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ON THE 2-VARIABLE SUBNORMAL COMPLETION PROBLEM

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ABSTRACT. In this note we give a connection between the truncated moment problem and the 2-variable subnormal completion problem.

1. Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. Recall that a bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$. An operator T is said to be *hyponormal* if $T^*T \geq TT^*$. For $S, T \in \mathcal{B}(\mathcal{H})$, let $[S,T] := ST - TS$. An n-tuple $\mathbf{T} := (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is said to be (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive semidefinite on the direct sum of n copies of $\mathcal{H}$ (cf. [1], [9]). For instance, if $n = 2$,

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix}.$$

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The n-tuple $\mathbf{T} \equiv (T_1, T_2, \cdots, T_n)$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. In particular, a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is said to be k-*hyponormal* $(k \geq 1)$ ([10]) if

$$\mathbf{T}(k) := (T_1, T_2, T_1^2, T_2T_1, T_2^2, \cdots, T_1^k, T_2T_1^{k-1}, \cdots, T_2^k)$$

is hyponormal, or equivalently

$$[\mathbf{T}(k)^*, \mathbf{T}(k)] = ([(T_2^qT_1^p)^*, T_2^mT_1^n])_{\substack{1\leq n+m\leq k \\ 1\leq p+q\leq k}} \geq 0.$$

Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ k-hyponormal. We now review results for one variable subnormal completion. For $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. For a weighted shift W_α, *the moments of* α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1, & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2, & \text{if } k > 0. \end{cases}$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \cdots$. C. Berger's characterization of subnormality for unilateral weighted shifts (cf. [4], [2, II.6.10]) states that W_α is subnormal if and only if there exists a Borel probability measure (so called Berger measure) μ supported in $[0, ||W_\alpha||]$, with $||W_\alpha|| \in \operatorname{supp}\mu$, such that

$$\gamma_n = \int t^{2n} d\mu(t) \quad \text{for all } n \geq 0.$$

In 1966, Stampfli [16] explicitly exhibited for a subnormal weighted shift A_0 its minimal normal extension

$$N := \begin{pmatrix} A_0 & B_1 & & 0 \\ & A_1 & B_2 & \\ & & A_2 & \ddots \\ 0 & & & \ddots \end{pmatrix},$$

where A_n is a weighted shift with weights $\{a_0^{(n)}, a_1^{(n)}, \cdots\}$, $B_n := \operatorname{diag}\{b_0^{(n)}, b_1^{(n)}, \cdots\}$, and these entries satisfy:

(I) $(a_j^{(n)})^2 - (a_{j-1}^{(n)})^2 + (b_j^{(n)})^2 \geq 0$ ($b_j^{(0)} = 0$ for all j);

(II) $b_j^{(n)} = 0 \Longrightarrow b_{j+1}^{(n)} = 0$;

(III) there exists a constant M such that $|a_j^{(n)}| \le M$ and $|b_j^{(n)}| \le M$ for $n = 0, 1, \cdots$ and $j = 0, 1, \cdots$.

Here $b_j^{(n+1)} := [(a_j^{(n)})^2 - (a_{j-1}^{(n)})^2 + (b_j^{(n)})^2]^{\frac{1}{2}}$ and $a_j^{(n+1)} := a_j^{(n)} \dfrac{b_{j+1}^{(n+1)}}{b_j^{(n+1)}}$

(if $b_{j_0}^{(n)} = 0$, then $a_{j_0}^{(n)}$ is taken to be 0).

Thus, we have W_α is subnormal if and only if conditions (I), (II), (III) hold for W_α.

Given an initial segment of weights $\alpha : \alpha_0, \cdots \alpha_m$, a sequence $\widehat{\alpha} \in \ell^\infty(\mathbb{Z}_+)$ such that $\widehat{\alpha}_i = \alpha_i$ $(i = 0, \cdots, m)$ is said to be *recursively generated* by α if there exist $r \ge 1$ and $\varphi_0, \cdots, \varphi_{r-1} \in \mathbb{R}$ such that $\gamma_{n+r} = \varphi_0 \gamma_n + \cdots + \varphi_{r-1} \gamma_{n+r-1}$ (all $n \ge 0$), where $\gamma_0 := 1$, $\gamma_n := \alpha_0^2 \cdots \alpha_{n-1}^2$ $(n \ge 1)$; in this case $W_{\widehat{\alpha}}$ is said to be *recursively generated*. If the associated recursively generated weighted shift $W_{\widehat{\alpha}}$ is subnormal, then its Berger measure is a finitely atomic measure of the form $\mu := \rho_0 \delta_{s_0} + \cdots + \rho_{r-1} \delta_{s_{r-1}}$. Let $\alpha : \alpha_0, \cdots, \alpha_m$ $(m \ge 0)$ be an initial segment of positive weights and let $\omega = \{\omega_n\}_{n=0}^\infty$ be a bounded sequence of positive numbers. We say that W_ω is a *completion* of α if $\omega_n = \alpha_n$ $(0 \le n \le m)$, and we write $\alpha \subset \omega$. The *completion problem* for a property (P) entails finding necessary and sufficient conditions on α to ensure the existence of a weight sequence $\omega \supset \alpha$ such that W_ω satisfies (P). In [2, Theorem 3.5], the following criterion was established.

Theorem 1. *(Subnormal Completion Criterion) If $\alpha : \alpha_0, \cdots, \alpha_n$ $(n \ge 0)$ is an initial segment of positive weights then the following are equivalent:*

(i) *α has a subnormal completion;*

(ii) *α has a recursively generated subnormal completion;*

(iii) *the Hankel matrices $H(l)$ and $H_x(m-1)$ are both positive* ($l := [\frac{n+1}{2}]$ and $m := [\frac{n}{2}] + 1$) and the vector

$$\begin{pmatrix} \gamma_{l+1} \\ \vdots \\ \gamma_{2l+1} \end{pmatrix} \quad \left(\text{resp. } \begin{pmatrix} \gamma_{m+1} \\ \vdots \\ \gamma_{2m} \end{pmatrix}\right)$$

is in the range of $H(l)$ (resp. $H_x(m-1)$) when n is even (resp. odd). Here

$$H(j) := \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_j \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{j+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_j & \gamma_{j+1} & \cdots & \gamma_{2j} \end{pmatrix}, \ H_x(j) := \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{j+1} \\ \gamma_2 & \gamma_3 & \cdots & \gamma_{j+2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{j+1} & \gamma_{j+2} & \cdots & \gamma_{2j+1} \end{pmatrix}.$$

We now consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$. Define the 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \quad \text{and} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1,0)$ and $\varepsilon_2 := (0,1)$. Clearly,

$$T_1T_2 = T_2T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}} \ \ (\text{all } \mathbf{k} \in \mathbb{Z}_+^2). \tag{1.1}$$

In an entirely similar way one can define multivariable weighted shifts. Given $\mathbf{k} \in \mathbb{Z}_+^2$, the *moments* of (α, β) of order $\mathbf{k}$ is

$$\begin{aligned} \gamma_{\mathbf{k}} &\equiv \gamma_{\mathbf{k}}(\alpha,\beta) \\ &:= \begin{cases} 1 & \text{if } \mathbf{k} \equiv (k_1,k_2) = (0,0) \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases} \end{aligned}$$

We remark that, due to the commutativity condition (1.1), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0,0)$ to (k_1,k_2). We then recall basic results for 2-variable weighted shifts. The following is a criterion on hyponormality for 2-variable weighted shifts.

Lemma 2. ([3])(*Six-point Test (see Figure 1-(i))*) *Let* $\mathbf{T} \equiv (T_1, T_2)$ *be a 2-variable weighted shift, with weight sequences* α *and* β. *Then*

$$[\mathbf{T}^*, \mathbf{T}] \geq 0 \Leftrightarrow (([T_j^*, T_i]e_{\mathbf{k}+\varepsilon_j}, e_{\mathbf{k}+\varepsilon_i}))_{i,j=1}^2 \geq 0 \ (\textit{all } \mathbf{k} \in \mathbb{Z}_+^2)$$

$$\Leftrightarrow \begin{pmatrix} \alpha_{\mathbf{k}+\varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta_{\mathbf{k}+\varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geq 0 \ (\textit{all } \mathbf{k} \in \mathbb{Z}_+^2).$$

We now recall a well known characterization of subnormality for multivariable weighted shifts [14], due to C. Berger (cf. [2, II.6.10]) and independently established by Gellar and Wallen [13]) in the single variable case:

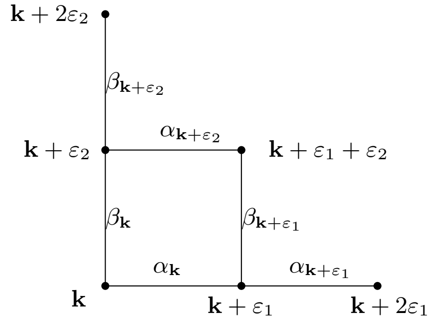

FIGURE 1. Weight diagram used in the Six-point Test

THEOREM 3 (Berger's Theorem). *$W_{\alpha,\beta} \equiv (T_1, T_2)$ is subnormal if and only if there is a probability measure μ (which we call the Berger measure of $W_{\alpha,\beta}$) defined on the 2-dimensional rectangle $R = [0, \|T_1\|^2] \times [0, \|T_2\|^2]$ such that*

$$\gamma_{\mathbf{k}} = \int_R s^{k_1} t^{k_2} d\mu(s,t), \quad \text{for all } \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2.$$

In this paper we consider the following problem.

PROBLEM 4. (*2-variable Subnormal Completion Problem*) *Given $m \geq 0$ and a finite collection of pairs of positive numbers $\Omega_m \equiv \{(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})\}_{|\mathbf{k}| \leq m}$ satisfying (1.1) for all $|\mathbf{k}| \leq m$ (where $|\mathbf{k}| := k_1 + k_2$), find necessary and sufficient conditions to guarantee the existence of a subnormal 2-variable weighted shift whose initial weights are given by Ω_m.*

Problem 4 is closely related to the truncated real moment problems.

PROBLEM 5. *Given real numbers*

$$\gamma \equiv \gamma^{(2n)} := \gamma_{00}, \gamma_{10}, \gamma_{01}, \gamma_{20}, \gamma_{11}, \gamma_{02}, \cdots, \gamma_{2n,0}, \cdots, \gamma_{0,2n} \tag{1.2}$$

with $\gamma_{00} > 0$, the truncated real moment problem for γ entails finding conditions for the existence of a positive Borel measure μ, supported in $\mathbb{R}^2$, such that

$$\gamma_{ij} = \int x^i y^j d\mu, \quad 0 \leq i + j \leq n.$$

Given the truncated moment sequence γ in (2) and $0 \leq i, j \leq n$, we define the $(i+1) \times (j+1)$ matrix $M[i, j]$ whose entries are the moments of order $i + j$:

$$M[i,j](\gamma) \equiv M[i,j] := \begin{pmatrix} \gamma_{i+j,0} & \gamma_{i+j-1,1} & \cdots & \gamma_{i,j} \\ \gamma_{i+j-1,1} & \gamma_{i+j-2,2} & \cdots & \gamma_{i-1,j+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{j,i} & \gamma_{j-1,i+1} & \cdots & \gamma_{0,i+j} \end{pmatrix},$$

where we note that $M[i,j]$ has the Hankel-like property of being constant on each skew-diagonal; in particular, $M[i,i]$ is a self-adjoint Hankel matrix. We now define the *moment matrix* $M(n) \equiv M(n)(\gamma)$ via the block decomposition $M(n) := (M[i,j])_{0\le i,j\le n}$. For example, if $n = 1$, the *quadratic moment problem* for $\gamma : \gamma_{00}, \gamma_{10}, \gamma_{01}, \gamma_{20}, \gamma_{11}, \gamma_{02}$ corresponds to

$$M(1) = \begin{pmatrix} M[0,0] & M[0,1] \\ M[1,0] & M[1,1] \end{pmatrix} = \begin{pmatrix} \gamma_{00} & \gamma_{10} & \gamma_{01} \\ \gamma_{10} & \gamma_{20} & \gamma_{11} \\ \gamma_{01} & \gamma_{11} & \gamma_{02} \end{pmatrix}$$

and if $n = 2$, the *quartic moment problem* for

$$\gamma : \gamma_{00}, \gamma_{10}, \gamma_{01}, \gamma_{20}, \gamma_{11}, \gamma_{02}, \gamma_{30}, \gamma_{21}, \gamma_{12}, \gamma_{03}, \gamma_{40}, \gamma_{31}, \gamma_{22}, \gamma_{13}, \gamma_{04}$$

corresponds to

$$M(2) = \begin{pmatrix} M[0,0] & M[0,1] & M[0,2] \\ M[1,0] & M[1,1] & M[1,2] \\ M[2,0] & M[2,1] & M[2,2] \end{pmatrix}$$
$$= \begin{pmatrix} \gamma_{00} & \gamma_{10} & \gamma_{01} & \gamma_{20} & \gamma_{11} & \gamma_{02} \\ \gamma_{10} & \gamma_{20} & \gamma_{11} & \gamma_{30} & \gamma_{21} & \gamma_{12} \\ \gamma_{01} & \gamma_{11} & \gamma_{02} & \gamma_{21} & \gamma_{12} & \gamma_{03} \\ \gamma_{20} & \gamma_{30} & \gamma_{21} & \gamma_{40} & \gamma_{31} & \gamma_{22} \\ \gamma_{11} & \gamma_{21} & \gamma_{12} & \gamma_{31} & \gamma_{22} & \gamma_{13} \\ \gamma_{02} & \gamma_{12} & \gamma_{03} & \gamma_{22} & \gamma_{13} & \gamma_{04} \end{pmatrix}.$$

(For basic results about truncated moment problems we refer to [4] and [6].)

We conclude this section some notations. For $n \ge 1$, let $m \equiv m(n) := \frac{(n+1)(n+2)}{2}$. For $A \in M_m(\mathbb{R})$ (the set of $m \times m$ real matrices), we denote the successive rows and columns according to the following lexicographic- functional ordering :

$$\mathbf{1}, X, Y, X^2, XY, Y^2, \cdots, X^n, \cdots, Y^n.$$

For A a positive $N \times N$ matrix, if $1 \le n_1 < \cdots < n_k \le N$ we let

$$[A]_{\{n_1,\ldots,n_k\}}$$

denote the compression of A to the rows and columns indexed by $\{n_1, \dots, n_k\}$. We let $\mathcal{C}_{\{n_1,\cdots,n_k\}}$ and $\mathcal{R}_{\{n_1,\cdots,n_k\}}$ denote the column space and the row space, respectively of A, i.e., the subspaces of $\mathbb{R}^N$ spanned by the columns and the rows indexed by $\{n_1, \cdots, n_k\}$ of A, respectively. We also denote the entry of $A \in M_{m(n)}(\mathbb{R})$ in row X^kY^l and column X^iY^j by $A_{(k,l)(i,j)}$. Then it is easy to see that $M(n)_{(k,l)(i,j)} = \gamma_{k+i,l+j}$.

2. Main results

We start this section recalling the main results in [10]. In [10], it was shown:

THEOREM 6. [10] $\mathbf{T} \equiv (T_1, T_2)$ *is subnormal if and only if* $\mathbf{T} \equiv (T_1, T_2)$ *is k-hyponormal for all* $k \geq 1$.

THEOREM 7. [10] *For a 2-variable weighted shift* $W_{\alpha,\beta}$*, the following are equivalent:*

(a) $W_{\alpha,\beta}$ *is k-hyponormal;*
(b) $(\gamma_{\mathbf{k}}\gamma_{\mathbf{k}+(n,m)+(p,q)} - \gamma_{\mathbf{k}+(n,m)}\gamma_{\mathbf{k}+(p,q)})_{\substack{1\leq n+m\leq k\\ 1\leq p+q\leq k}} \geq 0, \quad \forall \mathbf{k} \in \mathbb{Z}_+^2;$
(c) $M_{\mathbf{k}}(k) := (\gamma_{\mathbf{k}+(n,m)+(p,q)})_{\substack{0\leq n+m\leq k\\ 0\leq p+q\leq k}} \geq 0, \quad \forall \mathbf{k} \in \mathbb{Z}_+^2.$

We thus obtain:

COROLLARY 8. $W_{\alpha,\beta}$ *is subnormal if and only if*

$$M_{\mathbf{k}}(\infty) := (\gamma_{\mathbf{k}+(m,n)+(p,q)})_{\substack{m+n\geq 0\\ p+q\geq 0}} \geq 0$$

for $\mathbf{k} = (0,0), (1,0), (0,1), (1,1)$.

Proof. From Theorems 6 and 7, we can see that $W_{\alpha,\beta}$ is subnormal if and only if $M_{\mathbf{k}}(\infty) := (\gamma_{\mathbf{k}+(m,n)+(p,q)})_{\substack{m+n\geq 0\\ p+q\geq 0}} \geq 0$ for all $\mathbf{k} \in \mathbb{Z}_+^2$. But since $M_{\mathbf{u}}(\infty)$ is a principal submatrix of one of $M_{\mathbf{k}}(\infty)$ for $\mathbf{k} = (0,0), (1,0), (0,1), (1,1)$, it follows from well-known result in the matrix analysis which states that a principal submatrix of a positive matrix is positive. □

Corollary 8 provide a solution of the Stieltjes full power moment problem. Recall one variable Stieltjes full power moment problem. In 1894 Thomas Jan Stieltjes (1856-1894) published an extremely influential paper: *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse, 8, 1-122; 9, 5-47. He introduced what is now known as the Stieltjes integral with respect to an increasing function ϕ, the latter describing a distribution of mass (a measure μ) via the convention that the mass in an interval $[a,b]$ is $\mu([a,b]) = \phi(b) - \phi(a)$. This integral was used to solve the following problem which he called the moment problem:

PROBLEM 9. *Given $\gamma \equiv \{\gamma_n\}_{n=0}^{\infty}$, find necessary and sufficient conditions on γ for the existence of positive measure μ with supp$\mu \subset \mathbb{R}^+$ such that*

$$\gamma_n = \int x^n d\mu(x) \quad (\forall n \geq 0). \tag{2.1}$$

The number γ_n is called the nth moment of μ, and the sequence $\{\gamma_n\}$ is called the moment sequence of μ. Stieltjes was led to the Stieltjes moment problem above via a study of continued fractions. But now we can give an answer to the Problem 9 using Theorem 1.

PROPOSITION 10. *The following statements are equivalent:*
(i) There exists positive measure μ on $\mathbb{R}^+$ such that $\gamma_n = \int x^n d\mu(x) \quad (\forall n \geq 0)$.
(ii) $\Lambda(p) \geq 0$ for all $p \geq 0$ on $\mathbb{R}^+$ where $\Lambda(p) := \sum_{i=0}^n a_i\gamma_i$ for $p(x) = \sum_{i=0}^n a_i x^i$.
(iii) $H(\infty) := \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots \\ \gamma_1 & \gamma_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0, H_x(\infty) := \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots \\ \gamma_2 & \gamma_3 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0.$

Proof. (i) $\Longrightarrow$ (ii): Suppose there exist a positive measure μ with *supp* $\mu \subset \mathbb{R}_+$ satisfying (2.1). Then we have

$$\Lambda(p) = \sum_i a_i\gamma_i = \sum_i a_i \int x^i d\mu(x) = \int p(x) d\mu(x) \geq 0$$

for all $p(x) \geq 0$ on $\mathbb{R}_+$.
(ii) $\Longrightarrow$ (iii): Suppose $\Lambda(p) \geq 0$ for all $p \geq 0$ on $\mathbb{R}^+$. Let $p(x) := |a_0 + a_1 x + \cdots + a_n x^n|^2$ and $q(x) := x|a_0 + a_1 x + \cdots + a_n x^n|^2$. Then $p(x) \geq 0$ and $q(x) \geq 0$ on $\mathbb{R}^+$. Thus, $\Lambda(p) = \sum_{i,j}^n a_i\overline{a_j}\gamma_{i+j} \geq 0$ and $\Lambda(q) = \sum_{i,j}^n a_i\overline{a_j}\gamma_{i+j+1} \geq 0$. Since n and a_i are arbitrary, it follows that $H(\infty) \geq 0$ and $H_x(\infty) \geq 0$.
(iii) $\Longrightarrow$ (i): Suppose $H(\infty) \geq 0$ and $H_x(\infty) \geq 0$. Since $H(\infty) \geq 0$, we can see that $\gamma_0 > 0$. Let $\gamma' \equiv \{\gamma'_n\}_{n=0}^{\infty}$ with $\gamma'_n := \frac{\gamma_n}{\gamma_0}$. Then $H(\infty) \geq 0$ and $H_x(\infty) \geq 0$ for γ'. Now we can reproduce the weight sequence α form γ' via $\alpha_n := \sqrt{\frac{\gamma'_{n+1}}{\gamma'_n}}$. From Theorem 1, we can see that W_α is subnormal. Hence γ' has a Berger measure ν. Therefore $\mu = \gamma_0\nu$. □

The two-variable Stieltjes power moment problem states that:

Problem 11. *Given $\gamma \equiv \{\gamma_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$, find necessary and sufficient conditions on γ for the existence of positive measure μ with supp$\mu \subset \mathbb{R}_+^2$ such that*

$$\gamma_{\mathbf{k}} = \int x^{k_1} y^{k_2} d\mu(x,y) \quad (\forall \mathbf{k} := (k_1,k_2) \in \mathbb{Z}_+^2).$$

For given $\gamma \equiv \{\gamma_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$, define a linear functional Γ_γ on $\mathbb{R}[x,y]$ by

$$\Gamma_\gamma(p) := \sum_{i,j} a_{ij}\gamma_{ij},$$

where $p(x,y) = \sum_{i,j} a_{ij}x^iy^j$. Then, from Corollary 8, we have an answer to the two-variable Stieltjes power moment problem.

Proposition 12. *Given $\gamma \equiv \{\gamma_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$, the following statements are equivalent:*
(i)There exists positive measure μ with supp $\mu \subset \mathbb{R}_+^2$ such that

$$\gamma_{\mathbf{k}} = \int x^{k_1} y^{k_2} d\mu(x,y) \quad (\forall \mathbf{k} := (k_1,k_2) \in \mathbb{Z}_+^2). \tag{2.2}$$

(ii) $\Gamma_\gamma(p) \geq 0$ for all $p(x,y) \geq 0$ on $\mathbb{R}_+^2$.
(iii) $M_{\mathbf{k}}(\infty) := (\gamma_{\mathbf{k}+(m,n)+(p,q)})_{\substack{m+n\geq 0\\ p+q\geq 0}} \geq 0$ for $\mathbf{k} = (0,0)$, $(1,0)$, $(0,1)$, $(1,1)$.

Proof. (i) $\Longrightarrow$ (ii): Suppose there exist a positive measure μ with *supp* $\mu \subset \mathbb{R}_+^2$ satisfying (2.2). Then we have

$$\Gamma_\gamma(p) = \sum_{i,j} a_{ij}\gamma_{ij} = \sum_{i,j} a_{ij} \int x^iy^j d\mu(x,y) = \int p(x,y)d\mu(x,y) \geq 0$$

for all $p(x,y) \geq 0$ on $\mathbb{R}_+^2$.
(ii) $\Longrightarrow$ (iii): Suppose $\Gamma_\gamma(p) \geq 0$ for all $p(x,y) \geq 0$ on $\mathbb{R}_+^2$. Let $p(x,y) := |a_{00} + a_{10}x + a_{01}y + \cdots + a_{n0}x^n + a_{n-1,1}x^{n-1}y + \cdots + a_{0n}y^n|^2$ and $q(x,y) := xp(x,y)$ and $r(x,y) := yp(x,y)$ and $s(x,y) := xyp(x,y)$. Then $p(x,y), q(x,y), r(x,y), s(x,y) \geq 0$ on $\mathbb{R}_+^2$. Thus, $\Gamma_\gamma(p) = \sum a_{ij}\overline{a_{kl}}\gamma_{i+k,j+l} \geq 0$, $\Gamma_\gamma(q) = \sum a_{ij}\overline{a_{kl}}\gamma_{i+k+1,j+l} \geq 0$, $\Gamma_\gamma(r) = \sum a_{ij}\overline{a_{kl}}\gamma_{i+k,j+l+1} \geq 0$ and $\Gamma_\gamma(s) = \sum a_{ij}\overline{a_{kl}}\gamma_{i+k+1,j+l+1} \geq 0$. Thus, it follows that $M_{\mathbf{k}}(\infty) \geq 0$ for $\mathbf{k} = (0,0)$, $(1,0)$, $(0,1)$, $(1,1)$.
(iii) $\Longrightarrow$ (i): If $M_{\mathbf{k}}(\infty) \geq 0$ for $\mathbf{k} = (0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, then by Corollary 8, we can construct a 2-variable weighted shift $W_{\alpha,\beta}$ which is subnormal. Thus, it has the Berger measure μ with *supp* $\mu \subset \mathbb{R}_+^2$ satisfying (2.2). □

For the truncated moment problem, the direct analogue of the equivalence $(i) \iff (ii)$ above proposition need not hold. For example, let $\gamma : \gamma_{00} = \gamma_{10} = \gamma_{01} = \gamma_{20} = \gamma_{11} = 1$ and $\gamma_{02} = 2$. Then

(1) $\Gamma_\gamma(p) \geq 0$ for all $p(x,y) \geq 0$ with $\deg p \leq 2$.
(2) γ has no representing measure.
(3) $M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{10} & \gamma_{01} \\ \gamma_{10} & \gamma_{20} & \gamma_{11} \\ \gamma_{01} & \gamma_{11} & \gamma_{02} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \geq 0.$

Next we consider the main theorem in [6]. Although [6, Theorem 1.6] deals with truncated complex moment problems, there is an entirely equivalent version for the case of two real variables, which we now state. Let $K_{\mathcal{P}} := \{(x,y) \in \mathbb{R}^2 : p_i(x,y) \geq 0, \forall p_i \in \mathcal{P}\}$ for $\mathcal{P} \equiv \{p_1, \cdots, p_m\} \subseteq \mathbb{R}[x,y]$ and define k_i by $\deg p_i = 2k_i$ or $\deg p_i = 2k_i - 1$ $(1 \leq i \leq m)$.

In [6], Curto and Fialkow showed that the following statements are equivalent:

(i) There exists a rank $M(n)$-atomic representing measure μ for $\gamma^{(2n)}$ with supp$\mu \subseteq K_{\mathcal{P}}$.
(ii) $M(n) \geq 0$ and there is some flat (i.e., rank-preserving) extension $M(n+1)$ for which $M_{p_i}(n+k_i) \geq 0$ $(1 \leq i \leq m)$.

In this case, the representing measure for $M(n+1)$ is rank $M(n)$-atomic, supported in $K_{\mathcal{P}}$, and with precisely rank$M(n)-$rank$M_{p_i}(n+k_i)$ atoms in $\mathcal{Z}(p_i) := \{(x,y) \in \mathbb{R}^2 : p_i(x,y) = 0\}$ $(1 \leq i \leq m)$.

If we let $p_1(x,y) := x$ and $p_2(x,y) := y$, then $K_{\mathcal{P}} = \mathbb{R}^2_+$, $M_{p_1}(n+k_1) = M_x(n+1)$ and $M_{p_2}(n+k_2) = M_y(n+1)$. Thus, we have an abstract solution of Problem 4(Odd case):

Theorem 13. *If m is odd, then the following statements are equivalent:*

(i) *Ω_m has a subnormal completion.*
(ii) *There exists a rank $M(k)$-atomic representing measure μ for $\gamma(\Omega_m)$ supported in $\mathbb{R}^2_+$ where $k := [\frac{m+1}{2}]$.*
(iii) *$M(\Omega_m) = M(k) \geq 0$ and Ω_m admits a commutative extension $\widehat{\Omega}_{m+2}$ such that the moment matrix $M(\widehat{\Omega}_{m+2}) = M(k+1)$ is a flat extension of $M(k)$, $M_x(k+1) \geq 0$ and $M_y(k+1) \geq 0$.*

In this case, the Berger measure μ of a subnormal completion $\widehat{\Omega}_\infty$ of Ω_m has rank$M(k)-$rank$M_x(k+1)$ atoms in $\{0\} \times \mathbb{R}_+$ (resp. rank$M(k)-$rank$M_y(k+1)$ atoms in $\mathbb{R}_+ \times \{0\}$). We remark that the Theorem 13 resembles the Corollary 8 in the sense that $M = M_{00}, M_x = M_{10}, M_y = M_{01}$.

In 1966, Stampfli [16] showed that $\alpha : \alpha_0 < \alpha_1 < \alpha_2$ has always subnormal completions, directly constructing the normal extension. It was highly nontrivial that time. But it is easy now from Theorem 1.

Similarly, we can give a a concrete criterion of the Problem 4 for the case $m = 1$.

THEOREM 14. *Given* $\Omega_1 := \{(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}) : |\mathbf{k}| \leq 1\}$ *satisfying* $\beta_{10}\alpha_{00} = \alpha_{01}\beta_{00}$, *the following statement are equivalent:*

(i) Ω_1 *has a subnormal completion;*

(ii) Ω_1 *has a hyponormal completion;*

(iii) $(\alpha_{10}^2 - \alpha_{00}^2)(\beta_{01}^2 - \beta_{00}^2) \geq (\alpha_{01}\beta_{10} - \alpha_{00}\beta_{00})^2$;

(iv) $M(1) := \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \end{pmatrix} = \begin{pmatrix} 1 & \alpha_{00}^2 & \beta_{00}^2 \\ \alpha_{00}^2 & \alpha_{00}^2\alpha_{10}^2 & \alpha_{00}^2\beta_{10}^2 \\ \beta_{00}^2 & \alpha_{00}^2\beta_{10}^2 & \beta_{00}^2\beta_{01}^2 \end{pmatrix} \geq 0.$

Proof. See [12]. □

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