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ON THE GENERALIZED HYERS-ULAM STABILITY OF A BI-JENSEN FUNCTIONAL EQUATION

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Abstract. In this paper, we study the generalized Hyers-Ulam stability of a bi-Jensen functional equation

$$4f(\frac{x+y}{2}, \frac{z+w}{2}) = f(x,z) + f(x,w) + f(y,z) + f(y,w).$$

Moreover, we establish stability results on the punctured domain.

1. Introduction

The stability problem of functional equations originated from a question of S.M. Ulam [17] concerning the stability of group homomorphisms: Given a group G_1, a metric group (G_2, d) and $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $h : G_1 \to G_2$ satisfies

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$, then a homomorphism $H : G_1 \to G_2$ exists with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$? If the answer is affirmative, we would say the equation of homomorphism $H(xy) = H(x)H(y)$ stable.
In 1941, D.H. Hyers [5] gave a first affirmative answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [1] for additive mappings and by Th.M.Rassias [16] for linear mappings by considering an unbounded Cauchy difference(See the recent Maligranda's paper [13]). Since then, a further generalization of the Hyers-Ulam theorem has been extensively investigated by a number of mathematicians [3, 4, 6, 9, 11, 12, 14].

Throughout this paper, let X be a vector space and Y a Banach space. A mapping $g : X \to Y$ is called a Cauchy mapping (respectively,

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a Jensen mapping) if g satisfies the functional equation $g(x+y) = g(x)+g(y)$ (respectively, $2g(\frac{x+y}{2}) = g(x)+g(y)$).

For a given mapping $f : X \times X \to Y$, we define

$$Jf(x,y,z,w) := 4f(\frac{x+y}{2}, \frac{z+w}{2}) - f(x,z) - f(x,w) - f(y,z) - f(y,w),$$
$$C_1 f(x,y,z) := f(x+y,z) - f(x,z) - f(y,z),$$
$$C_2 f(x,y,z) := f(x,y+z) - f(x,y) - f(x,z),$$
$$J_1 f(x,y,z) := 2f(\frac{x+y}{2}, z) - f(x,z) - f(y,z),$$
$$J_2 f(x,y,z) := 2f(x, \frac{y+z}{2}) - f(x,y) - f(x,z)$$

for all $x,y,z,w \in X$. A mapping $f : X \times X \to Y$ is called a biadditive (Cauchy-Jensen, Jensen-Cauchy, bi-Jensen, respectively) mapping if f satisfies the functional equations $C_1 f = 0$ and $C_2 f = 0$($C_1 f = 0$ and $J_2 f = 0$, $C_2 f = 0$ and $J_1 f = 0$,$J_1 f = 0$ and $J_2 f = 0$, respectively).

When $X = Y = \mathbb{R}$, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x,y) = axy + bx + cy + d$ is a solution of $J_1 f(x,y,z) = 0$ and $J_2 f(x,y,z) = 0$. It is easy to see that a mapping $f : X \times X \to Y$ is a bi-Jensen mapping if and only if the mapping f satisfies the functional equation $Jf(x,y,z,w) = 0$ for all $x,y,z,w \in X$.

Park and Bae [15] obtained the generalized Hyers-Ulam stability of Cauchy-Jensen mapping. Jun, Lee and Cho [7] improved the Park and Bae's results of Cauchy-Jensen functional equation. Bae and Park [2] investigated the stability of a bi-Jensen mapping in the following theorem.

THEOREM 1.1. *Let $\varphi, \psi : X \times X \times X \to [0,\infty)$ be two functions such that*

$$\tilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{3^{j+1}} [\varphi(3^j x, 3^j y, z) + \varphi(x, y, 3^j z)] < \infty,$$
$$\tilde{\psi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{3^{j+1}} [\psi(x, 3^j y, 3^j z) + \psi(3^j x, y, z)] < \infty$$

for all $x,y,z \in X$. Let $f : X \times X \to Y$ be a mapping such that

$$\|J_1 f(x,y,z)\| \le \varphi(x,y,z),$$
$$\|J_2 f(x,y,z)\| \le \psi(x,y,z)$$

for all $x, y, z \in X$. *Then there exist two bi-Jensen mappings* $F, F' : X \times X \to Y$ *such that*

$$\|f(x,y) - f(0,y) - F(x,y)\| \le \tilde{\varphi}(x,-x,y) + \tilde{\varphi}(-x,3x,y),$$
$$\|f(x,y) - f(x,0) - F'(x,y)\| \le \tilde{\psi}(x,y,-y) + \tilde{\psi}(x,-y,3y)$$

for all $x, y \in X$. *The mappings* $F, F' : X \times X \to Y$ *are given by*

$$F(x,y) := \lim_{j\to\infty} \frac{1}{3^j} f(3^j x, y) \quad \text{and} \qquad F'(x,y) := \lim_{j\to\infty} \frac{1}{3^j} f(x, 3^j y)$$

for all $x, y \in X$.

In this paper, we improve Bae and Park's stability results for the bi-Jensen functional equation by adopting the direct method of proof and prove the uniqueness of a bi-Jensen mapping. Moreover we establish new results for the stability of a bi-Jensen functional equation.

2. Stability of a bi-Jensen functional equation

One can easily prove the basic properties of a bi-Jensen mapping in the following lemma([8]).

Lemma 2.1. *Let* $f : X \times X \to Y$ *be a bi-Jensen mapping. Then*

$$f(x,y) = \frac{1}{2^n} f(x, 2^n y) + (1 - \frac{1}{2^n}) f(x,0),$$
$$f(x,y) = \frac{1}{2^n} f(2^n x, y) + \frac{1}{2^n}(1 - \frac{1}{2^n}) f(0, 2^n y) + (1 - \frac{1}{2^n})^2 f(0,0),$$
$$f(x,y) = 2^n f(\frac{x}{2^n}, y) + 2^n(1 - 2^n) f(0, \frac{y}{2^n}) + (1 - 2^n)^2 f(0,0),$$
$$f(x,y) = \frac{1}{2^n} f(2^n x, y) + (1 - \frac{1}{2^n}) 2^n f(0, \frac{y}{2^n}) + (1 - \frac{1}{2^n})(1 - 2^n) f(0,0)$$

for all $x, y \in X$ *and* $n \in \mathbb{N}$.

Theorem 2.2. *Let* $\varphi, \psi : X \times X \times X \to [0, \infty)$ *be two functions satisfying*

$$\sum_{j=1}^{\infty} \frac{1}{2^j} (\varphi(2^j x, 2^j y, z) + \varphi(x, y, 2^j z) + \psi(2^j x, y, z) + \psi(x, 2^j y, 2^j z)) < \infty$$

for all $x, y, z \in X$. *Let* $f : X \times X \to Y$ *be a mapping such that*

$$\|J_1 f(x,y,z)\| \le \varphi(x,y,z), \tag{2.1}$$
$$\|J_2 f(x,y,z)\| \le \psi(x,y,z) \tag{2.2}$$

for all $x, y, z \in X$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \le \sum_{j=1}^{\infty} \frac{\varphi(2^j x, 0, y) + \psi(0,0,2^j y)}{2^j} \tag{2.3}$$

for all $x, y \in X$ with $F(0,0) = f(0,0)$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty} \frac{f(2^j x, y) + f(0, 2^j y)}{2^j} + f(0,0)$$

for all $x, y \in X$.

Proof. By (2.1) and (2.2), we get

$$\begin{aligned}
&\|\frac{f(2^j x, y) - f(0,y)}{2^j} - \frac{f(2^{j+1}x, y) - f(0,y)}{2^{j+1}}\| \\
&\quad = \|\frac{J_1 f(2^{j+1}x, 0, y)}{2^{j+1}}\| \le \frac{\varphi(2^{j+1}x, 0, y)}{2^{j+1}}, \\
&\|\frac{f(0, 2^j y) - f(0,0)}{2^j} - \frac{f(0, 2^{j+1}y) - f(0,0)}{2^{j+1}}\| \\
&\quad = \|\frac{J_2 f(0,0,2^{j+1}y)}{2^{j+1}}\| \le \frac{\psi(0,0,2^{j+1}y)}{2^{j+1}}
\end{aligned}$$

for all $x, y \in X$ and $j \in \mathbb{N}$. For given integers l, m $(0 \le l < m)$,

$$\begin{aligned}
&\|\frac{f(2^l x, y) - f(0,y)}{2^l} - \frac{f(2^m x, y) - f(0,y)}{2^m}\| \\
&\qquad \le \sum_{j=l}^{m-1} \frac{\varphi(2^{j+1}x, 0, y)}{2^{j+1}},
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&\|\frac{f(0, 2^l y) - f(0,0)}{2^l} - \frac{f(0, 2^m y) - f(0,0)}{2^m}\| \\
&\qquad \le \sum_{j=l}^{m-1} \frac{\psi(0,0,2^{j+1}y)}{2^{j+1}}
\end{aligned} \tag{2.5}$$

for all $x, y \in X$. By the hypotheses of φ and ψ, the sequences $\{\frac{1}{2^j}(f(2^j x, y) - f(0,y))\}$ and $\{\frac{1}{2^j}(f(0, 2^j y) - f(0,0))\}$ are Cauchy sequences for all $x, y \in X$. Since Y is complete, the sequences $\{\frac{1}{2^j}(f(2^j x, y) - f(0,y))\}$ and $\{\frac{1}{2^j}(f(0, 2^j y) - f(0,0))\}$ converge for all $x, y \in X$. Define $F_1, F_2 :$

$X \times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty} \frac{f(2^j x, y)}{2^j},$$
$$F_2(x,y) := \lim_{j\to\infty} \frac{f(0, 2^j y)}{2^j}$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (2.4) and (2.5), one can obtain the inequalities

$$\|f(x,y) - f(0,y) - F_1(x,y)\| \le \sum_{j=1}^{\infty} \frac{\varphi(2^j x, 0, y)}{2^j},$$
$$\|f(0,y) - f(0,0) - F_2(x,y)\| \le \sum_{j=1}^{\infty} \frac{\psi(0, 0, 2^j y)}{2^j}$$

for all $x, y \in X$. By (2.1), (2.2), the hypotheses of φ, ψ and the definitions of F_1 and F_2, we get

$$J_1F_1(x,y,z) = \lim_{j\to\infty} \frac{J_1 f(2^j x, 2^j y, z)}{2^j} = 0,$$
$$J_2F_1(x,y,z) = \lim_{j\to\infty} \frac{J_2 f(2^j x, y, z)}{2^j} = 0,$$
$$J_1F_2(x,y,z) = 0,$$
$$J_2F_2(x,y,z) = \lim_{j\to\infty} \frac{J_2 f(0, 2^j y, 2^j z)}{2^j} = 0$$

for all $x, y, z \in X$ and so F is a bi-Jensen mapping satisfying (2.3), where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y) + f(0,0).$$

Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (2.3) with $F'(0,0) = f(0,0)$. By Lemma 2.1, we have

$$\begin{aligned}
&\|F(x,y) - F'(x,y)\| \\
&\quad = \frac{1}{2^n}\|(F - F')(2^n x, y) + (1 - \frac{1}{2^n})(F - F')(0, 2^n y)\| \\
&\quad \le \frac{1}{2^n}\|(F - f)(2^n x, y))\| + \frac{1}{2^n}\|(F - f)(0, 2^n y))\| \\
&\quad + \frac{1}{2^n}\|(f - F')(2^n x, y)\| + \frac{1}{2^n}\|(f - F')(0, 2^n y)\|
\end{aligned}$$

$$\le \sum_{j=1}^{\infty} \frac{2}{2^{j+n}} [\varphi(2^{j+n}x, 0, y) + \psi(0, 0, 2^j y)$$
$$+ \varphi(0, 0, 2^n y) + \psi(0, 0, 2^{j+n} y)]$$

for all $x, y \in X$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x, y) = F'(x, y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique. □

THEOREM 2.3. *Let $\varphi, \psi : X \times X \times X \to [0, \infty)$ be two functions satisfying*

$$\sum_{j=0}^{\infty} 2^j \big(\varphi(\frac{x}{2^j}, \frac{y}{2^j}, z) + \varphi(x, y, \frac{z}{2^j}) + \psi(\frac{x}{2^j}, y, z) + \psi(x, \frac{y}{2^j}, \frac{z}{2^j})\big) < \infty$$

for all $x, y, z \in X$. Let $f : X \times X \to Y$ be a mapping satisfying (2.1) and (2.2) for all $x, y, z \in X$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x, y) - F(x, y)\| \le \sum_{j=0}^{\infty} 2^j \varphi(\frac{x}{2^j}, 0, y) \tag{2.6}$$

for all $x, y \in X$. The mapping F is given by

$$F(x, y) := \lim_{j\to\infty} 2^j \big(f(\frac{x}{2^j}, y) - f(0, y)\big) + f(0, y)$$

for all $x, y \in X$.

Proof. Letting $y = 0$, $z = y$ in (2.1), we get

$$\|2^j (f(\frac{x}{2^j}, y) - f(0, y)) - 2^{j+1}(f(\frac{x}{2^{j+1}}, y) - f(0, y))\| \le 2^j \varphi(\frac{x}{2^j}, 0, y),$$

for all $x, y \in X$. Applying the similar method as in the proof of Theorem 2.2, we can define the map $F_1 : X \times X \to Y$ by

$$F_1(x, y) := \lim_{j\to\infty} 2^j (f(\frac{x}{2^j}, y) - f(0, y))$$

for all $x, y \in X$ and get the inequality

$$\|f(x, y) - f(0, y) - F_1(x, y)\| \le \sum_{j=0}^{\infty} 2^j \varphi(\frac{x}{2^j}, 0, y)$$

for all $x, y \in X$. By (2.1), (2.2), the hypotheses of φ, ψ and the definition of F_1, we get

$$J_1F_1(x,y,z) = \lim_{j\to\infty} 2^j J_1 f(\frac{x}{2^j}, \frac{y}{2^j}, z) = 0,$$
$$J_2F_1(x,y,z) = \lim_{j\to\infty} 2^j (J_2 f(\frac{x}{2^j}, y, z) - J_2 f(0,y,z)) = 0$$

for all $x, y, z \in X$. By the hypotheses of φ and ψ, we have $\varphi(0,0,z) = 0$ and $\psi(0,y,z) = 0$ for all $y, z \in X$. And so we have $J_1 f(0,0,z) = 0$ and $J_2 f(0,y,z) = 0$ for all $y, z \in X$. Hence F is a bi-Jensen mapping satisfying (2.6), where F is given by

$$F(x,y) = F_1(x,y) + f(0,y).$$

Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (2.6) with $F'(0,0) = f(0,0)$. By Lemma 2.1 and $\varphi(0,0,y) = 0$ for all $y \in X$, we have

$$\begin{aligned}\|F(x,y) - F'(x,y)\| &= 2^n \|(F - F')(\frac{x}{2^n}, y) + (1 - 2^n)(F - F')(0, \frac{y}{2^n})\| \\ &\le 2^n \|(F - f)(\frac{x}{2^n}, y)\| + 2^n \|(f - F')(\frac{x}{2^n}, y)\| \\ &+ 4^n \|(F - f)(0, \frac{y}{2^n})\| + 4^n \|(f - F')(0, \frac{y}{2^n})\| \\ &\le \sum_{j=0}^{\infty} 2^{j+n} \varphi(\frac{x}{2^{j+n}}, 0, y)\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique. $\square$

THEOREM 2.4. *Let $\varphi, \psi : X \times X \times X \to [0, \infty)$ be two functions satisfying*

$$\sum_{j=1}^{\infty} \frac{\varphi(2^j x, 2^j y, z) + \psi(2^j x, y, z)}{2^j} + \sum_{j=0}^{\infty} 2^j (\varphi(x, y, \frac{z}{2^j}) + \psi(x, \frac{y}{2^j}, \frac{z}{2^j})) < \infty$$

for all $x, y, z \in X$. Let $f : X \times X \to Y$ be a mapping satisfying (2.1) and (2.2) for all $x, y, z \in X$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \le \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi(2^j x, 0, y) \tag{2.7}$$

for all $x, y \in X$. The mapping F is given by

$$F(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y) + f(0,y)$$

for all $x, y \in X$.

Proof. By the similar method as in Theorem 2.2 and Theorem 2.3, we get a bi-Jensen map F which satisfies (2.7). Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (2.7) with $F'(0,0) = f(0,0)$. By Lemma 2.1 and $\varphi(0,0,y) = 0$ for all $y \in X$, we have

$$\begin{aligned}
\|F(x,y) - F'(x,y)\| &= \frac{1}{2^n}\|(F-F')(2^n x, y)) + (2^n - 1)(F-F')(0, \frac{y}{2^n})\| \\
&\le \frac{1}{2^n}\|(F-f)(2^n x, y))\| + \|(F-f)(0, \frac{y}{2^n})\| \\
&+ \frac{1}{2^n}\|(f-F')(2^n x, y)\| + \|(f-F')(0, \frac{y}{2^n})\| \\
&\le \sum_{j=1}^{\infty} \frac{2}{2^{j+n}} \varphi(2^{j+n} x, 0, y)
\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique. □

Corollary 2.5. *Let p, q, θ be fixed positive real numbers with $p, q \neq 1$ and let X a normed space. If $f : X \times X \to Y$ is a mapping satisfying*

$$\begin{aligned}
\|J_1 f(x,y,z)\| &\le \theta(\|x\|^p + \|y\|^p)\|z\|^q, \\
\|J_2 f(x,y,z)\| &\le \theta\|x\|^p(\|y\|^q + \|z\|^q)
\end{aligned}$$

for all $x, y, z \in X$, then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \le \frac{2^p \theta}{|2 - 2^p|}\|x\|^p\|y\|^q$$

for all $x, y \in X$.

3. Stability of a bi-Jensen functional equation on the punctured domain

The following lemma can be found in [10].

LEMMA 3.1. *Let a set A be a subset of X satisfying the following condition: for every $x \neq 0$, there exists a positive integer n_x such that $nx \notin A$ for all integers n with $|n| \geq n_x$ and $nx \in A$ for all integers n with $|n| < n_x$. Let $f : X \times X \to Y$ be a mapping such that*

$$J_1 f(x,y,z) = 0, \quad J_2 f(x,y,z) = 0$$

for all $x, y, z \in X \backslash A$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$F(x,y) = f(x,y)$$

for all $x, y \in X \backslash A$. Moreover,

$$F(x,y) = f(x,y)$$

holds for all $(x,y) \in (X \times X) \backslash (A \times A)$.

THEOREM 3.2. *Let $\varphi, \psi : X \times X \times X \to [0,\infty)$ be as in Theorem 2.2. Let $f : X \times X \to Y$ be a mapping satisfying (2.1) and (2.2) for all $x, y, z \in X \backslash A$ and let $x_0 \in X \backslash A$. Then there exists a bi-Jensen mapping F such that*

$$\|f(x,y) - F(x,y)\| \leq \sum_{j=0}^{\infty} \Big(\frac{\Phi_1(2^j x, y)}{2^{j+2}} + \frac{\Phi_2(x, 2^j y)}{2^{j+4}}\Big) + \Phi_3(x_0, y) \tag{3.1}$$

for all $x, y \in X \backslash A$, where

$$\begin{aligned}
\Phi_1(x,y) &= \varphi(3x,-x,y) + \varphi(3x,x,y) + \varphi(x,-x,y), \\
\Phi_2(x,y) &= 4\varphi(x,-x,y) + 4\varphi(x,-x,-y) + 2\varphi(x,-x,2y) \\
&\quad + 2\varphi(x,-x,-2y) + \psi(x,3y,y) + \psi(-x,3y,y) + \psi(x,-3y,-y) \\
&\quad + \psi(-x,-3y,-y) + \psi(x,3y,-y) + \psi(-x,3y,-y) \\
&\quad + \psi(x,-3y,y) + \psi(-x,-3y,y), \\
\Phi_3(x_0,y) &= \frac{1}{4}(\varphi(x_0,-x_0,y) + \varphi(x_0,-x_0,-y) \\
&\quad + \psi(x_0,y,-y) + \psi(-x_0,y,-y)).
\end{aligned}$$

The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty} \Big(\frac{f(2^j x, y)}{2^j} + \frac{f(0, 2^j y)}{2^j}\Big) + \frac{f(x_0,0) + f(-x_0,0)}{2}$$

for all $x, y \in X$.

Proof. Let $c = \frac{f(x_0,0)+f(-x_0,0)}{2}$. For an arbitrary $y \in X\backslash\{0\}$, using (2.1), (2.2) and the following inequality

$$\begin{aligned}
&\|\frac{c}{2^j} - \frac{f(0,2^jy)+f(0,-2^jy)}{2^{j+1}}\| = \frac{1}{2^{j+2}}\|J_1f(x_0,-x_0,2^jy) \\
&\qquad + J_1f(x_0,-x_0,-2^jy) - J_2f(x_0,2^jy,-2^jy) - J_2f(-x_0,2^jy,-2^jy)\|
\end{aligned}$$

for sufficiently large $j \in \mathbb{N}$, we get

$$\lim_{j\to\infty} \frac{f(0,2^jy)+f(0,-2^jy)}{2^{j+1}} = 0 \tag{3.2}$$

for all $y \in X\backslash\{0\}$. In particular, for $y \in X\backslash A$, the inequality

$$\|\frac{f(0,y)+f(0,-y)}{2} - c\| \le \Phi_3(x_0,y) \tag{3.3}$$

holds. By (2.1) and (2.2), we get

$$\begin{aligned}
&\|\frac{f(2^jx,y)-f(0,y)}{2^j} - \frac{f(2^{j+1}x,y)-f(0,y)}{2^{j+1}}\| \\
&= \frac{1}{2^{j+2}}\|J_1f(3\cdot 2^jx,-2^jx,y) - J_1f(3\cdot 2^jx,2^jx,y) - J_1f(2^jx,-2^jx,y)\| \\
&\le \frac{\Phi_1(2^jx,y)}{2^{j+2}},
\end{aligned}$$

$$\begin{aligned}
&\|\frac{f(0,2^jy)-f(0,-2^jy)}{2^{j+1}} - \frac{f(0,2^{j+1}y)-f(0,-2^{j+1}y)}{2^{j+2}}\| \\
&= \frac{1}{2^{j+4}}\|4J_1f(x,-x,2^jy) - 4J_1f(x,-x,-2^jy) - 2J_1f(x,-x,2^{j+1}y) \\
&+ 2J_1f(x,-x,-2^{j+1}y) - J_2f(x,3\cdot 2^jy,2^jy) - J_2f(-x,3\cdot 2^jy,2^jy) \\
&+ J_2f(x,-3\cdot 2^jy,-2^jy) + J_2f(-x,-3\cdot 2^jy,-2^jy) \\
&+ J_2f(x,3\cdot 2^jy,-2^jy) + J_2f(-x,3\cdot 2^jy,-2^jy) - J_2f(x,-3\cdot 2^jy,2^jy) \\
&- J_2f(-x,-3\cdot 2^jy,2^jy)\| \le \frac{\Phi_2(x,2^jy)}{2^{j+4}}
\end{aligned}$$

for all $x, y \in X\backslash A$ and $j \in \mathbb{N}$. By the hypotheses of φ, ψ and the similar method as in Theorem 2.2, the sequences $\{\frac{f(2^jx,y)-f(0,y)}{2^j}\}$ and $\{\frac{f(0,2^jy)-f(0,-2^jy)}{2^{j+1}}\}$ are Cauchy sequences for all $x, y \in X\backslash A$. Since Y is complete, the sequences $\{\frac{f(2^jx,y)-f(0,y)}{2^j}\}$ and $\{\frac{f(0,2^jy)-f(0,-2^jy)}{2^{j+1}}\}$ converge for all $x, y \in X\backslash A$. Note that if $x \in X\backslash\{0\}$, then $2^jx \in X\backslash A$ for sufficiently large $j \in \mathbb{N}$. Hence the limit $\lim_{j\to\infty} \frac{f(2^jx,y)}{2^j} =$

$\lim_{j\to\infty} \frac{f(2^j x,y)-f(0,y)}{2^j}$ exists for all $x \in X$ and $y \in X\backslash A$. Since the equality (3.2) and the inequalities

$$\begin{aligned}
&\|\frac{f(2^j x,y)}{2^{j-1}} - \frac{f(2^j x,(k'+2)y)}{2^j} - \frac{f(2^j x,-k'y)}{2^j}\| \\
&\qquad = \|\frac{J_2 f(2^j x,(k'+2)y,-k'y)}{2^j}\| \le \frac{\psi(2^j x,(k'+2)y,-k'y)}{2^j}, \\
&\|\frac{f(2^j x,0)}{2^{j-1}} - \frac{f(2^j x,k'y)}{2^j} - \frac{f(2^j x,-k'y)}{2^j}\| \\
&\qquad = \|\frac{J_2 f(2^j x,(k'+2)y,-k'y)}{2^j}\| \le \frac{\psi(2^j x,k'y,-k'y)}{2^j}
\end{aligned}$$

hold for all $x, y \neq 0$ and $j, k' \in \mathbb{N}$ with $2^j x, k'y \in X\backslash A$, we can define $F_1, F_2 : X \times X \to Y$ by

$$\begin{aligned}
F_1(x,y) &:= \lim_{j\to\infty} \frac{f(2^j x,y)}{2^j} = \lim_{j\to\infty} \frac{f(2^j x,y)-f(0,y)}{2^j}, \\
F_2(x,y) &:= \lim_{j\to\infty} \frac{f(0,2^j y)}{2^j} = \lim_{j\to\infty} \frac{f(0,2^j y)-f(0,-2^j y)}{2^{j+1}}.
\end{aligned}$$

By (2.1), (2.2) and the definition of F_1, we obtain

$$\begin{aligned}
J_1 F_1(x,y,z) &= \lim_{j\to\infty} \frac{J_1 f(2^j x,2^j y,z)}{2^j} = 0, \\
J_2 F_1(x,z,w) &= \lim_{j\to\infty} \frac{J_2 f(2^j x,z,w)}{2^j} = 0
\end{aligned}$$

for all $x, y \neq 0$ and $z, w \notin A$. By Lemma 3.1, there exists a bi-Jensen mapping $F_1 : X \times X \to Y$ such that $F_1'(x,y) = F_1(x,y)$ for all $(x,y) \in (X \times X)\backslash(A \times A)$. Since the equalities

$$\begin{aligned}
F_1(x,y) - F_1'(x,y) &= \frac{1}{2}[J_1 F_1((k+2)x,-kx,y) - J_1 F_1'((k+2)x,-kx,y)] \\
&= 0, \\
F_1(x,y) - F_1'(x,y) &= \frac{1}{2}[J_1 F_1(kx,-kx,y) - J_1 F_1'(kx,-kx,y)] = 0
\end{aligned}$$

hold for all $x \neq 0$ and $y \notin A$ with $kx \notin A$, the equalities

$$\begin{aligned} F_1(x,y) - F_1'(x,y) &= \frac{1}{2}[J_2F_1(x,(k'+2)y,-k'y) \\ &\quad - J_2F_1'(x,(k'+2)y,-k'y)] = 0, \\ F_1(x,0) - F_1'(x,0) &= \frac{1}{2}[J_2F_1(x,k'y,-k'y) - J_2F_1'(x,k'y,-k'y)] = 0, \\ F_1(0,y) - F_1'(0,y) &= J_2F_1(0,(k'+2)y,-k'y) \\ &\quad - J_2F_1'(0,(k'+2)y,-k'y) = 0 \end{aligned}$$

hold for all $x, y \neq 0$ with $k'y \notin A$. Hence F_1 is a bi-Jensen mapping. Since

$$\begin{aligned} &J_2F_2(x,y,-y) = 0, \\ &J_2F_2(x,y,z) = \lim_{j\to\infty}\Big[\frac{J_1f(w,-w,2^{j-1}(y+z))}{2^j} - \frac{J_1f(w,-w,2^jy)}{2^{j+1}} \\ &\qquad - \frac{J_1f(w,-w,2^jz)}{2^{j+1}} + \frac{J_2f(w,2^jy,2^jz)}{2^{j+1}} \\ &\qquad + \frac{J_2f(-w,2^jy,2^jz)}{2^{j+1}}\Big] \\ &\qquad = 0 \end{aligned}$$

for all $x, y, z \in X$ with $y, z, y+z \neq 0$ and $w \notin A$, we have

$$J_1F_2(x,y,z) = 0, \;\; J_2F_2(x,y,z) = 0$$

for all $x, y, z \in X$. Using the similar method as in Theorem 2.2, one can obtain the inequalities

$$\|f(x,y) - f(0,y) - F_1(x,y)\| \leq \sum_{j=0}^{\infty} \frac{\Phi_1(2^jx,y)}{2^{j+2}}, \tag{3.4}$$

$$\Big\|\frac{f(0,y) - f(0,-y)}{2} - F_2(x,y)\Big\| \leq \sum_{j=0}^{\infty} \frac{\Phi_2(x,2^jy)}{2^{j+4}} \tag{3.5}$$

for all $x, y \in X \backslash A$. By (3.3), (3.4), (3.5) and the inequality

$$\begin{aligned} \|f(x,y) - F(x,y)\| &\leq \|f(x,y) - f(0,y) - F_1(x,y)\| \\ &\quad + \Big\|\frac{f(0,y) + f(0,-y)}{2} - c\Big\| + \Big\|\frac{f(0,y) - f(0,-y)}{2} - F_2(x,y)\Big\|, \end{aligned}$$

we see that F is a bi-Jensen mapping satisfying (3.1) for all $x, y \in X \backslash A$, where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y) + c$$

for all $x, y \in X$. □

Corollary 3.3. *Let p, q, θ be fixed positive real numbers with $p, q < 1$ and let X a normed space. If $f : X \times X \to Y$ is a mapping satisfying*

$$\|J_1 f(x, y, z)\| \le \theta(\|x\|^p + \|y\|^p)\|z\|^q,$$
$$\|J_2 f(x, y, z)\| \le \theta\|x\|^p(\|y\|^q + \|z\|^q)$$

for all $\|x\|, \|y\|, \|z\| \ge 1$ and $x_0 \in X$ with $\|x_0\| > 1$, then there exists a unique bi-Jensen mapping F such that

$$\|f(x, y) - F(x, y)\| \le (\frac{2 + 2 \cdot 3^p}{2 - 2^p} + \frac{3 + 2^q + 3^q}{2 - 2^q})\theta\|x\|^p\|y\|^q + 2\theta\|x_0\|^p\|y\|^q$$

for all $\|x\|, \|y\| \ge 1$ with $F(0, 0) = \frac{f(x_0,0)+f(-x_0,0)}{2}$.

Theorem 3.4. *Let $\varphi, \psi : X \times X \times X \to [0, \infty)$ be two functions satisfying*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\varphi(2^i x, 2^i y, 2^j z) < \infty, \quad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\psi(2^i x, 2^j y, 2^j z) < \infty$$

for all $x, y, z \in X$. Let $f : X \times X \to Y$ be a mapping satisfying (2.1) and (2.2) for all $x, y, z \in X\backslash A$. Then there exists a unique bi-Jensen mapping F such that

$$f(x, y) = F(x, y)$$

for all $x, y \in X\backslash A$.

Proof. Putting $d = \frac{f(0,x)+f(0,-x)}{2}$ for a fixed $x \in X\backslash A$ and using the equality

$$\begin{aligned}
\frac{f(0, x) + f(0, -x)}{2} &- \frac{f(0, y) + f(0, -y)}{2} \\
&= \frac{1}{4}\lim_{j\to\infty}[-J_1 f(2^j z, -2^j z, x) - J_1 f(2^j z, -2^j z, -x) \\
&+ J_2 f(2^j z, x, -x) + J_2 f(-2^j z, x, -x) \\
&+ J_1 f(2^j z, -2^j z, y) + J_1 f(2^j z, -2^j z, -y) \\
&- J_2 f(2^j z, y, -y) - J_2 f(-2^j z, y, -y)] \\
&= 0
\end{aligned}$$

for all $x, y, z \in X\backslash A$, we get

$$\frac{f(0, y) + f(0, -y)}{2} = d$$

for all $y \in X\backslash A$. By (3.4), (3.5) and the following inequality

$$\begin{aligned}\|f(x,y)-F(x,y)\| &\le \|f(x,y)-f(0,y)-F_1(x,y)\| \\ &\quad + \|\frac{f(0,y)+f(0,-y)}{2}-d\| + \|\frac{f(0,y)-f(0,-y)}{2}-F_2(x,y)\|,\end{aligned}$$

F is a bi-Jensen mapping satisfying

$$\|f(x,y)-F(x,y)\| \le \sum_{j=0}^{\infty}\big(\frac{\Phi_1(2^j x,y)}{2^{j+2}}+\frac{\Phi_2(x,2^j y)}{2^{j+4}}\big) \tag{3.6}$$

for all $x, y \in X\backslash A$, where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y) + d$$

for all $x, y \in X$. From (3.6), we know

$$\begin{aligned}&\|(f-F)(x,y)\| \\ &= \frac{1}{2}\|(J_1 f - J_1 F)((k+2)x,-kx,y) + (f-F)((k+2)x,y) \\ &\qquad + (f-F)(-kx,y)\| \\ &\le \frac{1}{2}\|J_1 f((k+2)x,-kx,y)\| + \frac{1}{2}\|(f-F)((k+2)x,y)\| \\ &\qquad + \frac{1}{2}\|(f-F)(-kx,y)\| \\ &\le \frac{1}{2}\varphi((k+2)x,-kx,y) + \sum_{j=0}^{\infty}\big(\frac{\Phi_1(2^j(k+2)x,y)}{2^{j+3}} + \frac{\Phi_2((k+2)x,2^j y)}{2^{j+5}}\big) \\ &+ \sum_{j=0}^{\infty}\big(\frac{\Phi_1(-2^j kx,y)}{2^{j+3}} + \frac{\Phi_2(-kx,2^j y)}{2^{j+5}}\big)\end{aligned}$$

for all $x \neq 0$, $y \in X\backslash A$ and $k \in \mathbb{N}$. As taking $k \to \infty$, we get

$$f(x,y) = F(x,y)$$

for all $x \neq 0$, $y \in X\backslash A$. Using the following three equalities

$$\begin{aligned}(f-F)(0,y) = \frac{1}{2}\big((J_1 f - J_1 F)(kx,-kx,y) + (f-F)(kx,y) \\ + (f-F)(-kx,y)\big)\end{aligned}$$

for all $y \in X\backslash A$ and $k \in \mathbb{N}$ with $kx \in X\backslash A$,

$$\begin{aligned}(f-F)(x,y) &= \frac{1}{2}\big((J_2f - J_2F)(x,(k'+2)y,-k'y) \\ &\quad + (f-F)(x,(k'+2)y) + (f-F)(x,-k'y)\big),\\ (f-F)(x,0) &= \frac{1}{2}\big((J_2f - J_2F)(x,k'y,-k'y) \\ &\quad + (f-F)(x,k'y) + (f-F)(x,-k'y)\big)\end{aligned}$$

for all $x \in X\backslash A$, $y \neq 0$ and $k \in \mathbb{N}$ with $k'y \in X\backslash A$, we easily get the desired result. □

Corollary 3.5. *Let p, q, θ be fixed positive real numbers with $p, q < 0$ and let X a normed space. If $f : X \times X \to Y$ is a mapping satisfying*

$$\|J_1 f(x,y,z)\| \leq \theta(\|x\|^p + \|y\|^p)\|z\|^q, \tag{3.7}$$

$$\|J_2 f(x,y,z)\| \leq \theta\|x\|^p(\|y\|^q + \|z\|^q) \tag{3.8}$$

for all $\|x\|, \|y\|, \|z\| \geq 1$, then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$f(x,y) = F(x,y)$$

for all $\|x\|, \|y\| \geq 1$.

Remark 3.6. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the map defined by

$$f(x,y) := \begin{cases} 0, & \text{for} \quad \|x\| \geq 1 \text{ or } \quad \|y\| \geq 1, \\ 1, & \text{for} \quad \|x\|, \|y\| < 1. \end{cases}$$

Then f is not a bi-Jensen map, but satisfies (3.7) and (3.8) for all $\|x\|, \|y\|, \|z\| > 1$.

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