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$\mathcal{N}$-IDEALS OF BCK/BCI-ALGERBAS

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Abstract. The notions of $\mathcal{N}$-subalgebras, (closed, commutative, retrenched) $\mathcal{N}$-ideals, θ-negative functions, and α-translations are introduced, and related properties are investigated. Characterizations of an $\mathcal{N}$-subalgebra and a (commutative) $\mathcal{N}$-ideal are given. Relations between an $\mathcal{N}$-subalgebra, an $\mathcal{N}$-ideal and commutative $\mathcal{N}$-ideal are discussed. We verify that every α-translation of an $\mathcal{N}$-subalgebra (resp. $\mathcal{N}$-ideal) is a retrenched $\mathcal{N}$-subalgebra (resp. retrenched $\mathcal{N}$-ideal).

1. Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, we introduce and use a new function which is called negative-valued function. The important achievement of this article is that one can deal with positive and negative information simultaneously by combining ideas in this article and already well known positive information.

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [9], and have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory

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and topology. Such algebras generalize Boolean rings as well as Boolean D-posets ($= MV$-algebras). Also, Iséki introduced the notion of a BCI-algebra which is a generalization of a BCK-algerba (see [10]). Several properties on BCK/BCI-algebras are investigated in the papers [1–7], [11–14], [16] and [18]. Soft set theory is applied to BCK/BCI-algebra by Y. B. Jun [15] and Y. B. Jun and C. H. Park [17]. Fuzzy set theory in BCK/BCI-algebras is discussed by several researchers. In this paper, we discuss the ideal theory of BCK/BCI-algebras based on negative-valued functions. We introduce the notions of $\mathcal{N}$-subalgebras, (closed, commutative, retrenched) $\mathcal{N}$-ideals, θ-negative functions, and α-translations, and then we investigate several properties. We give characterizations of an $\mathcal{N}$-subalgebra and a (commutative) $\mathcal{N}$-ideal. We discuss relations between an $\mathcal{N}$-subalgebra, an $\mathcal{N}$-ideal and commutative $\mathcal{N}$-ideal. We show that every α-translation of an $\mathcal{N}$-subalgebra (resp. $\mathcal{N}$-ideal) is a retrenched $\mathcal{N}$-subalgebra (resp. retrenched $\mathcal{N}$-ideal).

2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BCI-algebra* we mean a system $X := (X, *, \theta) \in K(\tau)$ in which the following axioms hold:

$$((x * y) * (x * z)) * (z * y) = \theta, \tag{2.1}$$

$$(x * (x * y)) * y = \theta, \tag{2.2}$$

$$x * x = \theta, \tag{2.3}$$

$$x * y = y * x = \theta \Rightarrow x = y, \tag{2.4}$$

for all $x, y, z \in X$. We can define a partial ordering $\preceq$ by

$$(\forall x, y \in X)\,(x \preceq y \Leftrightarrow x * y = \theta).$$

In a BCK/BCI-algebra X, the following hold:

$$x * \theta = x, \tag{2.5}$$

$$(x * y) * z = (x * z) * y, \tag{2.6}$$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $\theta * x = \theta$ for all $x \in X$, then we say that X is a *BCK-algebra*. A BCK-algebra X is said to be *commutative* if it satisfies the following equality:

$$(\forall x, y \in X)\,(x \vec{\wedge} y = y \vec{\wedge} x) \tag{2.7}$$

where $x \vec{\wedge} y = x * (x * y)$.

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A non-empty subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$\theta \in A, \tag{2.8}$$

$$(\forall x, y \in X)\,(x * y \in A \ \&\ y \in A \ \Rightarrow\ x \in A). \tag{2.9}$$

A non-empty subset A of a BCK-algebra X is called a *commutative ideal* of X (see [18]) if it satisfies (2.8) and

$$(\forall x, y, z \in X)\,((x * y) * z \in A \ \&\ z \in A \ \Rightarrow\ x * (y \vec{\wedge} x) \in A). \tag{2.10}$$

Note that any commutative ideal in a BCK-algebra is an ideal, but the converse is not valid (see [18]). We refer the reader to the books [8] and [19] for further information regarding BCK/BCI-algebras.

3. $\mathcal{N}$-subalgebras and (commutative) $\mathcal{N}$-ideals

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, *$\mathcal{N}$-function* on X.) By an *$\mathcal{N}$-structure* we mean an ordered pair (X, φ) of X and an $\mathcal{N}$-function φ on X. In what follows, let X denote a BCK/BCI-algebra and φ an $\mathcal{N}$-function on X unless otherwise specified.

Definition 3.1. By a *subalgebra* of X based on $\mathcal{N}$-function φ (briefly, *$\mathcal{N}$-subalgebra* of X), we mean an $\mathcal{N}$-structure (X, φ) in which φ satisfies the following assertion:

$$(\forall x, y \in X)(\varphi(x * y) \leq \max\{\varphi(x), \varphi(y)\}). \tag{3.1}$$

For any $\mathcal{N}$-function φ on X and $t \in [-1, 0)$, the set

$$C(\varphi; t) := \{x \in X \mid \varphi(x) \leq t\}$$

is called a *closed* (φ, t)*-cut* of φ, and the set

$$O(\varphi; t) := \{x \in X \mid \varphi(x) < t\}$$

is called an *open* (φ, t)*-cut* of φ.

Theorem 3.2. *Let (X, φ) be an $\mathcal{N}$-structure of X and φ. Then (X, φ) is an $\mathcal{N}$-subalgebra of X if and only if every non-empty closed (φ, t)-cut of φ is a subalgebra of X for all $t \in [-1, 0)$.*

Proof. Assume that (X, φ) is an $\mathcal{N}$-subalgebra of X and let $t \in [-1, 0)$ be such that $C(\varphi; t) \neq \emptyset$. Let $x, y \in C(\varphi; t)$. Then $\varphi(x) \leq t$ and $\varphi(y) \leq t$. It follows from (3.1) that $\varphi(x * y) \leq \max\{\varphi(x), \varphi(y)\} \leq t$ so that $x * y \in C(\varphi; t)$. Hence $C(\varphi; t)$ is a subalgebra of X.

Conversely, suppose that every non-empty closed (φ, t)-cut of X is a subalgebra of X for all $t \in [-1, 0)$. If (X, φ) is not an $\mathcal{N}$-subalgebra of X, then $\varphi(a * b) > t_0 \geq \max\{\varphi(a), \varphi(b)\}$ for some $a, b \in X$ and $t_0 \in [-1, 0)$. Hence $a, b \in C(\varphi; t_0)$ and $a * b \notin C(\varphi; t_0)$. This is a contradiction. Thus $\varphi(x * y) \leq \max\{\varphi(x), \varphi(y)\}$ for all $x, y \in X$. □

COROLLARY 3.3. *If (X, φ) is an $\mathcal{N}$-subalgebra of X, then every non-empty open (φ, t)-cut of X is a subalgebra of X for all $t \in [-1, 0)$.*

Proof. Straightforward. □

LEMMA 3.4. *Every $\mathcal{N}$-subalgebra (X, φ) of X satisfies the following inequality:*

$$(\forall x \in X)(\varphi(x) \geq \varphi(\theta)). \tag{3.2}$$

Proof. Note that $x * x = \theta$ for all $x \in X$. Using (3.1), we have $\varphi(\theta) = \varphi(x * x) \leq \max\{\varphi(x), \varphi(x)\} = \varphi(x)$ for all $x \in X$. □

PROPOSITION 3.5. *If every $\mathcal{N}$-subalgebra (X, φ) of X satisfies the following inequality:*

$$(\forall x, y \in X)(\varphi(x * y) \leq \varphi(y)), \tag{3.3}$$

then φ is a constant function.

Proof. Let $x \in X$. Using (2.5) and (3.3), we have $\varphi(x) = \varphi(x * \theta) \leq \varphi(\theta)$. It follows from Lemma 3.4 that $\varphi(x) = \varphi(\theta)$, and so φ is a constant function. □

DEFINITION 3.6. By an *ideal* of X based on $\mathcal{N}$-function φ (briefly, $\mathcal{N}$-*ideal* of X), we mean an $\mathcal{N}$-structure (X, φ) in which φ satisfies the following assertion:

$$(\forall x, y \in X)(\varphi(\theta) \leq \varphi(x) \leq \max\{\varphi(x * y), \varphi(y)\}). \tag{3.4}$$

EXAMPLE 3.7. *Let $X = \{\theta, a, b, c\}$ be a set with the following Cayley table:*

$*$	θ	a	b	c
θ	θ	θ	θ	θ
a	a	θ	θ	a
b	b	a	θ	b
c	c	c	c	θ

Then $(X, *, \theta)$ *is a BCK-algebra. Define an* $\mathcal{N}$*-function* φ *by*

X	θ	a	b	c
φ	-0.7	-0.5	-0.5	-0.3

It is easily verified that (X, φ) *is both an* $\mathcal{N}$*-subalgebra and an* $\mathcal{N}$*-ideal of* X.

Example 3.8. *Consider a BCI-algebra* $X := Y \times \mathbb{Z}$ *where* $(Y, *, \theta)$ *is a BCI-algebra and* $(\mathbb{Z}, -, 0)$ *is the adjoint BCI-algebra of the additive group* $(\mathbb{Z}, +, 0)$ *of integers (see* [8]*). Let* φ *be an* $\mathcal{N}$*-function on* X *defined by*

$$\varphi(x) = \begin{cases} t & \text{if } x \in Y \times (\mathbb{N} \cup \{0\}), \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in X$ *where* $\mathbb{N}$ *is the set of all natural numbers and* t *is fixed in* $[-1, 0)$. *We can easily check that* φ *satisfies the condition (3.4), and so* (X, φ) *is an* $\mathcal{N}$*-ideal of* X.

Proposition 3.9. *If* (X, φ) *is an* $\mathcal{N}$*-ideal of* X, *then*

$$(\forall x, y \in X)\,(x \preceq y \;\Rightarrow\; \varphi(x) \leq \varphi(y)). \tag{3.5}$$

Proof. Let $x, y \in X$ be such that $x \preceq y$. Then $x * y = \theta$, and so

$$\varphi(x) \leq \max\{\varphi(x * y), \varphi(y)\} = \max\{\varphi(\theta), \varphi(y)\} = \varphi(y).$$

This completes the proof. □

Proposition 3.10. *Let* (X, φ) *be an* $\mathcal{N}$*-ideal of* X. *Then the following are equivalent:*

(i) $(\forall x, y \in X)\ (\varphi(x * y) \leq \varphi((x * y) * y))$,
(ii) $(\forall x, y, z \in X)\ (\varphi((x * z) * (y * z)) \leq \varphi((x * y) * z))$.

Proof. Assume that (i) is valid and let $x, y, z \in X$. Since

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \preceq (x * y) * z,$$

it follows from Proposition 3.9 that $\varphi(((x*(y*z))*z)*z) \leq \varphi((x*y)*z)$. Using (2.6) and (i), we have

$$\varphi((x*z)*(y*z)) = \varphi((x*(y*z))*z) \leq \varphi(((x*(y*z))*z)*z) \leq \varphi((x*y)*z).$$

Conversely suppose that (ii) holds. If we use z instead of y in (ii), then

$$\varphi(x * z) = \varphi((x * z) * \theta) = \varphi((x * z) * (z * z)) \leq \varphi((x * z) * z)$$

for all $x, z \in X$ by using (2.3) and (2.5). This proves (i). □

THEOREM 3.11. *For any subalgebra* (resp. *ideal*) U *of* X, *there exists an* $\mathcal{N}$*-function* φ *such that* (X, φ) *is an* $\mathcal{N}$*-subalgebra* (resp. $\mathcal{N}$*-ideal*) *of* X *and* $C(\varphi; t) = U$ *for some* $t \in [-1, 0)$.

Proof. Let U be a subalgebra (resp. ideal) of X and let φ be an $\mathcal{N}$-function on X defined by

$$\varphi(x) = \begin{cases} 0 & \text{if } x \notin U, \\ t & \text{if } x \in U \end{cases}$$

where t is fixed in $[-1, 0)$. Then (X, φ) is an $\mathcal{N}$-subalgebra (resp. $\mathcal{N}$-ideal) of X and $C(\varphi; t) = U$. $\square$

THEOREM 3.12. *Let* (X, φ) *be an* $\mathcal{N}$*-structure of* X *and* φ. *Then* (X, φ) *is an* $\mathcal{N}$*-ideal of* X *if and only if it satisfies:*

$$(3.6) \qquad (\forall t \in [-1, 0))\,(C(\varphi; t) \neq \emptyset \;\Rightarrow\; C(\varphi; t) \text{ is an ideal of } X).$$

Proof. Assume that (X, φ) is an $\mathcal{N}$-ideal of X. Let $t \in [-1, 0)$ be such that $C(\varphi; t) \neq \emptyset$. Obviously, $\theta \in C(\varphi; t)$. Let $x, y \in X$ be such that $x * y \in C(\varphi; t)$ and $y \in C(\varphi; t)$. Then $\varphi(x * y) \leq t$ and $\varphi(y) \leq t$. It follows from (3.4) that $\varphi(x) \leq \max\{\varphi(x * y), \varphi(y)\} \leq t$, so that $x \in C(\varphi; t)$. Hence $C(\varphi; t)$ is an ideal of X.

Conversely, suppose that (3.6) is valid. If there exists $a \in X$ such that $\varphi(\theta) > \varphi(a)$, then $\varphi(\theta) > t_a \geq \varphi(a)$ for some $t_a \in [-1, 0)$. Then $\theta \notin C(\varphi; t_a)$ which is a contradiction. Hence $\varphi(\theta) \leq \varphi(x)$ for all $x \in X$. Now, assume that there exists $a, b \in X$ such that $\varphi(a) > \max\{\varphi(a * b), \varphi(b)\}$. Then there exists $s \in [-1, 0)$ such that $\varphi(a) > s \geq \max\{\varphi(a * b), \varphi(b)\}$. It follows that $a * b \in C(\varphi; s)$ and $b \in C(\varphi; s)$, but $a \notin C(\varphi; s)$. This is impossible, and so $\varphi(x) \leq \max\{\varphi(x * y), \varphi(y)\}$ for all $x, y \in X$. Therefore (X, φ) is an $\mathcal{N}$-ideal of X. $\square$

COROLLARY 3.13. *If* (X, φ) *is an* $\mathcal{N}$*-ideal of* X, *then every non-empty open* (φ, t)*-cut of* X *is an ideal of* X *for all* $t \in [-1, 0)$.

Proof. Straightforward. $\square$

PROPOSITION 3.14. *Let* (X, φ) *be an* $\mathcal{N}$*-ideal of* X. *If* X *satisfies the following assertion:*

$$(3.7) \qquad (\forall x, y, z \in X)(x * y \preceq z),$$

then $\varphi(x) \leq \max\{\varphi(y), \varphi(z)\}$ *for all* $x, y, z \in X$.

Proof. Assume that (3.7) is valid in X. Then $\varphi(x * y) \leq \max\{\varphi((x * y) * z), \varphi(z)\} = \max\{\varphi(\theta), \varphi(z)\} = \varphi(z)$ for all $x, y, z \in X$. It follows that $\varphi(x) \leq \max\{\varphi(x * y), \varphi(y)\} \leq \max\{\varphi(y), \varphi(z)\}$ for all $x, y, z \in X$. This completes the proof. $\square$

THEOREM 3.15. *For any BCK-algebra X, every $\mathcal{N}$-ideal is an $\mathcal{N}$-subalgebra.*

Proof. Let (X, φ) be an $\mathcal{N}$-ideal of a BCK-algebra X and let $x, y \in X$. Then

$$\begin{aligned}\varphi(x*y) &\leq \max\{\varphi((x*y)*x), \varphi(x)\} = \max\{\varphi((x*x)*y), \varphi(x)\} \\ &= \max\{\varphi(\theta*y), \varphi(x)\} = \max\{\varphi(\theta), \varphi(x)\} \leq \max\{\varphi(x), \varphi(y)\}.\end{aligned}$$

Therefore (X, φ) is an $\mathcal{N}$-subalgebra of X. □

The converse of Theorem 3.15 may not be true in general as seen in the following example.

EXAMPLE 3.16. *Consider a BCK-algebra $X = \{\theta, 1, 2, 3, 4\}$ with the following Cayley table:*

$*$	θ	1	2	3	4
θ	θ	θ	θ	θ	θ
1	1	θ	θ	θ	θ
2	2	1	θ	1	θ
3	3	3	3	θ	θ
4	4	4	4	3	θ

Define an $\mathcal{N}$-function φ on X by

X	θ	1	2	3	4
φ	-0.8	-0.8	-0.2	-0.7	-0.4

Then (X, φ) is an $\mathcal{N}$-subalgebra of X. But it is not an $\mathcal{N}$-ideal of X since $\varphi(2) = -0.2 > -0.7 = \max\{\varphi(2*3), \varphi(3)\}$.

The following example shows that Theorem 3.15 is not valid in a BCI-algebra X, that is, if X is a BCI-algebra then an $\mathcal{N}$-ideal (X, φ) may not be an $\mathcal{N}$-subalgebra for some $\mathcal{N}$-function φ on X.

EXAMPLE 3.17. *Consider the $\mathcal{N}$-ideal (X, φ) which is described in Example 3.8. Take $x = (\theta, 0)$ and $y = (\theta, 1)$. Then $z := x*y = (\theta, 0)*(\theta, 1) = (\theta, -1)$, and so $\varphi(x*y) = \varphi(z) = 0 > t = \max\{\varphi(x), \varphi(y)\}$. Therefore (X, φ) is not an $\mathcal{N}$-subalgebra of X.*

For any element w of X, we consider the set

$$X_w := \{x \in X \mid \varphi(x) \leq \varphi(w)\}.$$

Obviously, $w \in X_w$, and so X_w is a non-empty subset of X.

THEOREM 3.18. *Let w be an element of X. If (X, φ) is an $\mathcal{N}$-ideal of X, then the set X_w is an ideal of X.*

Proof. Obviously, $\theta \in X_w$ by (3.4). Let $x, y \in X$ be such that $x * y \in X_w$ and $y \in X_w$. Then $\varphi(x*y) \leq \varphi(w)$ and $\varphi(y) \leq \varphi(w)$. Since (X, φ) is an $\mathcal{N}$-ideal of X, it follows from (3.4) that $\varphi(x) \leq \max\{\varphi(x*y), \varphi(y)\} \leq \varphi(w)$ so that $x \in X_w$. Hence X_w is an ideal of X. □

THEOREM 3.19. *Let w be an element of X and let (X, φ) be an $\mathcal{N}$-syructure of X and φ. Then*

(i) *If X_w is an ideal of X, then (X, φ) satisfies the following assertion:*

(3.8) $(\forall x, y, z \in X)(\varphi(x) \geq \max\{\varphi(y * z), \varphi(z)\} \;\Rightarrow\; \varphi(x) \geq \varphi(y))$.

(ii) *If (X, φ) satisfies $\varphi(\theta) \leq \varphi(x)$ for all $x \in X$ and* (3.8), *then X_w is an ideal of X.*

Proof. (i) Assume that X_w is an ideal of X for each $w \in X$. Let $x, y, z \in X$ be such that $\varphi(x) \geq \max\{\varphi(y * z), \varphi(z)\}$. Then $y * z \in X_x$ and $z \in X_x$. Since X_x is an ideal of X, it follows that $y \in X_x$, that is, $\varphi(y) \leq \varphi(x)$.

(ii) Suppose that (X, φ) satisfies $\varphi(\theta) \leq \varphi(x)$ for all $x \in X$ and (3.8). For each $w \in X$, let $x, y \in X$ be such that $x * y \in X_w$ and $y \in X_w$. Then $\varphi(x * y) \leq \varphi(w)$ and $\varphi(y) \leq \varphi(w)$, which imply that $\max\{\varphi(x * y), \varphi(y)\} \leq \varphi(w)$. Using (3.8), we have $\varphi(w) \geq \varphi(x)$ and so $x \in X_w$. Obviously $\theta \in X_w$. Therefore X_w is an ideal of X. □

DEFINITION 3.20. Let X be a BCI-algebra. An $\mathcal{N}$-ideal (X, φ) is said to be *closed* if it is also an $\mathcal{N}$-subalgebra of X.

EXAMPLE 3.21. *Let $X = \{\theta, 1, a, b, c\}$ be a BCI-algebra with the following Cayley table:*

$*$	θ	1	a	b	c
θ	θ	θ	a	b	c
1	1	θ	a	b	c
a	a	a	θ	c	b
b	b	b	c	θ	a
c	c	c	b	a	θ

Let φ be an $\mathcal{N}$-function on X defined by

X	θ	1	a	b	c
φ	-0.9	-0.7	-0.6	-0.2	-0.2

Then (X, φ) is a closed $\mathcal{N}$-ideal of X.

THEOREM 3.22. *Let X be a BCI-algebra and let φ be defined by*

$$\varphi(x) = \begin{cases} t_1 & \text{if } x \in X_+, \\ t_2 & \text{otherwise} \end{cases}$$

where $t_1, t_2 \in [-1, 0)$ with $t_1 < t_2$ and $X_+ = \{x \in X \mid \theta \preceq x\}$. Then (X, φ) is a closed $\mathcal{N}$-ideal of X.

Proof. Since $\theta \in X_+$, we have $\varphi(\theta) = t_1 \leq \varphi(x)$ for all $x \in X$. Let $x, y \in X$. If $x \in X_+$, then $\varphi(x) = t_1 \leq \max\{\varphi(x*y), \varphi(y)\}$. Assume that $x \notin X_+$. If $x * y \in X_+$ then $y \notin X_+$; and if $y \in X_+$ then $x * y \notin X_+$. In either case, we get $\varphi(x) = t_2 = \max\{\varphi(x*y), \varphi(y)\}$. For every $x, y \in X$, if any one of x and y does not belong to X_+, then

$$\varphi(x * y) \leq t_2 = \max\{\varphi(x), \varphi(y)\}.$$

If $x, y \in X_+$, then $x * y \in X_+$, and so $\varphi(x * y) = t_1 = \max\{\varphi(x), \varphi(y)\}$. Therefore (X, φ) is a closed $\mathcal{N}$-ideal of X. □

DEFINITION 3.23. Let X be a BCI-algebra. If an $\mathcal{N}$-function φ on X satisfies the following condition:

$$(\forall x \in X)(\varphi(\theta * x) \leq \varphi(x)),$$

then we say that φ is a *θ-negative function.*

PROPOSITION 3.24. *Let X be a BCI-algebra. If (X, φ) is a closed $\mathcal{N}$-ideal of X, then φ is a θ-negative function.*

Proof. For any $x \in X$, we have

$$\varphi(\theta * x) \leq \max\{\varphi(\theta), \varphi(x)\} \leq \max\{\varphi(x), \varphi(x)\} = \varphi(x).$$

Therefore φ is a θ-negative function. □

We provide a condition for an $\mathcal{N}$-ideal to be closed.

PROPOSITION 3.25. *Let X be a BCI-algebra. If (X, φ) is an $\mathcal{N}$-ideal of X in which φ is θ-negative, then (X, φ) is an $\mathcal{N}$-subalgebra of X*

Proof. Note that $(x * y) * x \preceq \theta * y$ for all $x, y \in X$. Using Proposition 3.14 and the θ-negativity of φ, we have

$$\varphi(x * y) \leq \max\{\varphi(x), \varphi(\theta * y)\} \leq \max\{\varphi(x), \varphi(y)\}.$$

Therefore (X, φ) is an $\mathcal{N}$-subalgebra of X. □

DEFINITION 3.26. Let X be a BCK-algebra. By a *commutative ideal of X based on φ* (briefly, *commutative $\mathcal{N}$-ideal* of X), we mean an $\mathcal{N}$-structure (X, φ) in which φ satisfies (3.2) and

(3.9) $$(\forall x, y, z \in X)(\varphi(x * (y \,\vec{\wedge}\, x)) \leq \max\{\varphi((x * y) * z), \varphi(z)\}).$$

EXAMPLE 3.27. *Consider a BCK-algebra $X = \{\theta, a, b, c\}$ which is given in Example 3.7. Let φ be defined by*

X	θ	a	b	c
φ	-0.6	-0.4	-0.3	-0.3

Routine calculations give that (X, φ) is a commutative $\mathcal{N}$-ideal of X.

THEOREM 3.28. *Every commutative $\mathcal{N}$-ideal of a BCK-algebra X is an $\mathcal{N}$-ideal of X.*

Proof. Let (X, φ) be a commutative $\mathcal{N}$-ideal of X. For any $x, y, z \in X$, we have

$$\varphi(x) = \varphi(x * (\theta \,\vec{\wedge}\, x)) \leq \max\{\varphi((x * \theta) * z), \varphi(z)\} = \max\{\varphi(x * z), \varphi(z)\}.$$

Hence (X, φ) is an $\mathcal{N}$-ideal of X. $\square$

The following example shows that the converse of Theorem 3.28 is not valid.

EXAMPLE 3.29. *Consider a BCK-algebra $X = \{\theta, 1, 2, 3, 4\}$ with the following Cayley table:*

$*$	θ	1	2	3	4
θ	θ	θ	θ	θ	θ
1	1	θ	1	θ	θ
2	2	2	θ	θ	θ
3	3	3	3	θ	θ
4	4	4	4	3	θ

Let φ be defined by

X	θ	1	2	3	4
φ	-0.7	-0.6	-0.4	-0.4	-0.4

Then (X, φ) is an $\mathcal{N}$-ideal of X. But it is not a commutative $\mathcal{N}$-ideal of X since

$$\varphi(2 * (3 \,\vec{\wedge}\, 2))) = -0.4 > -0.7 = \max\{\varphi((2 * 3) * \theta), \varphi(\theta)\}.$$

THEOREM 3.30. *If (X, φ) is an $\mathcal{N}$-ideal of a commutative BCK-algebra X, then it is a commutative $\mathcal{N}$-ideal of X.*

Proof. Assume that (X, φ) is an $\mathcal{N}$-ideal of a commutative BCK-algebra X. Using (2.1) and (2.6), we have

$$\begin{aligned} ((x * (y \vec{\wedge} x)) * ((x * y) * z)) * z &= ((x * (y \vec{\wedge} x)) * z) * ((x * y) * z) \\ &\preceq (x * (y \vec{\wedge} x)) * (x * y) \\ &= (x \vec{\wedge} y) * (y \vec{\wedge} x) \;=\; \theta, \end{aligned}$$

and so $((x * (y \vec{\wedge} x)) * ((x * y) * z)) * z = \theta$, i.e.,

$$(x * (y \vec{\wedge} x)) * ((x * y) * z) \preceq z$$

for all $x, y, z \in X$. Since (X, φ) is an $\mathcal{N}$-ideal, it follows from Proposition 3.14 that $\varphi(x * (y \vec{\wedge} x)) \leq \max\{\varphi((x * y) * z), \varphi(z)\}$. Hence (X, φ) is a commutative $\mathcal{N}$-ideal of X. $\square$

THEOREM 3.31. *Let (X, φ) be an $\mathcal{N}$-structure of a BCK-algebra X and φ. Then (X, φ) is a commutative $\mathcal{N}$-ideal of X if and only if it satisfies:* (3.10)

$$(\forall t \in [-1, 0))\,(C(\varphi; t) \neq \emptyset \;\Rightarrow\; C(\varphi; t) \text{ is a commutative ideal of } X).$$

Proof. Assume that (X, φ) is a commutative $\mathcal{N}$-ideal of X. Then (X, φ) is an $\mathcal{N}$-ideal of X, and so every non-empty closed (φ, t)-cut $C(\varphi; t)$ of φ is an ideal of X. Let $x, y, z \in X$ be such that $(x * y) * z \in C(\varphi; t)$ and $z \in C(\varphi; t)$. Then $\varphi((x * y) * z) \leq t$ and $\varphi(z) \leq t$. It follows from (3.9) that

$$\varphi(x * (y \vec{\wedge} x)) \leq \max\{\varphi((x * y) * z), \varphi(z)\} \leq t$$

so that $x * (y \vec{\wedge} x) \in C(\varphi; t)$. Hence $C(\varphi; t)$ is a commutative ideal of X.

Conversely, suppose that the condition (3.10) is valid. Obviously $\varphi(\theta) \leq \varphi(x)$ for all $x \in X$. Let $\varphi((x * y) * z) = t_1$ and $\varphi(z) = t_2$ for $x, y, z \in X$. Then $(x * y) * z \in C(\varphi; t_1)$ and $z \in C(\varphi; t_2)$. Without loss of generality, we may assume that $t_1 \geq t_2$. Then $C(\varphi; t_2) \subseteq C(\varphi; t_1)$, and so $z \in C(\varphi; t_1)$. Since $C(\varphi; t_1)$ is a commutative ideal of X by hypothesis, we have $x * (y \vec{\wedge} x) \in C(\varphi; t_1)$, and so

$$\varphi(x * (y \vec{\wedge} x)) \leq t_1 = \max\{t_1, t_2\} = \max\{\varphi((x * y) * z), \varphi(z)\}.$$

Therefore (X, φ) is a commutative $\mathcal{N}$-ideal of X. $\square$

COROLLARY 3.32. *If (X, φ) is a commutative $\mathcal{N}$-ideal of a BCK-algebra X, then every non-empty open (φ, t)-cut of X is a commutative ideal of X for all $t \in [-1, 0)$.*

Proof. Straightforward. $\square$

4. Translations of $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals

For any $\mathcal{N}$-function φ on X, we denote

$$\perp := -1 - \inf\{\varphi(x) \mid x \in X\}.$$

For any $\alpha \in [\perp, 0]$, we define ${\varphi_\alpha}^T(x) = \varphi(x) + \alpha$ for all $x \in X$. Obviously, ${\varphi_\alpha}^T$ is a mapping from X to $[-1, 0]$, that is, ${\varphi_\alpha}^T$ is an $\mathcal{N}$-function on X. We say that $(X, {\varphi_\alpha}^T)$ is an *α-translation* of (X, φ).

THEOREM 4.1. *For every $\alpha \in [\perp, 0]$, the α-translation $(X, {\varphi_\alpha}^T)$ of an $\mathcal{N}$-subalgebra* (resp. *$\mathcal{N}$-ideal*) *(X, φ) is an $\mathcal{N}$-subalgebra* (resp. *$\mathcal{N}$-ideal*) *of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned} {\varphi_\alpha}^T(x * y) &= \varphi(x * y) + \alpha \le \max\{\varphi(x), \varphi(y)\} + \alpha \\ &= \max\{\varphi(x) + \alpha,\ \varphi(y) + \alpha\} = \max\{{\varphi_\alpha}^T(x), {\varphi_\alpha}^T(y)\}. \end{aligned}$$

Therefore $(X, {\varphi_\alpha}^T)$ is an $\mathcal{N}$-subalgebra of X. Let $x, y \in X$. Then ${\varphi_\alpha}^T(\theta) = \varphi(\theta) + \alpha \le \varphi(x) + \alpha = {\varphi_\alpha}^T(x)$, and

$$\begin{aligned} {\varphi_\alpha}^T(x) &= \varphi(x) + \alpha \le \max\{\varphi(x * y), \varphi(y)\} + \alpha \\ &= \max\{\varphi(x * y) + \alpha, \varphi(y) + \alpha\} = \max\{{\varphi_\alpha}^T(x * y),\ {\varphi_\alpha}^T(y)\}. \end{aligned}$$

Hence $(X, {\varphi_\alpha}^T)$ is an $\mathcal{N}$-ideal of X. $\square$

THEOREM 4.2. *If there exists $\alpha \in [\perp, 0]$ such that the α-translation $(X, {\varphi_\alpha}^T)$ of (X, φ) is an $\mathcal{N}$-subalgebra* (resp. *$\mathcal{N}$-ideal*) *of X, then (X, φ) is an $\mathcal{N}$-subalgebra* (resp. *$\mathcal{N}$-ideal*) *of X.*

Proof. Assume that $(X, {\varphi_\alpha}^T)$ is an $\mathcal{N}$-subalgebra (resp. $\mathcal{N}$-ideal) of X for some $\alpha \in [\perp, 0]$. Let $x, y \in X$. Then

$$\begin{aligned} \varphi(x * y) + \alpha &= {\varphi_\alpha}^T(x * y) \ \le\ \max\{{\varphi_\alpha}^T(x), {\varphi_\alpha}^T(y)\} \\ &= \max\{\varphi(x) + \alpha, \varphi(y) + \alpha\} = \max\{\varphi(x), \varphi(y)\} + \alpha \end{aligned}$$

which implies that $\varphi(x * y) \le \max\{\varphi(x), \varphi(y)\}$. Therefore (X, φ) is an $\mathcal{N}$-subalgebra of X. Now suppose that there exists $\alpha \in [\perp, 0]$ such that $(X, {\varphi_\alpha}^T)$ is an $\mathcal{N}$-ideal of X. Let $x, y \in X$. Then $\varphi(\theta) + \alpha = {\varphi_\alpha}^T(\theta) \le {\varphi_\alpha}^T(x) = \varphi(x) + \alpha$, and so $\varphi(\theta) \le \varphi(x)$. Finally,

$$\begin{aligned} \varphi(x) + \alpha &= {\varphi_\alpha}^T(x) \le \max\{{\varphi_\alpha}^T(x * y),\ {\varphi_\alpha}^T(y)\} \\ &= \max\{\varphi(x * y) + \alpha,\ \varphi(y) + \alpha\} = \max\{\varphi(x * y), \varphi(y)\} + \alpha, \end{aligned}$$

which implies that $\varphi(x) \le \max\{\varphi(x * y), \varphi(y)\}$. Thus (X, φ) is an $\mathcal{N}$-ideal of X. $\square$

For any $\mathcal{N}$-function φ on X, $\alpha \in [\perp, 0]$ and $t \in [-1, \alpha)$, let

$$L_\alpha(\varphi; t) := \{x \in X \mid \varphi(x) \le t - \alpha\}.$$

PROPOSITION 4.3. *Let (X, φ) be an $\mathcal{N}$-structure of X and φ, and let $\alpha \in [\perp, 0]$. If (X, φ) is an $\mathcal{N}$-subalgebra* (resp. *$\mathcal{N}$-ideal*) *of X, then $L_\alpha(\varphi; t)$ is a subalgebra* (resp. *ideal*) *of X for all $t \in [-1, \alpha)$.*

Proof. Assume that (X, φ) is an $\mathcal{N}$-subalgebra of X. Let $x, y \in L_\alpha(\varphi; t)$. Then $\varphi(x) \le t - \alpha$ and $\varphi(y) \le t - \alpha$. It follows that

$$\varphi(x * y) \le \max\{\varphi(x), \varphi(y)\} \le t - \alpha$$

so that $x * y \in L_\alpha(\varphi; t)$. Hence $L_\alpha(\varphi; t)$ is a subalgebra of X. Now suppose that (X, φ) is an $\mathcal{N}$-ideal of X and let $x, y \in X$ be such that $x * y \in L_\alpha(\varphi; t)$ and $y \in L_\alpha(\varphi; t)$. Then $\varphi(x * y) \le t - \alpha$ and $\varphi(y) \le t - \alpha$. Thus

$$\varphi(x) \le \max\{\varphi(x * y), \varphi(y)\} \le t - \alpha,$$

and hence $x \in L_\alpha(\varphi; t)$. Clearly, $\theta \in L_\alpha(\varphi; t)$. Therefore $L_\alpha(\varphi; t)$ is an ideal of X. $\square$

If we do not give a condition that (X, φ) is an $\mathcal{N}$-subalgebra (resp. $\mathcal{N}$-ideal) of X then $L_\alpha(\varphi; t)$ may not be a subalgebra (resp. ideal) of X as seen in the following example.

EXAMPLE 4.4. *Consider a BCK-algebra $X = \{\theta, a, b, c, d\}$ with the following Cayley table:*

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	θ	θ	θ
b	b	a	θ	θ	θ
c	c	a	a	θ	θ
d	d	c	c	a	θ

Define an $\mathcal{N}$-function φ on X by

X	θ	a	b	c	d
φ	-0.7	-0.4	-0.6	-0.3	-0.5

Then $\perp = -0.3$ and (X, φ) is not an $\mathcal{N}$-subalgebra of X because

$$\varphi(d * b) = \varphi(c) = -0.3 > -0.5 = \max\{\varphi(d), \varphi(b)\}.$$

For $\alpha = -0.1 \in [-0.3, 0]$ and $t = -0.5$, we obtain $L_\alpha(\varphi; t) = \{\theta, a, b, d\}$ which is not a subalgebra of X since $d * b = c \notin L_\alpha(\varphi; t)$.

EXAMPLE 4.5. *Consider a BCI-algebra $X = \{\theta, a, b, c, d\}$ with the following Cayley table:*

$*$	θ	a	b	c	d
θ	θ	d	c	b	a
a	a	θ	d	c	b
b	b	a	θ	d	c
c	c	b	a	θ	d
d	d	c	b	a	θ

Define an $\mathcal{N}$-function φ on X by

X	θ	a	b	c	d
φ	-0.6	-0.5	-0.6	-0.3	-0.2

Then $\perp = -0.4$ and (X, φ) is not an $\mathcal{N}$-ideal of X since

$$\varphi(d) = -0.2 > -0.6 = \max\{\varphi(d * b), \varphi(b)\}.$$

For $\alpha = -0.15 \in [\perp, 0]$ and $t = -0.5$ we have $L_\alpha(\varphi; t) = \{\theta, a, b\}$ which is not an ideal of X since $c * b = a \in L_\alpha(\varphi; t)$ and $c \notin L_\alpha(\varphi; t)$.

THEOREM 4.6. *Let (X, φ) be an $\mathcal{N}$-structure and $\alpha \in [\perp, 0]$. Then the α-translation $(X, {\varphi_\alpha}^T)$ of (X, φ) is an $\mathcal{N}$-subalgebra* (resp. *$\mathcal{N}$-ideal*) *of X if and only if $L_\alpha(\varphi; t)$ is a subalgebra* (resp. *ideal*) *of $X$for all $t \in [-1, \alpha]$.*

Proof. Assume that $(X, {\varphi_\alpha}^T)$ is an $\mathcal{N}$-subalgebra of X. Let $x, y \in L_\alpha(\varphi; t)$. Then $\varphi(x) \leq t - \alpha$ and $\varphi(y) \leq t - \alpha$. Hence

$$\begin{aligned} \varphi(x * y) + \alpha &= {\varphi_\alpha}^T(x * y) \ \leq \ \max\{{\varphi_\alpha}^T(x), {\varphi_\alpha}^T(y)\} \\ &= \max\{\varphi(x) + \alpha, \varphi(y) + \alpha\} = \max\{\varphi(x), \varphi(y)\} + \alpha \\ &\leq t - \alpha + \alpha \ = \ t, \end{aligned}$$

and so $\varphi(x * y) \leq t - \alpha$, i.e., $x * y \in L_\alpha(\varphi; t)$. Therefore $L_\alpha(\varphi; t)$ is a subalgebra of X. Suppose that $L_\alpha(\varphi; t)$ is a subalgebra of X for all $t \in [-1, \alpha]$. We claim that ${\varphi_\alpha}^T(x * y) \leq \max\{{\varphi_\alpha}^T(x), {\varphi_\alpha}^T(y)\}$ for all $x, y \in X$. If it is not valid, then

$${\varphi_\alpha}^T(a * b) > s \geq \max\{{\varphi_\alpha}^T(a), {\varphi_\alpha}^T(b)\}$$

for some $a, b \in X$ and $s \in [-1, \alpha]$. It follows that $\varphi(a) \leq s - \alpha$ and $\varphi(b) \leq s - \alpha$, but $\varphi(a * b) > s - \alpha$. Thus $a \in L_\alpha(\varphi; s)$ and $b \in L_\alpha(\varphi; s)$, but $a * b \notin L_\alpha(\varphi; s)$. This is a contradiction, and therefore $(X, {\varphi_\alpha}^T)$ is an $\mathcal{N}$-subalgebra of X. Suppose that $(X, {\varphi_\alpha}^T)$ is an $\mathcal{N}$-ideal of X. Let $t \in [-1, \alpha]$. For any $x \in L_\alpha(\varphi; t)$, we have $\varphi(\theta) \leq \varphi(x) \leq t - \alpha$, and thus $\theta \in L_\alpha(\varphi; t)$. Let $x, y \in X$ be such that $x * y \in L_\alpha(\varphi; t)$ and $y \in L_\alpha(\varphi; t)$. Then $\varphi(x * y) \leq t - \alpha$ and $\varphi(y) \leq t - \alpha$, i.e., ${\varphi_\alpha}^T(x * y) \leq t$ and ${\varphi_\alpha}^T(y) \leq t$. It follows from (3.4) that

$$\varphi(x) + \alpha = {\varphi_\alpha}^T(x) \leq \max\{{\varphi_\alpha}^T(x * y), {\varphi_\alpha}^T(y)\} \leq t$$

so that $\varphi(x) \leq t - \alpha$, i.e., $x \in L_\alpha(\varphi; t)$. Hence $L_\alpha(\varphi; t)$ is an ideal of X. Finally assume that $L_\alpha(\varphi; t)$ is an ideal of X for all $t \in [-1, \alpha]$. We claim that

(i) ${\varphi_\alpha}^T(\theta) \leq {\varphi_\alpha}^T(x)$ for all $x \in X$.
(ii) ${\varphi_\alpha}^T(x) \leq \max\{{\varphi_\alpha}^T(x * y), {\varphi_\alpha}^T(y)\}$ for all $x, y \in X$.

If (i) is not valid, then ${\varphi_\alpha}^T(\theta) > s_0 \geq {\varphi_\alpha}^T(a)$ for some $a \in X$ and $s_0 \in [-1, \alpha]$. Thus $\varphi(a) + \alpha = {\varphi_\alpha}^T(a) \leq s_0$, i.e., $\varphi(a) \leq s_0 - \alpha$; and $\varphi(\theta) + \alpha = {\varphi_\alpha}^T(\theta) > s_0$, i.e., $\varphi(\theta) > s_0 - \alpha$. Therefore $a \in L_\alpha(\varphi; s_0)$, but $\theta \notin L_\alpha(\varphi; s_0)$, which is a contradiction. If (ii) is not true, then

$${\varphi_\alpha}^T(a) > s_1 \geq \max\{{\varphi_\alpha}^T(a * b), {\varphi_\alpha}^T(b)\}$$

for some $a, b \in X$ and $s_1 \in [-1, \alpha]$. It follows that $\varphi(a * b) + \alpha = {\varphi_\alpha}^T(a * b) \leq s_1$, $\varphi(b) + \alpha = {\varphi_\alpha}^T(b) \leq s_1$ and $\varphi(a) + \alpha = {\varphi_\alpha}^T(a) > s_1$ so that $a * b \in L_\alpha(\varphi; s_1)$ and $b \in L_\alpha(\varphi; s_1)$, but $a \notin L_\alpha(\varphi; s_1)$. This is a contradiction. Consequently $(X, {\varphi_\alpha}^T)$ is an $\mathcal{N}$-ideal of X. □

For any $\mathcal{N}$-functions φ and ϖ, we say that (X, ϖ) is a *retrenchment* of (X, φ) if $\varpi(x) \leq \varphi(x)$ for all $x \in X$.

Definition 4.7. Let φ and ϖ be $\mathcal{N}$-functions on X. We say that (X, ϖ) is a *retrenched $\mathcal{N}$-subalgebra* (resp. *retrenched $\mathcal{N}$-ideal*) of (X, φ) if the following assertions are valid:

(i) (X, ϖ) is a retrenchment of (X, φ).
(ii) If (X, φ) is an $\mathcal{N}$-subalgebra (resp. $\mathcal{N}$-ideal) of X, then (X, ϖ) is an $\mathcal{N}$-subalgebra (resp. $\mathcal{N}$-ideal) of X.

Theorem 4.8. *Let (X, φ) be an $\mathcal{N}$-subalgebra* (resp. *$\mathcal{N}$-ideal*) *of X. For every $\alpha \in [\perp, 0]$, the α-translation $(X, {\varphi_\alpha}^T)$ of (X, φ) is a retrenched $\mathcal{N}$-subalgebra* (resp. *retrenched $\mathcal{N}$-ideal*) *of (X, φ).*

Proof. Obviously, $(X, {\varphi_\alpha}^T)$ is a retrenchment of (X, φ). Using Theorem 4.1, we conclude that $(X, {\varphi_\alpha}^T)$ is a retrenched $\mathcal{N}$-subalgebra (resp. retrenched $\mathcal{N}$-ideal) of (X, φ). □

The converse of Theorem 4.8 is not true as seen in the following example.

EXAMPLE 4.9. *Consider a BCK-algebra* $X = \{\theta, a, b, c, d\}$ *with the following Cayley table:*

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	a	θ	θ
b	b	b	θ	b	θ
c	c	a	c	θ	a
d	d	d	d	d	θ

Define $\mathcal{N}$*-functions* φ_1 *and* φ_2 *on* X *by*

X	θ	a	b	c	d
φ_1	-0.9	-0.6	-0.4	-0.7	-0.3
φ_2	-0.8	-0.4	-0.6	-0.4	-0.1

Then (X, φ_1) *is an* $\mathcal{N}$*-subalgebra of* X, *and* (X, φ_2) *is an* $\mathcal{N}$*-ideal of* X. *Let* ϖ_1 *and* ϖ_2 *be* $\mathcal{N}$*-functions on* X *defined by*

X	θ	a	b	c	d
ϖ_1	-0.92	-0.65	-0.43	-0.71	-0.38
ϖ_2	-0.88	-0.45	-0.63	-0.45	-0.19

Then (X, ϖ_1) *is a retrenched* $\mathcal{N}$*-subalgebra of* (X, φ_1), *which is not an* α*-translation of* (X, φ_1) *for* $\alpha \in [\perp, 0]$. *Also,* (X, ϖ_2) *is a retrenched* $\mathcal{N}$*-ideal of* (X, φ_2), *which is not an* α*-translation of* (X, φ_2) *for* $\alpha \in [\perp, 0]$.

For two $\mathcal{N}$-structures (X, φ_1) and (X, φ_2), we define the union $\varphi_1 \cup \varphi_2$ and the intersection $\varphi_1 \cap \varphi_2$ of φ_1 and φ_2 as follows:

$$(\forall x \in X)((\varphi_1 \cup \varphi_2)(x) = \max\{\varphi_1(x), \varphi_2(x)\}),$$

$$(\forall x \in X)((\varphi_1 \cap \varphi_2)(x) = \min\{\varphi_1(x), \varphi_2(x)\}),$$

respectively. Obviously, $(X, \varphi_1 \cup \varphi_2)$ and $(X, \varphi_1 \cap \varphi_2)$ are $\mathcal{N}$-structures which are called the *union* and the *intersection* of (X, φ_1) and (X, φ_2), respectively.

LEMMA 4.10. *If* (X, φ_1) *and* (X, φ_2) *are* $\mathcal{N}$*-subalgebras* (resp. $\mathcal{N}$*-ideals*) *of* X, *then the union* $(X, \varphi_1 \cup \varphi_2)$ *is an* $\mathcal{N}$*-subalgebra* (resp. $\mathcal{N}$*-ideal*) *of* X.

Proof. Straightforward. □

EXAMPLE 4.11. *Consider a BCI-algebra* $X = \{\theta, 1, 2, a, b\}$ *with the following Cayley table:*

$*$	θ	1	2	a	b
θ	θ	θ	θ	b	a
1	1	θ	1	b	a
2	2	2	θ	b	a
a	a	a	a	θ	b
b	b	b	b	a	θ

Define $\mathcal{N}$*-functions* φ_1 *and* φ_2 *on* X *by*

X	θ	1	2	a	b
φ_1	-0.7	-0.2	-0.2	-0.5	-0.4
φ_2	-0.9	-0.6	-0.7	-0.3	-0.3

Then (X, φ_1) *is an* $\mathcal{N}$*-subalgebra of* X, *and* (X, φ_2) *is an* $\mathcal{N}$*-ideal of* X *which is also an* $\mathcal{N}$*-subalgebra of* X. *But* (X, φ_1) *is not an* $\mathcal{N}$*-ideal of* X *since* $\varphi(2) = -0.2 > -0.4 = \max\{\varphi(2 * a), \varphi(a)\}$. *The union* $\varphi_1 \cup \varphi_2$ *and the intersection* $\varphi_1 \cap \varphi_2$ *are given by*

X	θ	1	2	a	b
$\varphi_1 \cup \varphi_2$	-0.7	-0.2	-0.2	-0.3	-0.3
$\varphi_1 \cap \varphi_2$	-0.9	-0.6	-0.7	-0.5	-0.4

Then $(X, \varphi_1 \cup \varphi_2)$ *is an* $\mathcal{N}$*-subalgebra of* X, *but it is not an* $\mathcal{N}$*-ideal of* X *because* $(\varphi_1 \cup \varphi_2)(1) = -0.2 > -0.3 = \max\{(\varphi_1 \cup \varphi_2)(1 * b), (\varphi_1 \cup \varphi_2)(b)\}$. *This shows that the union of an* $\mathcal{N}$*-subalgebra and an* $\mathcal{N}$*-ideal may not be an* $\mathcal{N}$*-ideal. We see that*

$$\begin{aligned}(\varphi_1 \cap \varphi_2)(1 * a) &= (\varphi_1 \cap \varphi_2)(b) = -0.4 > -0.5 \\ &= \max\{(\varphi_1 \cap \varphi_2)(1), (\varphi_1 \cap \varphi_2)(a)\},\end{aligned}$$

and so $(X, \varphi_1 \cap \varphi_2)$ *is not an* $\mathcal{N}$*-subalgebra of* X. *For* $t \in [-0.5, 0)$, *we have* $C(\varphi_1 \cap \varphi_2; t) = \{\theta, 1, 2, a\}$ *which is not an ideal of* X *since* $b * a = a \in C(\varphi_1 \cap \varphi_2; t)$ *and* $b \notin C(\varphi_1 \cap \varphi_2; t)$. *Hence* $(X, \varphi_1 \cap \varphi_2)$ *is not an* $\mathcal{N}$*-ideal of* X *by Theorem 3.12.*

THEOREM 4.12. *Let* (X, φ) *be an* $\mathcal{N}$*-subalgebra* (resp. $\mathcal{N}$*-ideal*) *of* X. *If* (X, ϖ_1) *and* (X, ϖ_2) *are retrenched* $\mathcal{N}$*-subalgebras* (resp. *retrenched* $\mathcal{N}$*-ideals*) *of* (X, φ), *then the union* $(X, \varpi_1 \cup \varpi_2)$ *is a retrenched* $\mathcal{N}$*-subalgebra* (resp. *retrenched* $\mathcal{N}$*-ideal*) *of* (X, φ).

Proof. Clearly, $(X, \varpi_1 \cup \varpi_2)$ is a retrenchment of (X, φ). Since (X, ϖ_1) and (X, ϖ_2) are retrenched $\mathcal{N}$-subalgebras (resp. $\mathcal{N}$-ideals) of (X, φ), it follows from Lemma 4.10 that $(X, \varpi_1 \cup \varpi_2)$ is an $\mathcal{N}$-subalgebra (resp. $\mathcal{N}$-ideal) of X. Therefore $(X, \varpi_1 \cup \varpi_2)$ is a retrenched $\mathcal{N}$-subalgebra (resp. retrenched $\mathcal{N}$-ideal) of (X, φ). □

Let (X, φ) be an $\mathcal{N}$-subalgebra (resp. $\mathcal{N}$-ideal) of X and let $\alpha, \beta \in [\perp, 0]$. Then the α-translation $(X, {\varphi_\alpha}^T)$ and the β-translation $(X, {\varphi_\beta}^T)$ are $\mathcal{N}$-subalgebras (resp. $\mathcal{N}$-ideals) of X by Theorem 4.1. If $\alpha \leq \beta$, then ${\varphi_\alpha}^T(x) = \varphi(x) + \alpha \leq \varphi(x) + \beta = {\varphi_\beta}^T(x)$ for all $x \in X$, and hence $(X, {\varphi_\alpha}^T)$ is a retrenchment of $(X, {\varphi_\beta}^T)$. Therefore we have the following theorem.

Theorem 4.13. *Let (X, φ) be an $\mathcal{N}$-subalgebra* (resp. *$\mathcal{N}$-ideal*) *of X and let $\alpha, \beta \in [\perp, 0]$. If $\alpha \leq \beta$, then the α-translation $(X, {\varphi_\alpha}^T)$ of (X, φ) is a retrenched $\mathcal{N}$-subalgebra* (resp. *retrenched $\mathcal{N}$-ideal*) *of the β-translation $(X, {\varphi_\beta}^T)$ of (X, φ).*

For every $\mathcal{N}$-subalgebra (resp. $\mathcal{N}$-ideal) (X, φ) of X and $\beta \in [\perp, 0]$, the β-translation $(X, {\varphi_\beta}^T)$ is an $\mathcal{N}$-subalgebra (resp. $\mathcal{N}$-ideal) of X. If (X, ϖ) is a retrenched $\mathcal{N}$-subalgebra (resp. retrenched $\mathcal{N}$-ideal) of $(X, {\varphi_\beta}^T)$, then there exists $\alpha \in [\perp, 0]$ such that $\alpha \leq \beta$ and $\varpi(x) \leq {\varphi_\alpha}^T(x)$ for all $x \in X$. Thus we obtain the following theorem.

Theorem 4.14. *Let (X, φ) be an $\mathcal{N}$-subalgebra* (resp. *$\mathcal{N}$-ideal*) *of X and let $\beta \in [\perp, 0]$. For every retrenched $\mathcal{N}$-subalgebra* (resp. *retrenched $\mathcal{N}$-ideal*) *(X, ϖ) of the β-translation $(X, {\varphi_\beta}^T)$ of (X, φ), there exists $\alpha \in [\perp, 0]$ such that $\alpha \leq \beta$ and (X, ϖ) is a retrenched $\mathcal{N}$-subalgebra* (resp. *retrenched $\mathcal{N}$-ideal*) *of the α-translation $(X, {\varphi_\alpha}^T)$ of (X, φ).*

The following examples illustrate Theorem 4.14.

Example 4.15. *Consider a BCK-algebra $X = \{\theta, a, b, c, d\}$ with the following Cayley table:*

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	θ	a	a
b	b	b	θ	b	b
c	c	c	c	θ	c
d	d	d	d	d	θ

Define an $\mathcal{N}$-function φ on X by

X	θ	a	b	c	d
φ	-0.7	-0.4	-0.2	-0.5	-0.1

Then (X, φ) is an $\mathcal{N}$-subalgebra of X and $\perp = -0.3$. If we take $\beta = -0.15$, then the β-translation $(X, {\varphi_\beta}^T)$ of (X, φ) is given by

X	θ	a	b	c	d
${\varphi_\beta}^T$	-0.85	-0.55	-0.35	-0.65	-0.25

Let ϖ be an $\mathcal{N}$-function on X defined by

X	θ	a	b	c	d
${\varphi_\beta}^T$	-0.89	-0.57	-0.38	-0.66	-0.28

Then (X, ϖ) is clearly an $\mathcal{N}$-subalgebra of X which is a retrenchment of $(X, {\varphi_\beta}^T)$, and so (X, ϖ) is a retrenched $\mathcal{N}$-subalgebra of the β-translation $(X, {\varphi_\beta}^T)$ of (X, φ). If we take $\alpha = -0.23$, then $\alpha = -0.23 < -0.15 = \beta$ and the α-translation $(X, {\varphi_\alpha}^T)$ of (X, φ) is given as follows:

X	θ	a	b	c	d
${\varphi_\alpha}^T$	-0.93	-0.63	-0.43	-0.73	-0.33

Note that $\varpi(x) \leq {\varphi_\alpha}^T(x)$ for all $x \in X$, and hence (X, ϖ) is a retrenched $\mathcal{N}$-subalgebra of the α-translation $(X, {\varphi_\alpha}^T)$ of (X, φ).

EXAMPLE 4.16. *Consider a BCI-algebra $X = \{\theta, 1, a, b, c\}$ with the following Cayley table:*

$*$	θ	1	a	b	c
θ	θ	θ	c	b	a
1	1	θ	c	b	a
a	a	a	θ	c	b
b	b	b	a	θ	c
c	c	c	b	a	θ

Define an $\mathcal{N}$-function φ on X by

X	θ	1	a	b	c
φ	-0.65	-0.53	-0.22	-0.38	-0.22

Then (X, φ) is an $\mathcal{N}$-ideal of X and $\perp = -0.35$. If we take $\beta = -0.2$, then the β-translation $(X, {\varphi_\beta}^T)$ of (X, φ) is given by

X	θ	1	a	b	c
${\varphi_\beta}^T$	-0.85	-0.73	-0.42	-0.58	-0.42

Let ϖ be an $\mathcal{N}$-function on X defined by

X	θ	1	a	b	c
${\varphi_\beta}^T$	-0.87	-0.75	-0.45	-0.59	-0.45

Then (X, ϖ) is clearly an $\mathcal{N}$-ideal of X which is a retrenchment of $(X, {\varphi_\beta}^T)$, and so (X, ϖ) is a retrenched $\mathcal{N}$-ideal of the β-translation $(X, {\varphi_\beta}^T)$ of (X, φ). If we take $\alpha = -0.21$, then $\alpha = -0.21 < -0.2 = \beta$ and the α-translation $(X, {\varphi_\alpha}^T)$ of (X, φ) is given as follows:

X	θ	1	a	b	c
${\varphi_\alpha}^T$	-0.86	-0.74	-0.43	-0.59	-0.43

Note that $\varpi(x) \leq {\varphi_\alpha}^T(x)$ for all $x \in X$, and hence (X, ϖ) is a retrenched $\mathcal{N}$-ideal of the α-translation $(X, {\varphi_\alpha}^T)$ of (X, φ).

References

[1] A. Dvurečenskij, *Measures on commutative BCK-algebras,* Atti Sem. Mat. Fis. Univ. Modena **49** (2001), no. 1, 19–49.
[2] A. Dvurečenskij, *Commutative BCK-algebras and lattice ordered groups with universal property,* Indian J. Math. **42** (2000), no. 2, 119–151.
[3] A. Dvurečenskij, *Ideals of commutative BCK-algebras,* Ricerche Mat. **49** (2000), no. 2, 241–256.
[4] A. Dvurečenskij, *Commutative BCK-algebras and quantum structures,* Internat. J. Theoret. Phys. **39** (2000), no. 3, 653–664.
[5] A. Dvurečenskij, *Commutative BCK-algebras with product,* Demonstratio Math. **33** (2000), no. 1, 1–19.
[6] A. Dvurečenskij, *On categorical equivalences of commutative BCK-algebras,* Studia Logica **64** (2000), no. 1, 21–36.
[7] A. Dvurečenskij and M. G. Graziano, *Measures on Dedekind complete commutative BCK-algebras,* Atti Sem. Mat. Fis. Univ. Modena **49** (2001), no. 1, 51–72.
[8] Y. S. Huang, *BCI-algebra,* Science Press, Beijing, 2006.
[9] Y. Imai and K. Iséki, *On axiom systems of propositional calculi,* Proc. Jpn. Acad. **42** (1966), 19–21.
[10] K. Iséki, *An algebra related with a propositional calculus,* Proc. Japan Acad. **42** (1966), 26–29.

[11] K. Iséki, *Some examples of BCI-algebras,* Math. Seminar Notes **8** (1980), 237–240.
[12] K. Iséki, *On ideals in BCK-algebras,* Math. Seminar Notes **3** (1975), 1–12.
[13] K. Iséki, *On some ideals in BCK-algebras,* Math. Seminar Notes **3** (1975), 65–70.
[14] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras,* Math. Japonica **23** (1978), no. 1, 1–26.
[15] Y. B. Jun, *Soft BCK/BCI-algebras,* Comput. Math. Appl. **56** (2008), 1408–1413.
[16] Y. B. Jun, *Biideals in BCK/BCI-algebras,* Kyungpook Math. J. **48** (2008), 577–584.
[17] Y. B. Jun and C. H. Park, *Applications of soft sets in ideal theory of BCK/BCI-algebras,* Inform. Sci. **178** (2008), 2466–2475.
[18] J. Meng, *Commutative ideals in BCK-algerbas,* Pure Appl. Math. (in China) **9** (1991), 49–53.
[19] J. Meng and Y. B. Jun, *BCK-algebras,* Kyungmoon Sa Co. Seoul, 1994.
[20] L. A. Zadeh, *Fuzzy sets,* Inform. Control **8** (1965), 338–353.

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