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ON WEAKLY $s\gamma$-CONTINUOUS FUNCTIONS

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Abstract. In [6], the author introduced the concepts of $s\gamma$-open sets and $s\gamma$-continuous functions. In this paper, we introduce the concept of weak $s\gamma$-continuity which is a generalization of $s\gamma$-continuity and weak continuity and investigate characterizations for such functions.

1. Introduction

Let X be a topological space on which no separation axioms are assumed unless explicit stated. Let S be a subset of X. The closure (resp. interior) of S will be denoted by $cl(S)$ (resp. $int(S)$). A subset S of X is called a *semi-open* [3] (resp. *β-open* [7], *preopen* [4], *regular open* [1]) set if $S \subseteq cl(int(S))$ (resp. $S \subseteq cl(int(cl(S)))$, $S \subseteq int(cl(S))$, $S = int(cl(S))$). The complement of a semi-open (resp. β-open, preopen, regular open) set is called a *semi-closed* (resp. *β-closed*, *preclosed*, *regular closed*) set.

Let X be a nonempty set and $P(X)$ the power set of X. A subclass $\tau \subseteq P(X)$ is called a *supratopology* on X [5] if $X \in \tau$ and τ is closed under arbitrary union. (X,τ) is called a supratopological space. The members of τ are called *supraopen* sets. The complement of supraopen sets are called *supraclosed* sets. Let (X,τ) be a topological space. Then τ^* is called an *associated supratopology* with τ if $\tau \subseteq \tau^*$.

Let (X,τ) be a supratopological space and let $S(x) = \{A \in \tau : x \in A\}$ for each $x \in X$. Then we call $\mathbf{S}_x = \{A \subseteq X :$ there exists $\mu \subseteq S(x)$ such that μ is finite and $\cap\mu \subseteq A\}$ the *supra-neighborhood filter* at x [6]. A filter $\mathbf{F}$ on X *supra-converges* [6] to x if $\mathbf{F}$ is finer than the supra-neighborhood filter $\mathbf{S}_x$. A subset U of X is called an *$s\gamma$-open* set [6] in X if whenever a filter $\mathbf{F}$ on X supra-converges to x and $x \in U$, $U \in \mathbf{F}$. The

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class of all $s\gamma$-open sets in X will be denoted by $s\gamma(X)$. In particular, The class of all $s\gamma$-open sets induced by the supratopology τ will be denoted by $s\gamma_\tau(X)$.

For $A \subseteq X$, the *$s\gamma$-interior* of A in X, denoted by $s_\gamma I(A)$, is the union of all $s\gamma$-open sets contained in A.

$s_\gamma C(A) = \{x \in X : A \cap U \neq \emptyset$ for all $U \in \mathbf{S}_x\}$. We call $s_\gamma C(A)$ the *$s\gamma$-closure* of A.

THEOREM 1.1. [6] *Let (X, τ) be a supratopological space and $A \subseteq X$.*
(1) $s_\gamma I(A) \subseteq A$ and $A \subseteq s_\gamma C(A)$;
(2) A is $s\gamma$-open if and only if $A = s_\gamma I(A)$;
(3) A is $s\gamma$-closed if and only if $A = s_\gamma C(A)$;
(4) $s_\gamma I(A) = X - s_\gamma C(X - A)$ and $s_\gamma C(A) = X - s_\gamma I(X - A)$.

DEFINITION 1.2. Let (X, τ) and (Y, μ) be topological spaces and τ^* an associated supratopology with τ. A function $f : X \to Y$ is called

(1) *S-continuous* [5] if the inverse image of each open set of Y is supraopen in X;

(2) *$s\gamma$-continuous* [6] if the inverse image of each open set of Y is an $s\gamma$-open set in X;

(3) *weakly continuous* [2] if for each x and each open set V of $f(x)$, there exists an open set U in X such that $f(U) \subset cl(V)$.

2. Weakly $s\gamma$-continuous functions

DEFINITION 2.1. Let (X, τ) and (Y, μ) be topological spaces and τ^* an associated supratopology with τ. Then a function $f : (X, \tau) \to (Y, \mu)$ is said to be *weakly $s\gamma$-continuous* if for $x \in X$ and each open subset V containing $f(x)$, there is an $s\gamma$-open subset U containing x such that $f(U) \subseteq cl(V)$.

From Definition 1.2 and 2.1, we get the following implications but the converses are not always true as shown in the next examples:

continuous	$\to$	S-continuous	$\to$	$s\gamma$-continuous
$\downarrow$				$\downarrow$
weakly continuous		$\to$		weakly $s\gamma$-continuous

EXAMPLE 2.2. *Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}, X\}$, $\tau^* = \{\emptyset, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ be a topology and an associated supra-topology, respectively. Then $s\gamma(X) = \{\emptyset, \{c\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. Consider a function $f : X \to X$ defined as follows*

$f(a) = b$, $f(b) = f(d) = d$, $f(c) = c$. Then f is weakly $s\gamma$-continuous but it is not $s\gamma$-continuous.

EXAMPLE 2.3. *Let $X = \{a, b, c\}$. Let $\tau = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$, $\tau^* = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ be a topology and an associated supra-topology, respectively. Then $s\gamma(X) = P(X)$. Consider a function $f : X \to X$ defined as follows $f(a) = a$, $f(b) = c$, $f(c) = b$. Then f is weakly $s\gamma$-continuous. Take an open set $V = \{c\}$ containing $f(b) = c$ for $b \in X$. But there is no any open set U containing $\{b\}$ such that $f(U) \subseteq cl(\{c\}) = \{c\}$. Thus f is not weakly continuous.*

THEOREM 2.4. *Let (X, τ) and (Y, μ) be topological spaces and τ^* be an associated supratopology with τ. If $f : (X, \tau) \to (Y, \mu)$ is a function, then the following statements are equivalent:*
(1) f is weakly $s\gamma$-continuous.
(2) $f^{-1}(V) \subseteq s_\gamma I(f^{-1}(cl(V)))$ for every open subset V of Y.
(3) $s_\gamma C(f^{-1}(int(A))) \subseteq f^{-1}(A)$ for every closed set A of Y.
(4) $s_\gamma C(f^{-1}(int(cl(B)))) \subseteq f^{-1}(cl(B))$ for every set B of Y.
(5) $f^{-1}(int(B)) \subseteq s_\gamma I(f^{-1}(cl(int(B))))$ for every set B of Y.
(6) $s_\gamma C(f^{-1}(V)) \subseteq f^{-1}(cl(V))$ for every open subset V of Y.

Proof. (1) $\Rightarrow$ (2) Let V be an open subset in Y and $x \in f^{-1}(V)$. By hypothesis, there exists an $s\gamma$-open subset U of X containing x such that $f(U) \subseteq cl(V)$. Since $x \in U \subseteq f^{-1}(cl(V))$, we have $x \in s_\gamma I(f^{-1}(cl(V)))$. Hence $f^{-1}(V) \subseteq s_\gamma I(f^{-1}(cl(V)))$.

(2) $\Rightarrow$ (3) Let A be a closed subset in Y. From (2) and Theorem 1.1, it follows

$$\begin{aligned} f^{-1}(Y - A) &\subseteq s_\gamma I(f^{-1}(cl(Y - A))) \\ &= s_\gamma I(f^{-1}(Y - int(A))) \\ &\subseteq X - s_\gamma C(f^{-1}(int(A))). \end{aligned}$$

Hence $s_\gamma C(f^{-1}(int(A))) \subseteq f^{-1}(A)$.

(3) $\Rightarrow$ (4) For $B \subseteq Y$, it follows from (3),

(4) $\Rightarrow$ (5) For $B \subseteq Y$, from (4) and Theorem 1.1, it follows

$$\begin{aligned} f^{-1}(int(B)) &= X - f^{-1}(cl(Y - B)) \\ &\subseteq X - s_\gamma C(f^{-1}(int(cl(Y - B)))) \\ &= s_\gamma I(f^{-1}cl(int(B))). \end{aligned}$$

Thus (5) is obtained.

(5) ⇒ (6) Let V be an open subset of Y. Suppose $x \notin f^{-1}(cl(V))$. Then $f(x) \notin cl(V)$ and so there exists an open set U containing $f(x)$ such that $U \cap V = \emptyset$ and so $cl(U) \cap V = \emptyset$. By (5), $x \in f^{-1}(U) \subseteq s_\gamma I(f^{-1}(cl(U)))$. Then by definition of s_γ-interior, there exists an $s\gamma$ open set G containing x such that $x \in G \subseteq f^{-1}(cl(U))$. Since $cl(U) \cap V = \emptyset$ and $f(G) \subseteq cl(U)$, we have $G \cap f^{-1}(V) = \emptyset$ and so $x \notin s_\gamma C(f^{-1}(V))$. Hence $s_\gamma C(f^{-1}(V)) \subseteq f^{-1}(cl(V))$.

(6) ⇒ (1) Let $x \in X$ and V an open set in Y containing $f(x)$. Since $V = int(V) \subseteq int(cl(V))$, by (6),

$$\begin{aligned} x \in f^{-1}(V) &\subseteq f^{-1}(int(cl(V))) \\ &= X - f^{-1}(cl(Y - cl(V))) \\ &\subseteq X - s_\gamma C(f^{-1}(Y - cl(V))) \\ &= s_\gamma I(f^{-1}(cl(V))). \end{aligned}$$

This implies $x \in s_\gamma I(f^{-1}(cl(V)))$, and so there exists an $s\gamma$-open subset U in X such that $U \subseteq f^{-1}(cl(V))$. Hence f is weakly $s\gamma$-continuous. □

Remark 2.5. Let (X, τ) and (Y, μ) be topological spaces and τ^* be an associated supratopology with τ. If $\tau = \tau^*$, then obviously $s\gamma(X) = \tau$.

From Theorem 2.4 and Remark 2.5, we have the next corollary.

Corollary 2.6. *Let (X, τ) and (Y, μ) be topological spaces. If $f : (X, \tau) \to (Y, \mu)$ is a function, then the following statements are equivalent:*

(1) f is weakly continuous.
(2) $f^{-1}(V) \subseteq int(f^{-1}(cl(V)))$ for every open subset V of Y.
(3) $cl(f^{-1}(int(A))) \subseteq f^{-1}(A)$ for every closed set A of Y.
(4) $cl(f^{-1}(int(cl(B)))) \subseteq f^{-1}(cl(B))$ for every set B of Y.
(5) $f^{-1}(int(B)) \subseteq int(f^{-1}(cl(int(B))))$ for every set B of Y.
(6) $cl(f^{-1}(V)) \subseteq f^{-1}(cl(V))$ for every open subset V of Y.

Theorem 2.7. *Let (X, τ) and (Y, μ) be topological spaces and τ^* be an associated supratopology with τ. If $f : (X, \tau) \to (Y, \mu)$ is a function, then the following statements are equivalent:*

(1) f is weakly $s\gamma$-continuous.
(2) $s_\gamma C(f^{-1}(int(K))) \subseteq f^{-1}(K)$ for every regular closed set K of Y.
(3) $s_\gamma C(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$ for every β-open set G of Y.
(4) $s_\gamma C(f^{-1}(int(cl(U)))) \subseteq f^{-1}(cl(U))$ for every semiopen set U of Y.

Proof. (1) $\Rightarrow$ (2) Let K be any regular closed set of Y. Then K is also closed, by Theorem 2.4(3), we have $s_\gamma C(f^{-1}(int(K))) \subseteq f^{-1}(K)$.

(2) $\Rightarrow$ (3) Let G be any β-open set; then $cl(G) \subseteq cl(int(cl(G))) \subseteq cl(G)$, and so $cl(G)$ is regular closed. By (2), we have $s_\gamma C(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$.

(3) $\Rightarrow$ (4) It is obvious since every semiopen set is β-open.

(4) $\Rightarrow$ (1) Let V be any open set of Y. Then by (4),

$$s_\gamma C(f^{-1}(V)) \subseteq s_\gamma C(f^{-1}(int(cl(V)))) \subseteq f^{-1}(cl(V)).$$

Hence from Theorem 2.4, f is weakly $s\gamma$-continuous. □

COROLLARY 2.8. *Let (X,τ) and (Y,μ) be topological spaces. If $f : (X,\tau) \to (Y,\mu)$ is a function, then the following statements are equivalent:*
(1) f is weakly continuous.
(2) $cl(f^{-1}(int(K))) \subseteq f^{-1}(K)$ for every regular closed set K of Y.
(3) $cl(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$ for every β-open set G of Y.
(4) $cl(f^{-1}(int(cl(U)))) \subseteq f^{-1}(cl(U))$ for every semiopen set U of Y.

THEOREM 2.9. *Let (X,τ) and (Y,μ) be topological spaces and τ^* be an associated supratopology with τ. If $f : (X,\tau) \to (Y,\mu)$ is a function, then the following statements are equivalent:*
(1) f is weakly $s\gamma$-continuous.
(2) $s_\gamma C(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$ for every preopen subset G of Y.
(3) $s_\gamma C(f^{-1}(G)) \subseteq f^{-1}(cl(G))$ for every preopen subset G of Y.
(4) $f^{-1}(G) \subseteq s_\gamma I(f^{-1}(cl(G)))$ for every preopen subset G of Y.

Proof. (1) $\Rightarrow$ (2) For any preopen set G in Y, $cl(G) = cl(int(cl(G)))$. Let $A = int(cl(G))$. Then from hypothesis, it follows that $s_\gamma C(f^{-1}(A)) \subseteq f^{-1}(cl(A))$. This implies $s_\gamma C(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) Let G be any preopen set in Y. Then from (3), it follows that

$$\begin{aligned} f^{-1}(G) &\subseteq f^{-1}(int(cl(G))) \\ &= X - f^{-1}(cl(Y - cl(G))) \\ &\subseteq X - (s_\gamma C(f^{-1}(Y - cl(G)))) \\ &= s_\gamma I(f^{-1}(cl(G))). \end{aligned}$$

Hence we have (4).

(4) $\Rightarrow$ (1) Since every open set is preopen, from (4) and Theorem 2.4, f is weakly $s\gamma$-continuous. $\square$

Corollary 2.10. *Let (X, τ) and (Y, μ) be topological spaces. If $f : (X, \tau) \rightarrow (Y, \mu)$ is a function, then the following statements are equivalent:*

(1) f is weakly continuous.

(2) $cl(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$ for every preopen set G of Y.

(3) $cl(f^{-1}(G)) \subseteq f^{-1}(cl(G))$ for every preopen set G of Y.

(4) $f^{-1}(G) \subseteq int(f^{-1}(cl(G)))$ for every preopen set G of Y.

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