionals on a product abstract Wiener space and then established the above results for functionals in the generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ containing $\mathcal{F}(B)$.

In this paper, we investigate more generalized relationships, between the Fourier-Feynman transform and the convolution product for functionals in $\mathcal{F}_{A_1,A_2}$, than the above results.

1. Introduction

The concept of an L_1 analytic Fourier-Feynman transform for functionals on classical Wiener space $(C_0[0,T],m)$ was introduced by Brue in [2]. In [3], Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform on classical Wiener space. In [13], Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \le p \le 2$ that extended the results in [2,3] and gave various relationships between the L_1 and L_2 theories. Also Huffman, Park and Skoug defined a convolution product for functionals on classical Wiener space and they obtained various results on the Fourier-Feynman transform and the convolution product [10,11,12]. In [19], Park, Skoug and Storvick

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investigated various relationships among the first variation, the convolution product and the Fourier-Feynman transform for functionals on classical Wiener space which belong to the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [4].

The concept of abstract Wiener space (H, B, ν) was introduced by Gross in [9]. Also Lee [17,18] established the Fourier-Wiener transform (Fourier-Feynman transform) theory on abstract Wiener space and applied this transform to differential equations on infinite dimensional spaces. Also Chang, Kim and Yoo [7] obtained the relationships among the Fourier-Feynman transform, the convolution and the first variation for functionals in the Fresnel class $\mathcal{F}(B)$ which corresponds to the Banach algebra $\mathcal{S}$. Moreover they [6] introduced an L_p analytic Fourier-Feynman transform for functionals on a product abstract Wiener space and established the relationships between the Fourier-Feynman transform and the convolution for functionals in a generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ containing $\mathcal{F}(B)$ introduced by Kallianpur and Bromley [14].

In this paper, we shall continue to study the L_p analytic Fourier-Feynman transform and convolution for functionals on abstract Wiener space [6]. In particular, we investigate more generalized relationships, between the Fourier-Feynman transform and the convolution product for functionals in the generalized Fresnel class $\mathcal{F}_{A_1,A_2}$, than those in [6].

2. Preliminaries

Let (H, B, ν) be an abstract Wiener space and let $\{e_j\}$ be a complete orthonormal system in H such that the e_j's are in B^*, the dual of B. For each $h \in H$ and $x \in B$, we define a stochastic inner product $(h, x)^\sim$ as follows:

$$(2.1) \qquad (h,x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \langle h, e_j\rangle (x, e_j), & \text{if the limit exists} \\ 0, & \text{otherwise,} \end{cases}$$

where $(\cdot,\cdot)$ denotes the natural dual pairing between B and B^*. It is well known [14,15] that for each $h(\neq 0)$ in H, $(h,\cdot)^\sim$ is a Gaussian random variable on B with mean zero and variance $|h|^2$, that is,

$$(2.2) \qquad \int_B \exp\{i(h,x)^\sim\}\, d\nu(x) = \exp\Big\{-\frac{1}{2}|h|^2\Big\}.$$

A subset E of a product abstract Wiener space B^2 is said to be scale-invariant measurable provided $\{(\alpha x_1, \beta x_2) : (x_1, x_2) \in E\}$ is abstract Wiener measurable for every $\alpha > 0$ and $\beta > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $(\nu \times \nu)(\{(\alpha x_1, \beta x_2) : (x_1, x_2) \in N\}) = 0$ for every $\alpha > 0$ and $\beta > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equals s-a.e., we write $F \approx G$. For more details, see [6,7,9,14,15,16,21].

Let $\mathbb{C}$ denote the complex numbers and let

$$\Omega = \{\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \operatorname{Re}\lambda_k > 0 \text{ for } k = 1, 2\}$$

and

$$\tilde{\Omega} = \{\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \lambda_k \neq 0, \operatorname{Re}\lambda_k \geq 0 \text{ for } k = 1, 2\}.$$

Let F be a complex-valued function on B^2 such that the integral

$$J_F(\lambda_1, \lambda_2) = \int_{B^2} F(\lambda_1^{-1/2} x_1, \lambda_2^{-1/2} x_2)\, d(\nu \times \nu)(x_1, x_2)$$

exists as a finite number for all real numbers $\lambda_1 > 0$ and $\lambda_2 > 0$. If there exists a function $J_F^*(\lambda_1, \lambda_2)$ analytic on Ω such that $J_F^*(\lambda_1, \lambda_2) = J_F(\lambda_1, \lambda_2)$ for all $\lambda_1 > 0$ and $\lambda_2 > 0$, then $J_F^*(\lambda_1, \lambda_2)$ is defined to be the analytic Wiener integral of F over B^2 with parameter $\vec{\lambda} = (\lambda_1, \lambda_2)$, and for $\vec{\lambda} \in \Omega$ we write

$$\int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1, x_2)\, d(\nu \times \nu)(x_1, x_2) = J_F^*(\lambda_1, \lambda_2).$$

Let q_1 and q_2 be nonzero real numbers and F be a functional on B^2 such that $\int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1, x_2)\, d(\nu \times \nu)(x_1, x_2)$ exists for all $\vec{\lambda} \in \Omega$. If the following limit exists, then we call it the analytic Feynman integral of F over B^2 with parameter $\vec{q} = (q_1, q_2)$ and we write

$$\begin{aligned}&\int_{B^2}^{\mathrm{anf}_{\vec{q}}} F(x_1, x_2)\, d(\nu \times \nu)(x_1, x_2)\\ &= \lim_{\vec{\lambda} \to -i\vec{q}} \int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1, x_2)\, d(\nu \times \nu)(x_1, x_2),\end{aligned}$$

where $\vec{\lambda} = (\lambda_1, \lambda_2)$ approaches $-i\vec{q} = (-iq_1, -iq_2)$ through Ω.

NOTATION 2.1. (i) *For* $\vec{\lambda} = (\lambda_1, \lambda_2) \in \Omega$ *and* $(y_1, y_2) \in B^2$, *let*

$$(2.3)\qquad (T_{\vec{\lambda}}(F))(y_1, y_2) = \int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1 + y_1, x_2 + y_2)\, d(\nu \times \nu)(x_1, x_2).$$

(ii) *Let $1 < p < \infty$ and let $\{G_n\}$ and G be scale-invariant measurable functionals such that, for each $\alpha > 0$ and $\beta > 0$,*

$$\lim_{n\to\infty} \int_{B^2} |G_n(\alpha x_1, \beta x_2) - G(\alpha x_1, \beta x_2)|^{p'} \, d(\nu \times \nu)(x_1, x_2) = 0, \tag{2.4}$$

where p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$. Then we write

$$\underset{n\to\infty}{\mathrm{l.\,i.\,m.}}(w_s^{p'})(G_n) \approx G \tag{2.5}$$

and call G the scale-invariant limit in the mean of order p'. A similar definition is understood when n is replaced by the continuously varying parameter $\vec{\lambda}$.

Definition 2.2. Let q_1 and q_2 be nonzero real numbers. For $1 < p < \infty$, we define the L_p analytic Fourier-Feynman transform $T_{\vec{q}}^{(p)}(F)$ of F on B^2 by the formula ($\vec{\lambda} \in \Omega$)

$$(T_{\vec{q}}^{(p)}(F))(y_1, y_2) = \underset{\vec{\lambda}\to -i\vec{q}}{\mathrm{l.\,i.\,m.}}(w_s^{p'})(T_{\vec{\lambda}}(F))(y_1, y_2), \tag{2.6}$$

whenever this limit exists. We define the L_1 analytic Fourier-Feynman transform $T_{\vec{q}}^{(1)}(F)$ of F by ($\vec{\lambda} \in \Omega$)

$$(T_{\vec{q}}^{(1)}(F))(y_1, y_2) = \lim_{\vec{\lambda}\to -i\vec{q}} (T_{\vec{\lambda}}(F))(y_1, y_2), \tag{2.7}$$

for s-a.e. $(y_1, y_2) \in B^2$.

Let $M(H)$ denote the space of complex-valued countably additive Borel measures on H. Under the total variation norm $\|\cdot\|$ and with convolution as multiplication, $M(H)$ is a commutative Banach algebra with identity [1].

Now we state the generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ introduced by Kallianpur and Bromley [14]. Let A_1 and A_2 be bounded, non-negative self-adjoint operators on H. Let $\mathcal{F}_{A_1,A_2}$ be the space of all s-equivalence classes of functionals F on B^2 which have the form

$$F(x_1, x_2) = \int_H \exp\Big\{ i \sum_{j=1}^{2} (A_j^{1/2} h, x_j)^{\sim} \Big\} \, d\sigma(h) \tag{2.8}$$

for some complex-valued countably additive Borel measure σ on H.

As is customary, we will identify a functional with its s-equivalence class and think of $\mathcal{F}_{A_1,A_2}$ as a collection of functionals on B^2 rather than as a collection of equivalence classes. Moreover the map $\sigma \mapsto [F]$ defined by (2.8) sets up an algebra isomorphism between $M(H)$ and $\mathcal{F}_{A_1,A_2}$ if the range of $A_1 + A_2$ is dense in H. In this case, $\mathcal{F}_{A_1,A_2}$ becomes a Banach algebra under the norm $\|F\| = \|\sigma\|$ [14].

REMARK 2.3. Let $\mathcal{F}(B)$ denote the Fresnel class of functions F on B of the form

$$F(x) = \int_H \exp\{i(h,x)^{\sim}\}\,d\sigma(h) \tag{2.9}$$

for some $\sigma \in M(H)$. If A_1 is the identity operator on H and $A_2 = 0$, then $\mathcal{F}_{A_1,A_2}$ is essentially the Fresnel class $\mathcal{F}(B)$.

The following theorems are well known results in [6] which play an important role in this paper. We now state them without proof. In [6], the authors restricted to the case where $1 \le p \le 2$. But concerning to the functionals in $\mathcal{F}_{A_1,A_2}$, it is easy to see that the results can be extended to the case where $1 \le p < \infty$.

THEOREM 2.4. *Let $F \in \mathcal{F}_{A_1,A_2}$ be given by (2.8) with $\sigma \in M(H)$ and let $1 \le p < \infty$. Then, for all $\vec{q} = (q_1, q_2)$ with nonzero real numbers q_1 and q_2, the analytic Fourier-Feynman transform $T_{\vec{q}}^{(p)}(F)$ exists, belongs to $\mathcal{F}_{A_1,A_2}$ and is given by the formula*

$$(T_{\vec{q}}^{(p)}(F))(y_1,y_2) = \int_H \exp\Big\{i\sum_{j=1}^{2}(A_j^{1/2}h, y_j)^{\sim}\Big\}\,d\hat{\sigma}(h) \tag{2.10}$$

for s-a.e. $(y_1,y_2) \in B^2$ where $\hat{\sigma} \in M(H)$ is defined by

$$\hat{\sigma}(E) = \int_E \exp\Big\{-\sum_{j=1}^{2}\frac{i}{2q_j}|A_j^{1/2}h|^2\Big\}\,d\sigma(h) \tag{2.11}$$

for $E \in \mathcal{B}(H)$.

REMARK 2.5. (i) We adopt the convention $\frac{1}{\pm\infty} = 0$ throughout this paper. Thus if $q_1 = q_2 = \pm\infty$, then $T_{\vec{q}}^{(p)}(F)$ is F itself for $\vec{q} = (q_1, q_2)$.

(ii) For nonzero real numbers q_1 and q_2, we have

$$(T_{\vec{q}}^{(1)}(F))(y_1,y_2) = \int_{B^2}^{\text{anf}_{\vec{q}}} F(x_1+y_1, x_2+y_2)\,d(\nu\times\nu)(x_1,x_2)$$

where $\vec{q} = (q_1, q_2)$ and $(y_1, y_2) \in B^2$. In particular, if $F \in \mathcal{F}_{A_1,A_2}$, then

$$(T_{\vec{q}}^{(p)}(F))(0,0) = \int_{B^2}^{\text{anf}_{\vec{q}}} F(x_1,x_2)\,d(\nu\times\nu)(x_1,x_2)$$

for $1 \le p < \infty$.

DEFINITION 2.6. Let F and G be functionals on B^2. For $\vec{q} = (q_1, q_2)$ with nonzero real numbers q_1 and q_2, we define their convolution product (if it exists) by

$$(2.12)\ (F * G)_{\vec{q}}(y_1, y_2) = \int_{B^2}^{\text{anf}_{\vec{q}}} F\Big(\frac{y_1 + x_1}{\sqrt{2}}, \frac{y_2 + x_2}{\sqrt{2}}\Big)\, G\Big(\frac{y_1 - x_1}{\sqrt{2}}, \frac{y_2 - x_2}{\sqrt{2}}\Big) d(\nu \times \nu)(x_1, x_2).$$

THEOREM 2.7. *Let F and G be elements of $\mathcal{F}_{A_1,A_2}$ with corresponding finite Borel measures σ and ρ in $M(H)$ respectively. Then, for all $\vec{q} = (q_1, q_2)$ with nonzero real numbers q_1 and q_2, the convolution product $(F * G)_{\vec{q}}$ exists, belongs to $\mathcal{F}_{A_1,A_2}$ and is given by the formula*

$$(2.13)\qquad (F * G)_{\vec{q}}(y_1, y_2) = \int_{H^2} \exp\Big\{ i \sum_{j=1}^{2} (A_j^{1/2} h, y_j)^{\sim} \Big\}\, d\eta(h)$$

for s-a.e. $(y_1, y_2) \in B^2$ where $\eta = \mu \circ \phi^{-1} \in M(H)$ is defined by

$$(2.14)\qquad \mu(E) = \int_E \exp\Big\{ - \sum_{j=1}^{2} \frac{i}{4q_j} |A_j^{1/2}(h - k)|^2 \Big\}\, d\sigma(h)\, d\rho(k)$$

for $E \in \mathcal{B}(H)$ and $\phi : H^2 \to H$ is the Borel measurable function defined by $\phi(h, k) = \frac{1}{\sqrt{2}}(h + k)$.

3. Fourier-Feynman transform and convolution for functionals in a generalized Fresnel class

In this section, we investigate more generalized relationships, between the Fourier-Feynman transform and the convolution product for functionals in the generalized Fresnel class $\mathcal{F}_{A_1,A_2}$, than those in [6].

THEOREM 3.1. *Let $F \in \mathcal{F}_{A_1,A_2}$ be given by (2.8) with $\sigma \in M(H)$ and let $1 \le p < \infty$. Let q_{k1} and q_{k2} $(k = 1, 2)$ be in $\mathbb{R}^{\#} - \{0\}$ where $\mathbb{R}^{\#}$ is the set of extended real numbers. Then*

$$(3.1)\qquad T_{\vec{q}_2}^{(p)}(T_{\vec{q}_1}^{(p)}(F)) \approx T_{\vec{q}}^{(p)}(F)$$

where $\vec{q}_k = (q_{k1}, q_{k2})$, $\vec{q} = (q_1, q_2)$, and q_1 and q_2 are extended real numbers such that $\frac{1}{q_{1k}} + \frac{1}{q_{2k}} = \frac{1}{q_k}$ for $k = 1, 2$.

Proof. As mentioned in Theorem 2.4, $T_{\vec{q}_1}^{(p)}(F)$ belongs to $\mathcal{F}_{A_1,A_2}$ and is given by (2.10) with $\vec{q}$ replaced by $\vec{q}_1$. Applying Theorem 2.4 to the expression of $T_{\vec{q}_1}^{(p)}(F)$ and using (2.11), we obtain

$$
\begin{aligned}
(3.2)\quad & T_{\vec{q}_2}(T_{\vec{q}_1}^{(p)}(F))(y_1,y_2) \\
&= \int_H \exp\Big\{\sum_{j=1}^{2}\Big[i(A_j^{1/2}h,y_j)^{\sim} - \frac{i}{2q_{1j}}|A_j^{1/2}h|^2 - \frac{i}{2q_{2j}}|A_j^{1/2}h|^2\Big]\Big\}\,d\sigma(h) \\
&= \int_H \exp\Big\{\sum_{j=1}^{2}\Big[i(A_j^{1/2}h,y_j)^{\sim} - \frac{i}{2q_j}|A_j^{1/2}h|^2\Big]\Big\}\,d\sigma(h)
\end{aligned}
$$

for s-a.e. $(y_1,y_2)\in B^2$ as desired. □

If $\vec{q}_2 = -\vec{q}_1$ in (3.1), then we obtain the following inverse transform theorem.

COROLLARY 3.2. *(Theorem 3.2 in [6]) Let F be given as in Theorem 3.1. Then for all nonzero real numbers q_1 and q_2,*

$$
T_{-\vec{q}}^{(p)}(T_{\vec{q}}^{(p)}(F)) \approx F \tag{3.3}
$$

for $1\le p<\infty$ where $\vec{q}=(q_1,q_2)$.

Moreover, if n is a natural number, then we obtain the following result.

COROLLARY 3.3. *Let $F\in\mathcal{F}_{A_1,A_2}$ be given as in Theorem 3.1 and let $1\le p<\infty$. Let q_{k1} and q_{k2} $(k=1,\cdots,n)$ be in $\mathbb{R}^{\#}-\{0\}$. Then*

$$
T_{\vec{q}_n}^{(p)}(T_{\vec{q}_{n-1}}^{(p)}(\cdots(T_{\vec{q}_1}^{(p)}(F)))) \approx T_{\vec{q}}^{(p)}(F) \tag{3.4}
$$

where $\vec{q}_k=(q_{k1},q_{k2})$, $\vec{q}=(q_1,q_2)$, and q_1 and q_2 are extended real numbers such that $\frac{1}{q_{1k}}+\frac{1}{q_{2k}}+\cdots+\frac{1}{q_{nk}}=\frac{1}{q_k}$ for $k=1,2$.

Cameron and Storvick [5] introduced a new translation theorem for the analytic Feynman integral on classical Wiener space. Now we give a simple proof of a product abstract Wiener space version of the translation theorem.

THEOREM 3.4. *Let $F \in \mathcal{F}_{A_1,A_2}$ be given by (2.8) and let $w \in H$. Then, for all $\vec{q} = (q_1, q_2)$ with non-zero real numbers q_1 and q_2,*

$$\begin{aligned} &\int_{B^2}^{\mathrm{anf}_{\vec{q}}} F(x_1 + \frac{1}{q_1} A_1^{1/2} w, x_2 + \frac{1}{q_2} A_2^{1/2} w)\, d(\nu \times \nu)(x_1, x_2) \\ = \;& \exp\Big\{\sum_{j=1}^{2} \frac{i}{2q_j} |A_j^{1/2} w|^2\Big\} \int_{B^2}^{\mathrm{anf}_{\vec{q}}} F(x_1, x_2) \\ & \exp\Big\{-i \sum_{j=1}^{2} (A_j^{1/2} w, x_j)^{\sim}\Big\}\, d(\nu \times \nu)(x_1, x_2). \end{aligned}$$

Proof. Let

$$G(x_1, x_2) = F(x_1, x_2) \exp\Big\{-i \sum_{j=1}^{2} (A_j^{1/2} w, x_j)^{\sim}\Big\}.$$

Then, by (2.8), we have

$$\begin{aligned} G(x_1, x_2) &= \int_H \exp\Big\{i \sum_{j=1}^{2} (A_j^{1/2}(h - w), x_j)^{\sim}\Big\}\, d\sigma(h) \\ &= \int_H \exp\Big\{i \sum_{j=1}^{2} (A_j^{1/2} k, x_j)^{\sim}\Big\}\, d\hat{\sigma}(k) \end{aligned}$$

where $\hat{\sigma}(E) = \sigma(E + w)$ for $E \in \mathcal{B}(H)$. Using Theorem 2.4, we obtain

$$\begin{aligned} (3.5) \qquad & (T_{\vec{q}}^{(1)}(G))(0, 0) \\ &= \int_H \exp\Big\{-\sum_{j=1}^{2} \frac{i}{2q_j} |A_j^{1/2}(h - w)|^2\Big\} d\sigma(h) \\ &= \exp\Big\{-\sum_{j=1}^{2} \frac{i}{2q_j} |A_j^{1/2} w|^2\Big\} \\ &\quad \int_H \exp\Big\{\sum_{j=1}^{2} \Big[\frac{i}{q_j} (A_j^{1/2} h, A_j^{1/2} w) - \frac{i}{2q_j} |A_j^{1/2} h|^2\Big\} d\sigma(h) \\ &= \exp\Big\{-\sum_{j=1}^{2} \frac{i}{2q_j} |A_j^{1/2} w|^2\Big\} (T_{\vec{q}}^{(1)}(F))\Big(\frac{1}{q_1} A_1^{1/2} w, \frac{1}{q_2} A_2^{1/2} w\Big). \end{aligned}$$

By Remark 2.5, we have the result. □

COROLLARY 3.5. *Under the hypothesis of Theorem 3.4, we have*

$$\int_{B^2}^{\mathrm{anf}_{\vec{q}}} F(x_1,x_2)\,d(\nu\times\nu)(x_1,x_2)$$
$$= \exp\Big\{\sum_{j=1}^{2}\frac{i}{2q_j}|A_j^{1/2}w|^2\Big\}\int_{B^2}^{\mathrm{anf}_{\vec{q}}} F(x_1+\frac{1}{q_1}A_1^{1/2}w, x_2+\frac{1}{q_2}A_2^{1/2}w)$$
$$\exp\Big\{i\sum_{j=1}^{2}(A_j^{1/2}w,x_j)^{\sim}\Big\}\,d(\nu\times\nu)(x_1,x_2).$$

Also we easily obtain the following corollaries as special cases of Theorem 3.4.

COROLLARY 3.6. *(Theorem 3.9 in [22]) Let $F\in\mathcal{F}_{A_1,A_2}$ be given by (2.8) and let $w\in H$. Then, for all non-zero real number q,*

$$\int_{B^2}^{\mathrm{anf}_q} F((x_1,x_2)+(A_1^{1/2}w,A_2^{1/2}w))\,d(\nu\times\nu)(x_1,x_2)$$
$$= \exp\Big\{\frac{iq}{2}\sum_{j=1}^{2}\langle A_jw,w\rangle\Big\}\int_{B^2}^{\mathrm{anf}_q} F(x_1,x_2)$$
$$\exp\Big\{-iq\sum_{j=1}^{2}(A_j^{1/2}w,x_j)^{\sim}\Big\}\,d(\nu\times\nu)(x_1,x_2).$$

COROLLARY 3.7. *Let $F\in\mathcal{F}(B)$ be given by (2.9) and let $w\in H$. Then, for all non-zero real number q,*

$$\int_{B}^{\mathrm{anf}_q} F(x+w)\,d\nu(x) = \exp\Big\{\frac{iq}{2}|w|^2\Big\}\int_{B}^{\mathrm{anf}_q} F(x)\exp\{-iq(w,x)^{\sim}\}\,d\nu(x).$$

The following theorem shows the existence for the analytic Fourier-Feynman transform of the convolution product.

THEOREM 3.8. *Let F and G be elements of $\mathcal{F}_{A_1,A_2}$ with corresponding finite Borel measures σ and ρ in $M(H)$ respectively. Let q_{k1} and q_{k2} $(k=1,2)$ be in $\mathbb{R}^{\#}-\{0\}$. Then*

$$(3.6)\qquad (T_{\vec{q}_1}^{(p)}(F*G)_{\vec{q}_2})(y_1,y_2)$$
$$= \int_{H^2}\exp\Big\{\sum_{j=1}^{2}\Big[\frac{i}{\sqrt{2}}(A_j^{1/2}(h+k),y_j)^{\sim}-\frac{i}{4q_{1j}}|A_j^{1/2}(h+k)|^2$$
$$-\frac{i}{4q_{2j}}|A_j^{1/2}(h-k)|^2\Big]\Big\}\,d\sigma(h)\,d\rho(k)$$

for s-a.e. (y_1, y_2) *in* B^2 *where* $\vec{q}_k = (q_{k1}, q_{k2})$ *for* $k = 1, 2$ *and* $1 \le p < \infty$.

Proof. As mentioned in Theorem 2.7, $(F * G)_{\vec{q}_2}$ belongs to $\mathcal{F}_{A_1,A_2}$ and is given by (2.13) with $\vec{q}$ replaced by $\vec{q}_2$. Applying Theorem 2.4 to the expression of $(F * G)_{\vec{q}_2}$ and using (2.14), we have

$$(3.7) \qquad (T_{\vec{q}_1}(F * G)_{\vec{q}_2})(y_1, y_2) = \int_H \exp\Big\{ i \sum_{j=1}^{2} (A_j^{1/2} h, y_j)^{\sim} \Big\} \, d\hat{\eta}(h)$$

where $\hat{\eta}$ is given by (2.11) with σ replaced by η. Hence by the expressions (2.11) and (2.14) we have (3.6) as desired. □

Taking $\vec{q}_1 = \vec{q}_2 = \vec{q}$ in Theorem 3.8 and using the expression (2.10), we have the following corollary.

Corollary 3.9. *(Theorem 3.4 in* [6]*) Let* F *and* G *be given as in Theorem* 3.8. *Then for all* $\vec{q} = (q_1, q_2)$ *with nonzero real numbers* q_1 *and* q_2*,*

$$(3.8) \quad T_{\vec{q}}^{(p)}((F * G)_{\vec{q}})(y_1, y_2) = T_{\vec{q}}^{(p)}(F)\Big(\frac{y_1}{\sqrt{2}}, \frac{y_2}{\sqrt{2}}\Big) T_{\vec{q}}^{(p)}(G)\Big(\frac{y_1}{\sqrt{2}}, \frac{y_2}{\sqrt{2}}\Big)$$

for s-a.e. (y_1, y_2) *in* B^2 *and* $1 \le p < \infty$.

Corollary 3.10. *Let* F *be given as in Theorem* 3.1. *Then for all nonzero real numbers* q_{k1} *and* q_{k2} $(k = 1, 2)$*,*

$$T_{\vec{q}_1}^{(p)}((F * 1)_{\vec{q}_2})(y_1, y_2) = T_{\vec{q}'}^{(p)}(F)\Big(\frac{y_1}{\sqrt{2}}, \frac{y_2}{\sqrt{2}}\Big)$$

for s-a.e. (y_1, y_2) *in* B^2 *where* $\vec{q}_k = (q_{k1}, q_{k2})$, $\vec{q}' = (q_1', q_2')$, $\frac{2}{q_k'} = \frac{1}{q_{1k}} + \frac{1}{q_{2k}}$ *for* $k = 1, 2$ *and* $1 \le p < \infty$.

Proof. For a probability measure ρ whose support is $\{0\} \subset H$,

$$G(x_1, x_2) = \int_H \exp\Big\{ i \sum_{j=1}^{2} (A_j^{1/2} h, y_j)^{\sim} \Big\} \, d\rho(h) = 1.$$

Hence by Theorems 3.8 and 2.4, we have

$$\begin{aligned} & T_{\vec{q}_1}^{(p)}((F * 1)_{\vec{q}_2})(y_1, y_2) \\ &= \int_H \exp\Big\{ \sum_{j=1}^{2} \Big[\frac{i}{\sqrt{2}} (A_j^{1/2} h, y_j)^{\sim} - \frac{i}{4 q_{1j}} |A_j^{1/2} h|^2 - \frac{i}{4 q_{2j}} |A_j^{1/2} h|^2 \Big] \Big\} \, d\sigma(h) \\ &= T_{\vec{q}'}^{(p)}(F)\Big(\frac{y_1}{\sqrt{2}}, \frac{y_2}{\sqrt{2}}\Big) \end{aligned}$$

as desired. □

In the next theorem, we will show the existence for the convolution product of the analytic Fourier-Feynman transforms.

THEOREM 3.11. *Let F and G be given as in Theorem 3.8. Let q_{k1} and q_{k2} $(k=1,2)$ be in $\mathbb{R}^{\#}-\{0\}$. Then for all nonzero real numbers q_1 and q_2,*

$$(3.9)\qquad \begin{aligned} &(T^{(p)}_{\vec{q}_1}(F) * T^{(p)}_{\vec{q}_2}(G))_{\vec{q}}(y_1,y_2)\\ &= \int_{H^2} \exp\Big\{\sum_{j=1}^{2}\Big[\frac{i}{\sqrt{2}}(A_j^{1/2}(h+k),y_j)^{\sim} - \frac{i}{2q_{1j}}|A_j^{1/2}h|^2\\ &\quad -\frac{i}{2q_{2j}}|A_j^{1/2}k|^2 - \frac{i}{4q_j}|A_j^{1/2}(h-k)|^2\Big]\Big\}\, d\sigma(h)\, d\rho(k) \end{aligned}$$

for s-a.e. (y_1,y_2) in B^2 where $\vec{q}_k=(q_{k1},q_{k2})$, $\vec{q}=(q_1,q_2)$ for $k=1,2$ and $1\le p<\infty$.

Proof. As mentioned in Theorem 2.4, $T^{(p)}_{\vec{q}_1}(F)$ and $T^{(p)}_{\vec{q}_2}(G)$ belong to $\mathcal{F}_{A_1,A_2}$ and are given by (2.10) with corresponding measures $\hat{\sigma}$ and $\hat{\rho}$, respectively. Applying Theorem 2.7 to the expressions of $T^{(p)}_{\vec{q}_1}(F)$ and $T^{(p)}_{\vec{q}_2}(G)$, we obtain

$$(3.10)\qquad \begin{aligned} &(T^{(p)}_{\vec{q}_1}(F) * T^{(p)}_{\vec{q}_2}(G))_{\vec{q}}(y_1,y_2)\\ &= \int_{H^2} \exp\Big\{\sum_{j=1}^{2}\Big[i(A_j^{1/2}h,y_j)^{\sim} - \frac{i}{4q_j}|A_j^{1/2}(h-k)|^2\Big]\Big\}\, d\hat{\sigma}(h)\, d\hat{\rho}(k). \end{aligned}$$

Using the expression (2.11) for $\hat{\sigma}$ and $\hat{\rho}$, we obtain (3.9). □

COROLLARY 3.12. *(Theorem 3.5 in [6]) Let F and G be given as in Theorem 3.8. Then for all $\vec{q}=(q_1,q_2)$ with nonzero real numbers q_1 and q_2,*

$$(3.11)\qquad \begin{aligned} &(T^{(p)}_{\vec{q}}(F) * T^{(p)}_{\vec{q}}(G))_{-\vec{q}}(y_1,y_2)\\ &= T^{(p)}_{\vec{q}}\Big(F\Big(\frac{\cdot}{\sqrt{2}},\frac{\cdot}{\sqrt{2}}\Big)\, G\Big(\frac{\cdot}{\sqrt{2}},\frac{\cdot}{\sqrt{2}}\Big)\Big)(y_1,y_2) \end{aligned}$$

for s-a.e. (y_1,y_2) in B^2 and $1\le p<\infty$.

Proof. From the expression (3.9), we have

$$(T_{\vec{q}}^{(p)}(F) * T_{\vec{q}}^{(p)}(G))_{-\vec{q}}(y_1, y_2)$$
$$= \int_{H^2} \exp\Big\{\sum_{j=1}^{2}\Big[\frac{i}{\sqrt{2}}(A_j^{1/2}(h+k), y_j)^\sim - \frac{i}{4q_j}|A_j^{1/2}(h+k)|^2\Big]\Big\}\, d\sigma(h)\, d\rho(k).$$

On the other hand, FG belongs to $\mathcal{F}_{A_1,A_2}$ and has the form

$$F\Big(\frac{x_1}{\sqrt{2}}, \frac{x_2}{\sqrt{2}}\Big)\, G\Big(\frac{x_1}{\sqrt{2}}, \frac{x_2}{\sqrt{2}}\Big)$$
$$= \int_{H^2} \exp\Big\{\frac{i}{\sqrt{2}}\sum_{j=1}^{2}(A_j^{1/2}(h+k), y_j)^\sim\Big\}\, d\sigma(h)\, d\rho(k).$$

By applying Theorem 2.4 it is easy to see that

$$T_{\vec{q}}^{(p)}\Big(F\Big(\frac{\cdot}{\sqrt{2}}, \frac{\cdot}{\sqrt{2}}\Big)\, G\Big(\frac{\cdot}{\sqrt{2}}, \frac{\cdot}{\sqrt{2}}\Big)\Big)(y_1, y_2)$$

also has the same expression as $(T_{\vec{q}}^{(p)}(F) * T_{\vec{q}}^{(p)}(G))_{-\vec{q}}(y_1, y_2)$. □

THEOREM 3.13. *Let F and G be given as in Theorem 3.8. Let q_{k1} and q_{k2} $(k = 1, 2)$ be in $\mathbb{R}^\# - \{0\}$. Then for all $\vec{q} = (q_1, q_2)$ with nonzero real numbers q_1 and q_2,*

$$T_{\vec{q}}^{(p)}(T_{\vec{q_1}}^{(p)}(F) * T_{\vec{q_2}}^{(p)}(G))_{\vec{q}})(y_1, y_2)$$
$$= T_{\vec{q_1}'}^{(p)}(F)\Big(\frac{y_1}{\sqrt{2}}, \frac{y_2}{\sqrt{2}}\Big)\, T_{\vec{q_2}'}^{(p)}(G)\Big(\frac{y_1}{\sqrt{2}}, \frac{y_2}{\sqrt{2}}\Big) \tag{3.12}$$

for s-a.e. (y_1, y_2) in B^2 where $\vec{q_1'} = (q'_{11}, q'_{12})$, $\vec{q_2'} = (q'_{21}, q'_{22})$ and q'_{1k} and q'_{2k} are extended real numbers such that $\frac{1}{q'_{1k}} = \frac{1}{q_{1k}} + \frac{1}{q_k}$ and $\frac{1}{q'_{2k}} = \frac{1}{q_{2k}} + \frac{1}{q_k}$ for $k = 1, 2$ and $1 \le p < \infty$. Also both sides of the above expression are given by

$$\int_{H^2} \exp\Big\{\sum_{j=1}^{2}\Big[\frac{i}{\sqrt{2}}(A_j^{1/2}(h+k), y_j)^\sim$$
$$-\frac{i}{2q'_{1j}}|A_j^{1/2}h|^2 - \frac{i}{2q'_{2j}}|A_j^{1/2}k|^2\Big]\Big\}\, d\sigma(h)\, d\rho(k). \tag{3.13}$$

Proof. A simple calculation together with Theorem 2.4 and 3.11 shows that the left hand side of (3.12) is expressed as (3.13). On the other hand, using Theorem 2.4, the right hand side of (3.12) is also expressed as (3.13). □

Corollary 3.14. *Equation* (3.8) *in Corollary* 3.9 *holds.*

Proof. Using Theorem 3.13 together with (i) of Remark 2.5, we can easily obtain our result. □

Theorem 3.15. *Let $F, G, \sigma, \rho, \vec{q_1}$ and $\vec{q_2}$ be given as in Theorem* 3.8. *Then for all $\vec{q} = (q_1, q_2)$ with nonzero real numbers q_1 and q_2, the following Parseval's relation*

$$(3.14)\qquad \begin{aligned} &T_{-\vec{q}}^{(p)}(T_{\vec{q}}^{(p)}(T_{\vec{q_1}}^{(p)}(F) * T_{\vec{q_2}}^{(p)}(G))_{\vec{q}})(0,0) \\ = \;& T_{\vec{q}}^{(p)}\Big(T_{\vec{q_1}}^{(p)}(F)\Big(\frac{\cdot}{\sqrt{2}}, \frac{\cdot}{\sqrt{2}}\Big)\, T_{\vec{q_2}}^{(p)}(G)\Big(-\frac{\cdot}{\sqrt{2}}, -\frac{\cdot}{\sqrt{2}}\Big)\Big)(0,0) \end{aligned}$$

holds for $1 \le p < \infty$.

Proof. By Corollary 3.2 and Theorem 3.11, the left hand side of (3.14) is expressed as

$$(3.15)\qquad \begin{aligned} &\int_{H^2} \exp\Big\{\sum_{j=1}^{2}\Big[-\frac{i}{2q_{1j}}|A_j^{1/2}h|^2 - \frac{i}{2q_{2j}}|A_j^{1/2}k|^2 \\ &-\frac{i}{4q_j}|A_j^{1/2}(h-k)|^2\Big]\Big\}\, d\sigma(h)\, d\rho(k). \end{aligned}$$

On the other hand, we have that by Theorem 2.4,

$$(3.16)\qquad \begin{aligned} &T_{\vec{q_1}}^{(p)}(F)\Big(\frac{y_1}{\sqrt{2}}, \frac{y_2}{\sqrt{2}}\Big)\, T_{\vec{q_2}}^{(p)}(G)\Big(-\frac{y_1}{\sqrt{2}}, -\frac{y_2}{\sqrt{2}}\Big) \\ = \;& \int_{H^2} \exp\Big\{\sum_{j=1}^{2}\Big[\frac{i}{\sqrt{2}}(A_j^{1/2}(h-k), y_j)^{\sim} \\ &-\frac{i}{2q_{1j}}|A_j^{1/2}h|^2 - \frac{i}{2q_{2j}}|A_j^{1/2}k|^2\Big]\Big\}\, d\sigma(h)\, d\rho(k) \end{aligned}$$

and belongs to $\mathcal{F}_{A_1,A_2}$. By applying Theorem 2.4 once more to the above expression, we have that the Fourier-Feynman transform of (3.16) is also expressed as (3.15). □

Also we can express (3.14) alternately as

$$(3.17)\qquad \begin{aligned} &\int_{B^2}^{\mathrm{anf}_{-\vec{q}}} (T_{\vec{q}}^{(p)}(T_{\vec{q_1}}^{(p)}(F) * T_{\vec{q_2}}^{(p)}(G))_{\vec{q}})(x_1, x_2)\, d(\nu \times \nu)(x_1, x_2) \\ = \;& \int_{B^2}^{\mathrm{anf}_{\vec{q}}} T_{\vec{q_1}}^{(p)}(F)\Big(\frac{x_1}{\sqrt{2}}, \frac{x_2}{\sqrt{2}}\Big)\, T_{\vec{q_2}}^{(p)}(G)\Big(-\frac{x_1}{\sqrt{2}}, -\frac{x_2}{\sqrt{2}}\Big)\, d(\nu \times \nu)(x_1, x_2). \end{aligned}$$

COROLLARY 3.16. *(Theorem 3.6 in [6]) Let F and G be given as in Theorem 3.8. Then for all $\vec{q} = (q_1, q_2)$ with nonzero real numbers q_1 and q_2, the Parseval's relation*

$$\begin{aligned}(3.18)\quad &\int_{B^2}^{\mathrm{anf}_{-\vec{q}}} (T_{\vec{q}}^{(p)}(F))\Big(\frac{x_1}{\sqrt{2}}, \frac{x_2}{\sqrt{2}}\Big)(T_{\vec{q}}^{(p)}(G))\Big(\frac{x_1}{\sqrt{2}}, \frac{x_2}{\sqrt{2}}\Big)\, d(\nu \times \nu)(x_1, x_2) \\ = &\int_{B^2}^{\mathrm{anf}_{\vec{q}}} F\Big(\frac{x_1}{\sqrt{2}}, \frac{x_2}{\sqrt{2}}\Big)\, G\Big(-\frac{x_1}{\sqrt{2}}, -\frac{x_2}{\sqrt{2}}\Big)\, d(\nu \times \nu)(x_1, x_2)\end{aligned}$$

holds for $1 \le p < \infty$.

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