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LIGHTLIKE HYPERSURFACES WITH TOTALLY UMBILICAL SCREEN DISTRIBUTIONS

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Abstract. In this paper, we study the geometry of lightlike hypersurfaces of a semi-Riemannian manifold. We prove a classification theorem for lightlike hypersurfaces M with totally umbilical screen distributions of a semi-Riemannian space form.

1. Introduction

It is well known that the normal bundle $TM^{\perp}$ of the lightlike hypersurfaces (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is a vector subbundle of TM, of rank 1. A complementary vector bundle $S(TM)$ of $TM^{\perp}$ in TM is non-degenerate distribution on M, which called a *screen distribution* on M, such that

$$TM = TM^{\perp} \oplus_{orth} S(TM), \tag{1.1}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $(M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. For any null section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^{\perp}$ [2] satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \; \forall\, X \in \Gamma(S(TM)). \tag{1.2}$$

Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follows:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \; \oplus_{orth} \; S(TM). \tag{1.3}$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to $S(TM)$ respectively.

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The purpose of this paper is to prove a classification theorem for lightlike hypersurfaces M of a semi-Riemannian space form $(\bar{M}^{m+2}(c), \bar{g})$, $m > 2$, such that $S(TM)$ is totally umbilical in M. This theorem shows that the local second fundamental forms B and C of such a half lightlike submanifold and its screen distribution $S(TM)$ respectively satisfy $B = 0$ or $C = 0$. Using this theorem, we prove several additional theorems for lightlike hypersurfaces M of a semi-Riemannian space form $(\bar{M}^{m+2}(c), \bar{g})$, $m > 2$, such that $S(TM)$ is totally umbilical.

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.1). Then the local Gauss and Weingartan formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{1.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{1.5}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{1.6}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{1.7}$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are the induced linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively and τ is a 1-form on TM defined by $\tau(X) = \bar{g}(\bar{\nabla}_X N, \xi)$. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric. From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that B is independent of the choice of a screen distribution and satisfies

$$B(X,\xi) = 0, \quad \forall\, X \in \Gamma(TM). \tag{1.8}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y), \tag{1.9}$$

for any X, Y, $Z \in \Gamma(TM)$, where η is a 1-form such that

$$\eta(X) = \bar{g}(X,N), \quad \forall\, X \in \Gamma(TM). \tag{1.10}$$

But ∇^* is metric connection. The above local second fundamental forms B and C of M and on $S(TM)$ are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{1.11}$$

$$C(X,PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0. \tag{1.12}$$

From (1.11), A^*_ξ is $S(TM)$-valued and self-adjoint on TM such that

$$A^*_\xi \xi = 0. \tag{1.13}$$

We denote by $\bar{R}$, R and R^* the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$, the induced connection ∇ of M and the induced connection ∇^* on $S(TM)$ respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$ such that, for any vector fields X, Y, Z, $W \in \Gamma(TM)$,

$$\begin{aligned}(1.14)\quad \bar{g}(\bar{R}(X,Y)Z,\, PW) &= g(R(X,Y)Z,\, PW)\\ &\quad + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW),\end{aligned}$$

$$\begin{aligned}(1.15)\quad \bar{g}(\bar{R}(X,Y)Z,\, \xi) &= g(R(X,Y)Z,\, \xi)\\ &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)\\ &\quad + B(Y,Z)\tau(X) - B(X,Z)\tau(Y),\end{aligned}$$

$$(1.16)\quad \bar{g}(\bar{R}(X,Y)Z,\, N) = g(R(X,Y)Z,\, N),$$

$$\begin{aligned}(1.17)\quad g(R(X,Y)PZ,\, PW) &= g(R^*(X,Y)PZ,\, PW)\\ &\quad + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW),\end{aligned}$$

$$\begin{aligned}(1.18)\quad g(R(X,Y)PZ,\, N) &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ)\\ &\quad + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X).\end{aligned}$$

2. Totally umbilical screen distributions

DEFINITION 2.1. We say that $S(TM)$ is *totally umbilical* [2] in M if, on any coordinate neighborhood $\mathcal{U} \subset M$, there is a smooth function γ such that $A_N X = \gamma PX$ for any $X \in \Gamma(TM)$, or equivalently,

$$(2.1)\qquad C(X,PY) = \gamma g(X,Y),\ \forall\, X,\, Y \in \Gamma(TM).$$

In case $\gamma = 0$ on $\mathcal{U}$, we say that $S(TM)$ is *totally geodesic*.

In general, $S(TM)$ is not necessarily integrable. The following result gives equivalent conditions for the integrability of $S(TM)$:

THEOREM 2.2. [2]. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the following are equivalent:*
(1) $S(TM)$ is integrable.
(2) C is symmetric on $\Gamma(S(TM))$.
(3) A_N is self-adjoint on $\Gamma(S(TM))$ with respect to g.

NOTE 2.3. If $S(TM)$ is totally umbilical in M, then the second fundamental form C on $S(TM)$ is symmetric on $\Gamma(S(TM))$. Thus, by Theorem 2.2, $S(TM)$ is integrable and M is locally a product manifold $L \times M^*$, where L is a null curve and M^* is a leaf of $S(TM)$ [2].

Let $\bar{M}(c)$ be a semi-Riemannian space form and $S(TM)$ a totally umbilical distribution of M. Then the equation (1.15) reduces to

$$(2.2)\quad (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = B(X,Z)\tau(Y) - B(Y,Z)\tau(X).$$

Using (1.9), (1.16), (1.18) and (2.1), for any X, Y, $Z \in \Gamma(TM)$, we get

$$\begin{aligned}&\gamma\, B(Y,PZ)\eta(X) - \{X[\gamma] - \gamma\tau(X) - c\eta(X)\}g(Y,PZ)\\ &= \gamma\, B(X,PZ)\eta(Y) - \{Y[\gamma] - \gamma\tau(Y) - c\eta(Y)\}g(X,PZ).\end{aligned}$$

Replacing Y by ξ in this equation and using (1.8), we have

$$(2.3)\quad \gamma\, B(X,Y) = \{\xi[\gamma] - \gamma\tau(\xi) - c\}g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

In the sequel, by a *totally umbilical* we shall mean a *totally umbilical in M* unless otherwise specified.

THEOREM 2.4. *Let $(M, g, S(TM))$ be an $(m+1)(m>2)$-dimensional lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c),\, \bar{g})$ such that $S(TM)$ is totally umbilical. Then $C = 0$ or $B = 0$. Moreover,*

1. *$C = 0$ implies that $S(TM)$ is totally geodesic and $c = 0$.*
2. *$B = 0$ implies that M is totally geodesic immersed in $\bar{M}(c)$ and the induced connection ∇ on M is a metric one.*

Proof. Assume that $C \neq 0$, i.e., $\gamma \neq 0$. Then, from (2.3), we have

$$(2.4)\quad B(X,Y) = \beta g(X,Y),\ \forall\, X,\, Y \in \Gamma(TM),$$

where $\beta = \gamma^{-1}(\xi[\gamma] - \gamma\tau(\xi) - c)$. Since $S(TM)$ is totally umbilical, M is locally a product manifold $L \times M^*$ where L is a null curve and M^* is a leaf of $S(TM)$. From (1.14), (1.17), (2.1) and (2.4), we have

$$R^*(X,Y)Z = (c + 2\beta\gamma)\{g(Y,Z)X - g(X,Z)Y\},$$

for any X, Y, $Z \in \Gamma(S(TM))$, where R^* is the curvature tensor of M^*. Let Ric^* be the symmetric Ricci tensor of M^*. Then we have

$$Ric^*(X,Y) = (c + 2\beta\gamma)(m-1)g(X,Y),\ \forall\, X,\, Y \in \Gamma(S(TM)).$$

Thus M^* is an Einstein manifold of constant curvature $(c+2\beta\gamma)$ due to $m > 2$. From (2.3), we have $\xi[\gamma] = \beta\gamma + \gamma\tau(\xi) + c$. Differentiating (2.4) and using (1.9) and (2.2), for all X, Y, $Z \in \Gamma(S(TM))$, we have

$$\begin{aligned}(2.5)\quad &\{X[\beta] + \beta\tau(X) - \beta^2\eta(X)\}g(Y,Z)\\ &= \{Y[\beta] + \beta\tau(Y) - \beta^2\eta(Y)\}g(X,Z).\end{aligned}$$

Replacing X by ξ in this equation, we have $\xi[\beta] = \beta^2 - \beta\tau(\xi)$. Since $(c + 2\beta\gamma)$ is a constant, we get $\xi[c + 2\beta\gamma] = 2\beta(c + 2\beta\gamma) = 0$. Therefore $\beta = 0$ or $c + 2\beta\gamma = 0$. If $c + 2\beta\gamma = 0$, then M^* is a semi-Euclidean space and the second fundamental form C of M^* satisfies $C = 0$. It is a contradiction to $C \neq 0$. Thus we have $\beta = 0$. Consequently, we get $B = 0$ by (2.4). Thus M is totally geodesic in $\bar{M}$. Also, from the equation (1.9), we see that $(\nabla_X g)(Y, Z) = 0$ for all $X, Y, Z \in \Gamma(TM)$, that is, the induced connection ∇ on M is a metric one. If $C = 0$, i.e., $\gamma = 0$, then, by (2.3), we have $c = 0$. Thus we have our main theorem. □

The induced Ricci type tensor $R^{(0,2)}$ of M is defined by

$$R^{(0,2)}(X, Y) = trace\{Z \to R(Z, X)Y\}, \ \forall X, Y \in \Gamma(TM). \tag{2.6}$$

Consider the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M such that $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$. Using this frame field and the equation (2.6), we obtain

$$R^{(0,2)}(X, Y) = \sum_{a=1}^{m} \epsilon_a \, g(R(W_a, X)Y, W_a) + \bar{g}(R(\xi, X)Y, N), \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$ and $\epsilon_a = g(W_a, W_a)$. In general, the induced Ricci type tensor $R^{(0,2)}$, defined by the method of the geometry of the non-degenerate submanifolds [7], is not symmetric [2, 3, 4]. Hence we need the following definition: A tensor field $R^{(0,2)}$ of lightlike hypersurfaces M is called its *induced Ricci tensor* of M if it is symmetric. A symmetric $R^{(0,2)}$ tensor will be denoted by Ric.

THEOREM 2.5. *Let $(M, g, S(TM))$ be an $(m+1)(m > 2)$-dimensional lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$ such that $S(TM)$ is totally umbilical. Then M admits an induced symmetric Ricci tensor Ric. Moreover, both M and the leaf M^* of $S(TM)$ are spaces of constant curvature c.*

Proof. Using (1.14), (1.15), (1.16) and (2.7), we have

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}, \quad R^{(0,2)}(X, Y) = mcg(X, Y),$$

for any $X, Y, Z \in \Gamma(TM)$, due to the fact $\beta\gamma = 0$ by Theorem 2.2. Thus $R^{(0,2)}$ is a symmetric Ricci tensor Ric and M is a space of constant curvature c. Also, from (1.14) and (1.17), we have

$$R^*(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}, \quad Ric^*(X, Y) = (m-1)cg(X, Y),$$

for $X, Y, Z \in \Gamma(S(TM))$. Also M^* is a space of constant curvature c. □

Combining Theorem 2.4 and 2.5 with Note 2.3, we have

Theorem 2.6. *Let $(M, g, S(TM))$ be an $(m+1)(m>2)$-dimensional lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$ such that $S(TM)$ is totally umbilical. Then M is a lightlike space form of constant curvature c and locally a product manifold $L \times M^*$, where L is a null curve in M and M^* is a semi-Riemannian space form of same constant curvature c.*

Recall the following notion of *null sectional curvature* [1, 2, 3, 5]. Let $x \in M$ and ξ be a null vector of T_xM. A plane H of T_xM is called a *null plane* directed by ξ if it contains ξ, $g_x(\xi, W) = 0$ for any $W \in H$ and there exists $W_o \in H$ such that $g_x(W_o, W_o) \neq 0$. Then, the null sectional curvature of H, with respect to ξ and the induced connection ∇ of M, is defined as a real number

$$K_\xi(H) = \frac{g_x(R(W, \xi)\xi, W)}{g_x(W, W)},$$

where $W \neq 0$ is any vector in H independent with ξ. It is easy to see that $K_\xi(H)$ is independent of W but depends in a quadratic fashion on ξ. An $n(\geq 3)$-dimensional Lorentzian manifold is of constant curvature if and only if its null sectional curvatures are everywhere zero [7].

Theorem 2.7. *Let $(M, g, S(TM))$ be an $(m+1)(m>2)$-dimensional lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$ such that $S(TM)$ is totally umbilical. Then every null plane H of T_xM directed by ξ has everywhere zero null sectional curvatures.*

Proof. From (1.14) and the fact that $\beta\gamma = 0$, we show that

$$g(R(X,Y)Z, PW) = c\{g(Y,Z)g(X,PW) - g(X,Z)g(Y,PW)\},$$

for any $X, Y, Z, W \in \Gamma(TM)$. Thus $K_\xi(H) = \frac{g_x(R(W,\xi)\xi, W)}{g_x(W,W)} = 0$ for any null plane H of T_xM directed by ξ.

□

Nomizu and Pinkall [6] defined an affine immersion as follows: Let $f : M \to \bar{M}$ be an immersion of a manifold M as a hypersurface of $\bar{M}$ and ∇ and $\bar{\nabla}$ be torsion-free connections on M and $\bar{M}$ respectively. Then f is an *affine immersion* if there exists locally a transversal vector field N along f such that

$$\bar{\nabla}_{f_*X} f_*Y = f_*(\nabla_X Y) + B(X,Y)N, \quad \forall\, X, Y \in \Gamma(TM),$$

where f_* is the differential map of f. Then, as usual, we put

$$\bar{\nabla}_{f_*X} N = -A_N(f_*X) + \tau(f_*X)N.$$

Clearly, by (1.4), any lightlike isometric immersion is an affine immersion. Suppose ∇ is a flat connection on M. Let $\psi : M \to \mathbf{R}^{m+1}$ such that every point $x \in M$ has a neighborhood $\mathcal{U}$ on which ψ is an affine connection preserving diffeomorphism with an open neighborhood $\mathcal{W}$ of $\psi(x)$ in $\mathbf{R}^{m+1}$. Consider $\mathbf{R}^{m+1}$ as a hyperplane of $\mathbf{R}^{m+2}$ and let N be a parallel vector field, transversal to $\mathbf{R}^{m+1}$. Then, for any differentiable function $F : M \to \mathbf{R}$, define

$$f : M \to \mathbf{R}^{m+2} \; ; \; f(x) = \psi(x) + F(x)N, \quad \forall\, x \in M.$$

Thus, f is an affine immersion with $A_N = 0$, called the *graph immersion* with respect to F. Now, we recall the following result.

THEOREM 2.8. [2]. *Let M be a lightlike hypersurface of $\mathbf{R}^{m+2}_q$ with a parallel screen distribution $S(TM)$. Then the immersion of M is affinely equivalent to the graph immersion of a certain function $F : M \to \mathbf{R}$.*

THEOREM 2.9. *Let $(M, g, S(TM))$ be an $(m+1)(m > 2)$-dimensional non-totally geodesic lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$ such that $S(TM)$ is totally umbilical. Then the immersion of M is affinely equivalent to the graph immersion of a certain function $F : M \to \mathbf{R}$.*

Proof. Since $B \neq 0$ on any $\mathcal{U} \subset M$, by Theorem 2.4, we have $C = 0$ on any $\mathcal{U} \subset M$, i.e., the screen distribution $S(TM)$ is totally geodesic and the sectional curvature c of the ambient space $\bar{M}(c)$ satisfies $c = 0$. Therefore, by (1.6), $C = 0$, on any $\mathcal{U} \subset M$, implies $S(TM)$ is parallel with respect to the induced connection ∇. The fact $c = 0$ implies that $\bar{M}(c)$ is $\mathbf{R}^{m+2}_q$. Thus, by Theorem 2.8, we have our theorem. □

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