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SOME FUZZY SEMIPRIME IDEALS OF SEMIGROUPS

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Abstract. The purpose of this paper is to study the some properties of fuzzy quasi-semiprime ideal, fuzzy prime ideals and to prove some fundamental properties of semigroups. In particular, we will establish a relation between fuzzy prime ideals and weakly completely semiprime ideals by using the some equivalent conditions of fuzzy semiprime ideals.

1. Introduction

The concept of fuzzy subset was introduced by Zadeh [16]. Since then, many researchers are engaged in extending the concepts and results of pure algebra to the broader framework of the fuzzy setting although not all the results in algebra can be fuzzified. Rosenfeld [12] applied the concept of fuzzy set to the theory of semigroups and groups. Fuzzy ideals in semigroups were introduced in [12] and discussed further in [4, 5, 15]. N. Kuroki has studied fuzzy ideals and bi-ideals in a semigroup [6]. Many classes of semigroups were studied and discussed further by using fuzzy ideals. For example, Kuroki characterized intra-regular semigroups by fuzzy semiprime ideals [7], completely regular semigroups and a semigroup that is a semilattice of groups in terms of fuzzy semiprime quasi-ideals[6]. Recently, Ahsan and others [1] have shown that a semigroup is semisimple if and only if each fuzzy ideal of semigroup is the intersection of fuzzy prime ideals containing it. Liu [4] introduced and studied the notation of fuzzy ideals in rings. N. Mukherjee and Sen [9, 10] defined and examined fuzzy prime ideals, fuzzy completely prime ideals and weakly completely prime ideals of a ring. In particular, they characterized all fuzzy prime ideals of the ring Z of integers. Fuzzy prime ideals were further investigated by Malik and Mordeson in [8] and

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in [14]. The reader may refer to [2], [3] and [11] for the basic theory of semigroup.

2. Definitions and preliminaries

Let S be a semigroup. A nonempty subset A of S is called a *left ideal* of S if $SA \subset A$. A is called a *right ideal* of S if $AS \subset A$ and A is called an *ideal* of S if A is both a left and a right ideal of S. A function f from S to the unit interval [0, 1] is a *fuzzy subset* of S. The semigroup S itself is a fuzzy subset of S such that $S(x) = 1$ for all $x \in S$, denoted also by S. Let A and B be two fuzzy subsets of S. Then the inclusion relation $A \subset B$ is defined $A(x) \leq B(x)$ for all $x \in S$. $A \cap B$ and $A \cup B$ are fuzzy subsets of S defined by $(A \cap B)(x) = min\{A(x), B(x)\}$, $(A \cup B)(x) = max\{A(x), B(x)\}$ for all $x \in S$. The product $A \circ B$ is defined as follows; $(A \circ B)(x) = sup_{x=yz}[min\{A(y), B(z)\}]$, 0 if x is not expressible as $x = yz$ [10]. As is well known [4], this operation " $\circ$ " is associative. We denote by a_t a fuzzy point of S, where $a_t(x) = t$, if $x = a$, 0 otherwise.

For any fuzzy subset f of S, it is obvious that $f = \cup_{a_t \in f} a_t$. Let tf_A is a fuzzy subset of S defined as following; $tf_A(x) = t$, if $x \in A$, 0 otherwise. Then, we have a fuzzy point a_t of S denoted by tf_a.

Fuzzy subset θ of S is called a *fuzzy subsemigroup* of S if $\theta(xy) \geq min\{\theta(x), \theta(y)\}$ for all x, y in S, and is called a *fuzzy left*(*right*) *ideal* of S if $\theta(xy) \geq \theta(y)(\theta(xy) \geq \theta(x))$ for all $x, y \in S$, if θ is both fuzzy left and right ideal of S, then θ is called a *fuzzy ideal* of S [4]. It is easy that θ is a fuzzy ideal of S if and only if $\theta(xy) \geq max\{\theta(x), \theta(y)\}$ for all x, y in S and any fuzzy left(right) ideal of S is a fuzzy subsemigroup of S. Equivalently, We can prove easily that A is a (left, right) ideal of S if and only if the characteristic function ψ_A of A is a fuzzy (left, right) ideal of S ([6], lemma 2.1.). For each semigroup S we shall define S^1 by $S \cup \{1\}$ where 1 is a symbol not in S and where multiplication on S is extended to S^1 by defining $x1 = 1x = x$ for all $x \in S^1$ with the operation so defined, S^1 is a semigroup with identity 1. Note that this definition for S^1 differs from the standard one. Throughout S will denote semigroup with or without 1, for every $a \in S$ we define $S^1a = Sa \cup \{a\}, aS^1 = aS \cup \{a\}$ and $S^1aS^1 = SaS \cup S^1a \cup aS$.

DEFINITION 2.1. [14] A fuzzy subset θ of a semigroup S is called *quasi-prime* if for any two fuzzy right ideals A and B of S, $A \circ B \subset \theta$ implies $A \subset \theta$ or $B \subset \theta$.

Definition 2.2. A fuzzy subset θ of a semigroup S is called *quasi-semiprime* if for any fuzzy right ideal A of S, $A \circ A \subset \theta$ implies $A \subset \theta$.

Definition 2.3. A fuzzy subset θ of a semigroup S is called *fuzzy prime* of S if for any two fuzzy ideals A and B of S, $A \circ B \subset \theta$ implies $A \subset \theta$ or $B \subset \theta$

Definition 2.4. A fuzzy subset θ of a semigroup S is called *fuzzy semiprime* of S if for any fuzzy ideal A of S $A \circ A \subset \theta$ implies $A \subset \theta$

Definition 2.5. A fuzzy subset θ of a semigroup of S is called *fuzzy weakly completely prime* if $max\{\theta(x), \theta(y)\} \geq \theta(xy)$ for all $x, y \in S$.

An ideal θ of S is fuzzy weakly completely prime if and only if $max\{\theta(x), \theta(y)\} = \theta(xy)$ for all $x, y \in S$ [10].

Definition 2.6. A fuzzy subset θ of a semigroup of S is called *fuzzy weakly completely semiprime* if $\theta(x) \geq \theta(x^2)$ for all $x \in S$.

If θ is fuzzy subsemigroup of S, then θ is fuzzy weakly completely semiprime if and only if $\theta(x) = \theta(x^2)$ for all $x \in S$ [7].

Theorem 2.7. *θ is fuzzy subsemigroup of S if and only if $1 - \theta$ is fuzzy weakly completely prime.*

Proof. Assume θ is a fuzzy subsemigroup of S. Since $\theta(xy) \geq min\{\theta(x), \theta(y)\}$, we have $1 - \theta(xy) \leq 1 - min\{\theta(x), \theta(y)\}$.

1) if $\theta(x) \leq \theta(y)$, then $1-\theta(x) \geq 1-\theta(y)$ and $max\{1-\theta(x), 1-\theta(y)\} = 1 - \theta(x)$. $1 - min\{\theta(x), \theta(y)\} = 1 - \theta(x) = max\{1 - \theta(x), 1 - \theta(y)\}$.

2) if $\theta(x) \geq \theta(y)$, we have the same result. Thus $max\{(1-\theta)(x), (1-\theta)(y)\} = max\{1 - \theta(x), 1 - \theta(y)\} = 1 - min\{\theta(x), \theta(y)\} \geq 1 - \theta(xy) = (1 - \theta)(xy)$

Conversely, let $1 - \theta$ be fuzzy weakly completely semiprime subset of S. Since $max\{1 - \theta(x), 1 - \theta(y)\} \geq 1 - \theta(xy)$, we have $1 - min\{\theta(x), \theta(y)\} \geq 1 - \theta(xy)$. Thus $min\{\theta(x), \theta(y)\} \leq \theta(xy)$ and so θ is fuzzy subsemigroup of S. □

Theorem 2.8. *If $P_i, i \in I$ are fuzzy weakly completely prime subset of S, then $\cup P_i$ is fuzzy weakly completely prime subset of S*

Proof. $P_i(xy) \leq max\{P_i(x), P_i(y)\}$ for all $x, y \in S$, and for $i \in I$. Since $max\{\cup P_i(x), \cup P_i(y)\} \geq P_i(xy)$ for all $i \in I$, $max\{\cup P_i(x), \cup P_i(y)\} \geq \cup P_i(xy)$ □

Different from ring, union of fuzzy ideals of semigroup S is also fuzzy ideal of S, we have the following corollary.

Corollary 2.9. *If $P_i, i \in I$ are fuzzy weakly completely prime ideals of S, then $\cup P_i$ is fuzzy weakly completely prime ideal of S*

Theorem 2.10. *Let I be an ideal (left, right ideal) of S, $m \in [0,1]$. Then If Q is fuzzy set of S such that $Q(x) = m$ if $x \in I, 0$ otherwise, then Q is a fuzzy ideal (fuzzy left, fuzzy right ideal) of S.*

3. Fuzzy prime ideals of semigroups

Theorem 3.1. *fuzzy right ideal θ of S is a quasi-semiprimel if and only if $\inf \theta(aSa) = \theta(a)$ for all $a \in S$*

Proof. " if " Let A be a fuzzy right ideal of S such that $A \circ A \subset \theta$. If $A \not\subseteq \theta$, then there exist $a \in S$ such that $\theta(a) < A(a)$. Since $\theta(a) = \inf \theta(aSa)$, there exists $u \in S$ such that $\theta(aua) < A(a)$. $A(a) > \theta(aua) \geq A \circ A(aua) = sup_{xy=aua}[min\{A(x), A(y)\}] \geq min\{A(au), A(a)\} = A(a)$, since A is fuzzy right ideal of S. But this leads to a contradiction.

" only if " If $\theta(a) \neq \inf \theta(aSa)$ for some $a \in S$, then since θ is fuzzy right ideal of S, $\theta(a) \leq \theta(axa) = \theta(a(xa))$ for all $x \in S$ we have $\theta(a) < \inf \theta(aSa)$. Let $\inf \theta(aSa) = m$ and Q be fuzzy subset of S such that $Q(x) = m$ if $x \in aS, 0$ otherwise. Then by above Theorem 2.10, Q is a fuzzy right ideal of S. If $Q \circ Q(x) = m$, then $m = sup_{x=yz}[min\{Q(y), Q(z)\}]$. This means there exist some $u, v \in aS$ such that $uv = x$. Put $u = at, v = aq$. Then $\theta(x) = \theta(uv) = \theta(ataq) \geq \theta(ata) \geq \inf \theta(aSa) = m$. Hence $Q \subset \theta$. Again let R be fuzzy set of S such that $R(x) = m$ if $x \in aS^1$, 0 otherwise, then by above theorem 2.10, R is a fuzzy right ideal of S. $R \circ R(x) = sup_{yz=x}[min\{R(y), R(z)\}] = m$ only if $y, z \in aS^1$. Put $y = ai, u = aj, i, j \in S^1$. Then $x = yz = aiaj \in aS$. Thus $R \circ R(x) = m$ implies $Q(x) = Q(yz) = Q(aiaj) = m$. Hence $R \circ R \subset Q \subset \theta$ and since θ is quasi-semiprime, $R \subset \theta$. But from $m = R(a) \leq \theta(a) < \inf \theta(aSa) = m$, we have a contradiction. □

Since right side of theorem 3.1 is left-right dual, we have following corollary.

Corollary 3.2. *fuzzy ideal θ of S is semiprime if and only if $\theta(a) = \inf \theta(aSa)$ for all $a \in S$*

Corollary 3.3. *If θ is fuzzy weakly completely semiprime ideal of S, then θ is a fuzzy semiprime ideal of S.*

Proof. for any $a \in S$, $\theta(a) = \theta(aa) = \theta(aaaa) \geq \inf \theta(aSa) \geq \theta(a)$. □

The converse of above corollary 3.3 does not hold in general semigroup. But the following corollary show that the converse also holds on commutative semigroup.

COROLLARY 3.4. *If S is commutative, then fuzzy ideal θ of S is semiprime if and only if θ is fuzzy weakly completely semiprime.*

Proof. Let θ be a fuzzy semiprime ideal of commutative semigroup S. If $\theta(a) \neq \theta(aa)$ for some $a \in S$, then $\theta(aa) > \theta(a) = \inf \theta(aSa)$. Thus there exists $t \in S$ such that $\theta(aa) > \theta(ata) = \theta(aat)$. But this contradicts to the fact that θ is fuzzy ideal. So $\theta(x) = \theta(xx)$ for all $x \in S$. □

THEOREM 3.5. *If fuzzy right ideal θ of S is quasi-prime, then* $\inf \theta(aSb) \leq max\{\theta(a), \theta(b)\}$ *for all $a, b \in S$.*

Proof. Put $\inf \theta(aSb) = m$. Let A and B be fuzzy sets of S such that $A(x) = m$ if $x \in aS$, 0 otherwise. $B(x) = m$ if $x \in bS$, 0 otherwise. Then A and B are fuzzy right ideals of S by theorem 2.10. If $A \circ B(x) = sup_{x=yz}[min\{A(y), B(z)\}] = m$, then there exists an $u \in aS, v \in bS$ such that $uv = x$. Put $u = at$ and $v = bq, t, q \in S$. Then $\theta(x) = \theta(uv) = \theta(atbq) \geq \theta(atb) \geq \inf \theta(aSb) = m$. So that $A \circ B(x) \leq \theta(x)$. Since θ is a quasi-prime ideal, $A \subset \theta$ or $B \subset \theta$. Assume $A \subset \theta$. Let R be a fuzzy subset of S such that $R(x) = m$ if $x \in aS^1$, 0 otherwise. Then we have $R \circ R \subset A \subset \theta$ and by the same way to the proof of above theorem 3.1, $R \subset \theta$. Thus $m = R(a) \leq \theta(a)$. Similarly in case of $B \subset \theta$, we have $m = R(b) \leq \theta(b)$. □

COROLLARY 3.6. *If fuzzy ideal θ of S is quasi-prime, then* $\inf \theta(aSb) = max\{\theta(a), \theta(b)\}$ *for all $a, b \in S$.*

By the same way, we can prove that if θ is fuzzy weakly completely prime or fuzzy prime ideal of S, then $max\{\theta(a), \theta(b)\} = \inf \theta(aSb)$ for all $a, b \in S$. Similar to semiprime, it is natural to think the fuzzy weakly completely prime ideal of S is also fuzzy prime. But it does not hold even when S is commutative.

EXAMPLE 3.7. Let $S = Z \times Z$, where Z is set of integers with multiplication. Define a fuzzy ideal A of S by $A(x) = 1$ if $x = (0, n)$, $1/2$ if $x = (2m, n)$, 0 if $x = (2n + 1, k)$. where $n, k \in Z, m$ are non-zero integers. Then we can easily check that A is a fuzzy weakly completely prime ideal, but A is not a fuzzy prime ideal of Z.

Example 3.7 also shows that the converse of theorem 3.5 does not hold. Mukherjee and Sen [10] have proved that range of non-constant fuzzy prime ideal of ring consists of exactly two points of $[0, 1]$. Following theorem shows that it is sufficient and essential for fuzzy subset θ of semigroup S become fuzzy prime ideal of S.

THEOREM 3.8. *Let θ is a fuzzy subset of S. Then θ is fuzzy prime ideal of S if and only if 1) $max\{\theta(a), \theta(b)\} = \inf \theta(aSb)$ for all $a, b \in S$ and 2) range of θ consists of exactly two points of* $[0, 1]$.

Proof. " only if " If there exist $a, b, c \in S$ such that $\theta(a) < \theta(b) < \theta(c)$. Let us choose $s, t \in [0, 1]$ such that $\theta(a) < s < \theta(b) < t < \theta(c)$. Let A, B be fuzzy subsets of S defined by $A(x) = s$ if $x \in S^1aS^1$, 0 otherwise. $B(x) = t$ if $x \in S^1bS^1$, 0 otherwise. Then A, B are fuzzy ideals of S by theorem 2.10 and $A \circ B(x) = max_{yz=x}min\{A(y), B(z)\} = s$ if $y \in S^1aS^1, z \in S^1bS^1$, 0 otherwise. If $A \circ B(x) = s$, then there exists an $u \in S^1aS^1, v \in S^1bS^1$ such that $uv = x$. Thus there are $m, n, p, q \in S^1$ such that $u = man, v = pbq$. Thus $\theta(x) = \theta(uv) = \theta(manpbq) \geq \theta(b)$. Hence $A \circ B(x) \leq \theta(x)$ for all $x \in S$. Since θ is a fuzzy prime ideal, this implies $A \subset \theta$ or $B \subset \theta$, and hence either $s = A(a) \leq \theta(a)$ or $t = B(b) \leq \theta(b)$ which is contradiction. consequently, the range of θ consists of exactly two points.

" if " Let range of $\theta = \{s, t\}, s < t, A, B$ be fuzzy ideals of S such that $A \circ B \subset \theta$. If $A \not\subseteq \theta$ and $B \not\subseteq \theta$, then $A(x) > \theta(x)$ and $B(x) > \theta(x)$ for some $x, y \in S$. Thus $A(x) = B(x) = t$ and $\theta(x) = \theta(y) = S$. From $max\{\theta(x), \theta(y)\} = \inf \theta(xSy)$, there exists an element $u \in S$ such that $\theta(xuy) = s$. Since $A \circ B \subset \theta$, $s = \theta(xuy) \geq A \circ B(xuy) \geq min\{A(xu), B(y)\} \geq min\{A(x), B(y)\} = t$. this contradicts to $s < t$ □

The following example show that condition 1) and 2) of theorem 3.8 are essential for fuzzy subset of S become prime ideal of S.

EXAMPLE 3.9. Let $S = Z$ with multiplication. Define a fuzzy subset A in S by $A(x) = 1$ if $x = 0$, $1/2$ if $x = 2m$, 0 if $x = 2k + 1$. where $k \in Z, m$ is nonzero integer. Then we can easily check $max\{\theta(a), \theta(b)\} = \theta(ab)$ for all $a, b \in Z$ and so that A is a fuzzy weakly completely prime ideal of Z. Since $|Im(\theta)| = 3, A$ is not a fuzzy prime ideal of Z.

EXAMPLE 3.10. Let $S = Z$ with multiplication, $B = 6Z$. Then the characteristic function ψ_B of B is not fuzzy prime ideal from [9, corollary 2.6]. Since $|Im(\psi_B)| = 2$, 1) of theorem does not holds. In fact, $0 = max\{\psi_B(2), \psi_B(3)\} \neq min\psi_B(2Z3) = 1$.

NOTE 3.11. Mukherjee and Sen suggest that the concept of fuzzy weakly completely prime ideal is more important in the setting of fuzzy ideals in a ring in view of characterization of fuzzy prime and fuzzy completely prime ideals. The following example shows that intersection of fuzzy prime ideals of S is not a fuzzy prime ideal in general. But from definition of fuzzy semiprime, the intersection of fuzzy prime ideals of S is fuzzy semiprime ideal of S.

EXAMPLE 3.12. Let A be a function from Z to [0, 1] such that $A(3x) = 1$ for all $x \in Z$, $A(x) = 0$ otherwise. Then $\inf A(mZn) = max\{A(m), A(n)\}$. Let B be a function from Z to [0, 1] such that $B(5x) = 1$ for all $x \in Z, A(x) = 0$ otherwise. Then also $\inf B(mZn) = max\{B(m), B(n)\}$. Thus A, B are fuzzy prime ideals of Z. Let $C = A \circ B$. Then C is a function from Z to [0, 1] such that $C(15x) = 1$ for all $x \in Z$, $C(x) = 0$ otherwise. But $\inf C(3Z5) = 1, max\{C(3), C(5)\} = 0$. Thus by above theorem 3.8, C is not a fuzzy prime ideal of Z.

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