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STRONGLY PRIME FUZZY IDEALS AND RELATED FUZZY IDEALS IN AN INTEGRAL DOMAIN

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Abstract. We introduce the concepts of strongly prime fuzzy ideals, powerful fuzzy ideals, strongly primary fuzzy ideals, and pseudo-strongly prime fuzzy ideals of an integral domain R and we provide characterizations of pseudo-valuation domains, almost pseudo-valuation domains, and pseudo-almost valuation domains in terms of these fuzzy ideals.

1. Introduction

In [9], K. H. Lee and J. N. Mordeson introduced the notions of fractionary fuzzy ideals and of fuzzy invertible fractionary fuzzy ideals. Using these notions they characterized Dedekind domains in terms of the invertibility of certain fractionary fuzzy ideals. In [5], we introduced the concept of fuzzy star-operations on an integral domain R and characterized Prüfer domains, pseudo-Dedekind domains and G-GCD domains and others in terms of the invertibility of certain fractionary fuzzy ideals. Our study of fuzzy multiplicative ideal theory has continued in [6], where we characterized UFD's, valuation domains, Prüfer domains, π-domains and Mori domains. Also we introduced the concept of nonfactorable fuzzy ideals and characterized Dedekind domains in terms of nonfactorable fuzzy ideals.

This article continues a study of fuzzy multiplicative ideal theory. That is, we introduce the concepts of several new fuzzy ideals and using these notions we characterize some integral domains which are important classes in multiplicative ideal theory. More precisely, in section 2, we study strongly prime fuzzy ideals and characterize pseudo-valuation

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domains. Section 3 contains the concept and properties of powerful fuzzy ideals. In section 4, we introduce the concept of strongly primary fuzzy ideals and characterize almost pseudo-valuation domains. In the last section, using the notion of pseudo-strongly prime fuzzy ideals, we characterize pseudo-almost valuation domains.

Throughout this article let R be an integral domain with quotient field K. A *fuzzy subset* of R is a function from R into $[0, 1]$. Let μ, ν be fuzzy subsets of R. We write $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \in R$. If $\mu \subseteq \nu$ and there exists $x \in R$ such that $\mu(x) < \nu(x)$, then we write $\mu \subset \nu$. We denote the image of μ by $\mathrm{Im}(\mu)$. We say that μ is *finite-valued* if $|Im(\mu)| < \infty$.

Let $\mu_t = \{ x \in R \mid \mu(x) \geq t \}$, a *level set*, for every $t \in [0, 1]$. For a subset W of R let $\chi_W^{(t)}$ be the fuzzy subset of K such that $\chi_W^{(t)}(x) = 1$ if $x \in W$ and $\chi_W^{(t)}(x) = t$ if $x \in K \setminus W$, where $t \in [0, 1)$.

A fuzzy subset μ of R is a *fuzzy ideal* of R if for every $x,\ y \in R$, $\mu(x - y) \geq \mu(x) \wedge \mu(y)$ and $\mu(xy) \geq \mu(x) \vee \mu(y)$. A fuzzy subset μ of R is a fuzzy ideal of R if and only if $\mu(0) \geq \mu(x)$ for every $x \in R$, and μ_t is an ideal of R for every $t \in [0, \mu(0)]$.

A fuzzy subset β of K is a *fuzzy R-submodule* of K if $\beta(x - y) \geq \beta(x) \wedge \beta(y)$, $\beta(rx) \geq \beta(x)$ and $\beta(0) = 1$ for every $x,\ y \in K$, $r \in R$. A fuzzy subset β of K is a fuzzy R-submodule of K if and only if $\beta(0) = 1$ and β_t is an R-submodule of K for every $t \in [0, 1]$. We let β_* denote $\{x \in K \mid \beta(x) = \beta(0)\}$. Let $\mathbb{N}$ denote the positive integers.

Let α and β be fuzzy subsets of K. Define the fuzzy subset $\alpha \circ \beta$ of K by, for every $x \in K$, $(\alpha \circ \beta)(x) = \bigvee\{\alpha(y) \wedge \beta(z) \mid y, z \in K\}$ if x is expressible as a product $x = yz$. The product of α and β, written by $\alpha\beta$, is defined by $\alpha\beta(x) = \bigvee\{\bigwedge_{i=1}^{n}(\alpha(y_i) \wedge \beta(z_i)) \mid y_i, z_i \in K, 1 \leq i \leq n, n \in \mathbb{N}, \sum_{i=1}^{n} y_i z_i = x\}$. Let $\{\alpha_i \mid i = 1, \cdots, n\}$ be a collection of fuzzy subsets of K; we define the fuzzy subset $\sum_{i=1}^{n}\alpha_i$ of K by for every $x \in K$, $(\sum_{i=1}^{n}\alpha_i)(x) = \bigvee\{\bigwedge\{\alpha_i(x_i) \mid i = 1, \cdots, n\} \mid x = \sum_{i=1}^{n} x_i, x_i \in K\}$. A fuzzy subset $\bigcap_{i \in I} \alpha_i$ of K is defined by $(\bigcap_{i \in I} \alpha_i)(x) = \bigwedge\{\alpha_i(x) \mid i \in I\}$ for every $x \in K$. For $d \in K$ and $t \in [0, 1]$, we let d_t denote the fuzzy subset of K defined by, for every $x \in K$, $d_t(x) = t$ if $x = d$ and $d_t(x) = 0$ otherwise. We call d_t a *fuzzy singleton*. Let σ be a fuzzy subset of K and let $\langle\sigma\rangle$ denote the intersection of all fuzzy submodules of K which contain σ. Then $\langle\sigma\rangle$ is called the fuzzy submodule of K generated by σ.

A fuzzy ideal ξ of a ring R is said to be *fuzzy prime* if it is non-constant and for any two fuzzy ideals μ and ν of R, the condition $\mu \circ \nu \subseteq \xi$ implies that either $\mu \subseteq \xi$ or $\nu \subseteq \xi$. It is well-known that ξ is a fuzzy prime of R if and only if $\xi(0) = 1$, ξ_* is a prime ideal of R and $|Im(\xi)| = 2$ ([10, Theorem 3.5.5]). A fuzzy ideal ω is called a *maximal fuzzy ideal* if ω is a maximal element in the set of all non-constant fuzzy ideals of R under pointwise partial ordering.

A fuzzy R-submodule β of K is called a *fractionary fuzzy ideal of* R if there exists $d \in R, d \neq 0$, such that $d_1 \circ \beta \subseteq \chi_R^{(t)}$ for some $t \in [0, 1)$. Let β be a fractionary fuzzy ideal of R. Then $\beta|_R$ is a fuzzy ideal of R. If $\beta|_R$ is a prime(maximal) fuzzy ideal of R, then β is called a *prime (maximal) fractionary fuzzy ideal of* R. If $\beta(x) = 0$ for all $x \in K \setminus R$, then β is called an *integral fractionary fuzzy ideal of* R. Thus, if β is a prime (maximal) integral fractionary fuzzy ideal of R, then $Im(\beta) = \{0, t, 1\}$ for some $t \in [0, 1)$. Any unexplained notation or terminology is standard like in [10].

2. Strongly prime fuzzy ideals

We recall from [4] that a prime ideal P of R is said to be *strongly prime* if $x, y \in K$ and $xy \in P$ imply that $x \in P$ or $y \in P$ and an integral domain R is called a *pseudo-valuation domain* if every prime ideal of R is strongly prime. In this section, we introduce the concept of strongly prime fuzzy ideals and characterize pseudo-valuation domains using this concept.

Definition 2.1. A prime integral fractionary fuzzy ideal β of R is said to be *strongly prime* if for any fractionary fuzzy ideals μ, ν of R, $\mu \circ \nu \subseteq \beta$ implies that $\mu \subseteq \beta$ or $\nu \subseteq \beta$.

Proposition 2.2. *If β is a strongly prime integral fractionary fuzzy ideal of R, then β_* is a strongly prime ideal of R.*

Proof. Let $x, y \in K$ and $xy \in \beta_*$. Then $< x_1 > \circ < y_1 >=< (xy)_1 >\subseteq \beta$. Since β is a strongly prime fuzzy ideal, we have $< x_1 >\subseteq \beta$ or $< y_1 >\subseteq \beta$. Hence $x \in \beta_*$ or $y \in \beta_*$. Thus β_* is a strongly prime ideal of R. □

Theorem 2.3. *Let β be a $\{0, 1\}$-valued prime integral fractionary fuzzy ideal of R. Then the following statements are equivalent:*

(1) *β is a strongly prime fuzzy ideal of R.*
(2) *β_* is a strongly prime ideal of R.*
(3) *If $x, y \in K$, $a, b \in [0, 1]$ and $< x_a > \circ < y_b >\subseteq \beta$, then either $< x_a >\subseteq \beta$ or $< y_b >\subseteq \beta$.*

Proof. (1) $\Rightarrow$ (2) This follows from Proposition 2.2.

(2) $\Rightarrow$ (1) Suppose that there exist fractionary fuzzy ideals μ, ν of R such that $\mu \circ \nu \subseteq \beta$, but $\mu \nsubseteq \beta$ and $\nu \nsubseteq \beta$. Then there exist $x, y \in K$ such that $\mu(x) > \beta(x)$ and $\nu(y) > \beta(y)$. Since β is $\{0,1\}$-valued, we have $\beta(x) = 0$ and $\beta(y) = 0$. Then $x \notin \beta_*$ and $y \notin \beta_*$ By hypothesis, $xy \notin \beta_*$, and so $\beta(xy) = 0$. But $0 = \beta(xy) \geq (\mu \circ \nu)(xy) \geq \mu(x) \wedge \nu(y) > 0$, which is a contradiction. Thus $\mu \subseteq \beta$ or $\nu \subseteq \beta$. Therefore β is a strongly prime fuzzy ideal of R.

(1) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Suppose that for any fractionary fuzzy ideals μ, ν of R, $\mu \circ \nu \subseteq \beta$. Let $x, y \in K$. Since $x_{\mu(x)} \subseteq \mu$ and $y_{\nu(y)} \subseteq \nu$, we have $x_{\mu(x)} \circ y_{\nu(y)} \subseteq \mu \circ \nu \subseteq \beta$. By hypothesis, $x_{\mu(x)} \subseteq \beta$ or $y_{\nu(y)} \subseteq \beta$. Hence $\mu(x) \subseteq \beta(x)$ or $\nu(y) \subseteq \beta(y)$. Thus $\mu \subseteq \beta$ or $\nu \subseteq \beta$. □

Corollary 2.4. *Let α and β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideals of R. If $\alpha \subseteq \beta$ and β is a strongly prime fuzzy ideal of R, then α is a strongly prime fuzzy ideal of R.*

Theorem 2.5. *The following statements are equivalent for an integral domain R:*

(1) *R is a pseudo-valuation domain.*
(2) *Every $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R is a strongly prime fuzzy ideal of R.*

Proof. (1) $\Rightarrow$ (2) Let β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R. Since R is a pseudo-valuation domain, we have $\beta_* = (\beta|_R)_*$ is a strongly prime ideal of R. Hence β is a strongly prime fuzzy ideal of R by Theorem 2.3.

(2)$\Rightarrow$ (1) Let P be a prime ideal of R, and assume that $x, y \in K$ and $xy \in P$. Then $< x_1 > \circ < y_1 > \subseteq \chi_P^{(0)}$. By hypothesis, $< x_1 > \subseteq \chi_P^{(0)}$ or $< y_1 > \subseteq \chi_P^{(0)}$. Hence $x \in P$ or $y \in P$. Thus P is a strongly prime ideal of R. Therefore R is a pseudo-valuation domain. □

The following corollary is well-known ([4, Proposition 1.1]). Here, we will give its proof in terms of fractional fuzzy ideals.

Corollary 2.6. *Every valuation domain is a pseudo-valuation domain.*

Proof. Let R be a valuation domain and let β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideals of R. Suppose that $x, y \in K$ and $< x_1 > \circ < y_1 > \subseteq \beta$. Then $xy \in \beta_*$. If $x \notin R$, $y \notin R$, then $\beta(x) = \beta(y) = 0$. Since R is a valuation domain, we have $x^{-1},\ y^{-1} \in R$.

Then $y = xyx^{-1} \in \beta_*$, $x = xyy^{-1} \in \beta_*$, which is a contradiction. Thus either $x \in R$ or $y \in R$. If $x, y \in R$, then we are done. If $x \notin R$, then $< x_1 > \nsubseteq \beta$, and so $y = xyx^{-1} \in \beta_*$. Thus $< y_1 > \subseteq \beta$. Hence β is a strongly prime fuzzy ideal of R. Therefore R is a pseudo-valuation domain. $\square$

The following result is the fuzzification of [4, Proposition 1.2].

PROPOSITION 2.7. *Let β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R. Then β is a strongly prime fuzzy ideal of R if and only if $(\frac{1}{x})_1 \circ \beta \subseteq \beta$ whenever $x \in K \setminus R$.*

Proof. Suppose that β is a strongly prime fuzzy ideal of R and let $x \in K \setminus R$ and $w \in K$. If $\beta(w) = 1$, then $((\frac{1}{x})_1 \circ \beta)(w) \leq 1 = \beta(w)$. If $\beta(w) = 0$, then we now show that $((\frac{1}{x})_1 \circ \beta)(w) = \beta(xw) = 0$. If $w \in R$ and $\beta(xw) = 1$, then $xw \in \beta_*$. Since β_* is a strongly prime ideal of R, we have $w \in \beta_*$. Then $\beta(w) = 1$, which is a contradiction. If $w \notin R$ and $\beta(xw) = 1$, then $< x_1 > \circ < w_1 > \subseteq \beta$. Since β is a strongly prime fuzzy ideal of R, $< x_1 > \subseteq \beta$ or $< w_1 > \subseteq \beta$. Hence $\beta(x) = 1$ or $\beta(w) = 1$, which is a contradiction. Therefore $(\frac{1}{x})_1 \circ \beta \subseteq \beta$.

Conversely, assume that $(\frac{1}{x})_1 \circ \beta \subseteq \beta$ for each $x \in K \setminus R$, and that $y, z \in K$ and $< y_1 > \circ < z_1 > \subseteq \beta$. If $y, z \in R$, then $\beta(yz) = \beta(y)$ or $\beta(yz) = \beta(z)$ since $\beta|_R$ is a prime fuzzy ideal of R. Then $y \in \beta_*$ or $z \in \beta_*$. Hence $< y_1 > \subseteq \beta$ or $< z_1 > \subseteq \beta$. If $y \notin R$ and $z \in R$, then $< y_1 > \nsubseteq \beta$. By hypothesis, $(\frac{1}{y})_1 \circ \beta \subseteq \beta$. Hence $< z_1 > = < (\frac{1}{y})_1 > \circ < y_1 > \circ < z_1 > \subseteq (\frac{1}{y})_1 \circ \beta \subseteq \beta$. Therefore β is a strongly prime fuzzy ideal of R . $\square$

COROLLARY 2.8. *If R is a pseudo-valuation domain, then every $\{0,1\}$-valued prime integral fractionary fuzzy ideals of R are linearly ordered.*

Proof. Let α, β be $\{0,1\}$-valued prime integral fractionary fuzzy ideals of R and assume that $\alpha \nsubseteq \beta$. Then there exists $x \in K$ such that $\alpha(x) \gneq \beta(x)$. Since α and β are $\{0,1\}$-valued, we have $\alpha(x) = 1$ and $\beta(x) = 0$, and so $x \in R$. We now show that $\beta \subseteq \alpha$. Suppose that $w \in R$ and $\beta(w) = 1$. If $\frac{x}{w} \in R$, then $0 = \beta(x) = \beta(w \cdot \frac{x}{w}) \geq \beta(w) = 1$, which is a contradiction. Hence we conclude $\frac{x}{w} \notin R$. Since α is a strongly prime fuzzy ideals of R, $(\frac{w}{x})_1 \circ \alpha \subseteq \alpha$ by Proposition 2.7. Then $\alpha(w) \geq ((\frac{w}{x})_1 \circ \alpha)(w) = \alpha(x) = 1$. Thus $\alpha(w) = 1$. Hence $\beta \subseteq \alpha$. $\square$

The following result is the fuzzification of [4, Theorem 1.4].

THEOREM 2.9. *Let (R, M) be a quasi-local domain. Then the following statements are equivalent:*

(1) *R is a pseudo-valuation domain.*
(2) *For any $\{0,1\}$-valued integral fractionary fuzzy ideals α, β of R, either $\alpha \subseteq \beta$ or $\chi_M^{(0)} \circ \beta \subseteq \chi_M^{(0)} \circ \alpha$.*
(3) *For any $\{0,1\}$-valued integral fractionary fuzzy ideals α, β of R, either $\alpha \subseteq \beta$ or $\chi_M^{(0)} \circ \beta \subseteq \alpha$.*
(4) *$\chi_M^{(0)}$ is a strongly prime integral fractional fuzzy ideal of R.*

Proof. (1) $\Rightarrow$ (2) Since R is a pseudo-valuation domain, we have either $\alpha_* \subseteq \beta_*$ or $M\beta_* \subseteq M\alpha_*$. Then $\alpha \subseteq \beta$ or $\chi_M^{(0)} \circ \beta \subseteq \chi_M^{(0)} \circ \alpha$.

(2) $\Rightarrow$ (3) If $\chi_M^{(0)} \circ \beta \subseteq \chi_M^{(0)} \circ \alpha$, then $M\beta_* \subseteq M\alpha_* \subseteq \alpha_*$. Hence $\chi_M^{(0)} \circ \beta \subseteq \alpha$.

(3) $\Rightarrow$ (4) Clearly $\chi_M^{(0)}$ is a $\{0,1\}$-valued prime integral fractional fuzzy ideal of R. Let $x, y \in R$ with $\frac{x}{y} \notin R$. Since $\frac{x}{y} \notin R$, we have $xR \nsubseteq yR$. Then $< x_1 > \nsubseteq < y_1 >$. By hypothesis, we have $< y_1 > \circ \chi_M^{(0)} \subseteq < x_1 > \subseteq < x_1 > \circ \chi_R^{(0)}$. Then $< (\frac{y}{x})_1 > \circ \chi_M^{(0)} \subseteq \chi_R^{(0)}$. Hence $\frac{y}{x}M \subseteq R$. If $\frac{y}{x}M = R$, then $M = \frac{x}{y}R$, and so $\frac{x}{y} \in M \subseteq R$, which is a contradiction. Thus $\frac{y}{x}M \subset M$, and so $< (\frac{y}{x})_1 > \circ \chi_M^{(0)} \subseteq \chi_M^{(0)}$. Therefore $\chi_M^{(0)}$ is a strongly prime fuzzy ideal of R by Proposition 2.7.

(4) $\Rightarrow$ (1) Since $\chi_M^{(0)}$ is a strongly prime integral fractional fuzzy ideal of R, $M = (\chi_M^{(0)})_*$ is a strongly prime ideal of R by Theorem 2.3. Hence R is a pseudo-valuation domain. □

Proposition 2.10. *Let R be a pseudo-valuation domain. If β is a $\{0,1\}$-valued integral fractionary fuzzy ideal of R, then $\mu = \cap\{ \beta^k \mid k = 1, 2, \cdots\}$ is a prime integral fractionary fuzzy ideal of R.*

Proof. Since R is a pseudo-valuation domain, $P = \bigcap\{ (\beta_*)^k \mid k = 1, 2, \cdots\}$ is a prime ideal of R. Since β is finite-valued, $(\beta_*)^k = (\beta^k)_*$ for each $k \in \mathbb{N}$ by [10, Theorem 3.1.16]. Then $P = \bigcap\{ (\beta_*)^k \mid k = 1, 2, \cdots\} = \bigcap\{ (\beta^k)_* \mid k = 1, 2, \cdots\} = (\bigcap\{ \beta^k \mid k = 1, 2, \cdots\})_* = \mu_*$. Thus μ is a prime integral fractionary fuzzy ideal of R. □

Let ξ be a fractionary fuzzy ideal of R and let $\mathscr{P}_\xi$ be the family of all prime fractionary fuzzy ideal of R such that $\xi \subseteq \mu$. The *fuzzy radical ideal of* ξ, denoted by $\sqrt{\xi}$, is defined by $\sqrt{\xi} = \cap\{ \mu \mid \mu \in \mathscr{P}_\xi\}$ if $\mathscr{P}_\xi \neq \emptyset$ and $\sqrt{\xi} = \chi_R^{(0)}$ if $\mathscr{P}_\xi = \emptyset$. Then $\xi \subseteq \sqrt{\xi}$ and $\sqrt{\xi_*} \subseteq (\sqrt{\xi})_*$. If ξ is a $\{0,1\}$-valued fractionary fuzzy ideal of R, then $\sqrt{\xi_*} = (\sqrt{\xi})_*$.

COROLLARY 2.11. *Let (R, M) be a pseudo-valuation domain. If α and β are $\{0,1\}$-valued integral fractionary fuzzy ideals of R with $\alpha \subsetneq \sqrt{\beta}$, then $\alpha^k \subseteq \beta$ for some $k > 0$.*

Proof. Suppose that $\alpha^k \nsubseteq \beta$ for each $k > 0$. Since R is a pseudo-valuation domain, $\chi_M^{(0)} \circ \beta \subseteq \alpha^k$ for each $k > 0$. Since $\beta \subseteq \chi_M^{(0)}$, we have $\beta^2 \subseteq \alpha^k$ for each $k > 0$. Then $\beta^2 \subseteq \bigcap \alpha^k$, and $\bigcap \alpha^k$ is a prime integral fractionary fuzzy ideal of R by Proposition 2.10. By [10, Theorem 3.5.3], we have $\beta \subseteq \bigcap \alpha^k$. Hence $\sqrt{\beta} \subseteq \bigcap \alpha^k \subseteq \alpha$, which is a contradiction. □

3. Powerful fuzzy ideals

We recall from [2] that a nonzero ideal I of R is said to be *powerful* if $x, y \in K$ and $xy \in I$ imply that $x \in R$ or $y \in R$. In this section, we introduce the concept of powerful fuzzy ideal of R.

DEFINITION 3.1. Let β be an integral fractionary fuzzy ideal of R. Then β is called a *powerful fuzzy ideal of R* if $x, y \in K$ and $< x_1 > \circ < y_1 > \subseteq \beta$ imply that $< x_1 > \subseteq \chi_R^{(0)}$ or $< y_1 > \subseteq \chi_R^{(0)}$.

PROPOSITION 3.2. *Let β be an integral fractionary fuzzy ideal of R. Then β is a powerful fuzzy ideal of R if and only if β_* is a powerful ideal of R.*

Proof. Suppose that β is a powerful fuzzy ideal of R and that $x, y \in K$ and $xy \in \beta_*$. Then $< x_1 > \circ < y_1 > = < (xy)_1 > \subseteq \beta$. Since β is a powerful fuzzy ideal of R, we have $< x_1 > \subseteq \chi_R^{(0)}$ or $< y_1 > \subseteq \chi_R^{(0)}$. Then $x \in R$ or $y \in R$. Hence β_* is powerful ideal of R. Conversely, assume that β_* is a powerful ideal of R and that $x, y \in K$ and $< x_1 > \circ < y_1 > \subseteq \beta$. Then $xy \in \beta_*$, and so $x \in R$ or $y \in R$. Hence $< x_1 > \subseteq \chi_R^{(0)}$ or $< y_1 > \subseteq \chi_R^{(0)}$. Therefore β is a powerful fuzzy ideal of R. □

The following corollary is well-known ([2]). Here, we will give its proof in terms of fractional fuzzy ideals.

COROLLARY 3.3. *An integral domain R is powerful if and only if R is a valuation domain.*

Proof. Suppose that $R = (\chi_R^{(0)})_*$ is powerful. Then $\chi_R^{(0)}$ is a powerful fuzzy ideal of R by Proposition 3.2. Let $0 \neq x \in K$. Then $< x_1 > \circ < (\frac{1}{x})_1 > \subseteq \chi_R^{(0)}$. By hypothesis, $< x_1 > \subseteq \chi_R^{(0)}$ or $< (\frac{1}{x})_1 > \subseteq \chi_R^{(0)}$. Thus $x \in R$ or $x^{-1} \in R$. Hence R is a valuation domain. Conversely, assume that $x, y \in K$ and $< x_1 > \circ < y_1 > \subseteq \chi_R^{(0)}$, but $< x_1 > \nsubseteq \chi_R^{(0)}$. Then $x \notin R$. Since R is a valuation domain, we have $x^{-1} \in R$. Thus $< y_1 > = < (\frac{1}{x})_1 > \circ < x_1 > \circ < y_1 > \subseteq \chi_R^{(0)} \circ \chi_R^{(0)} = \chi_R^{(0)}$. Hence $\chi_R^{(0)}$ is a powerful fuzzy ideal of R. Therefore R is powerful. □

COROLLARY 3.4. *Let β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R. Then β is a strongly prime fuzzy ideal of R if and only if β is a powerful fuzzy ideal of R.*

Proof. This follows from Theorem 2.3, Proposition 3.2, and the fact that a prime ideal of R is strongly prime if and only if it is powerful ([2, Proposition 1.3]). □

The following result is easy to prove and so we omit its proof.

COROLLARY 3.5. *Let α, β be integral fractionary fuzzy ideals of R. If $\alpha \subseteq \beta$ and β is a powerful fuzzy ideal of R, then α is a powerful fuzzy ideal of R.*

COROLLARY 3.6. *Let β be a $\{0,1\}$-valued powerful integral fractionary fuzzy ideal of R. If α is a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R, then α and β are comparable.*

Proof. This follows from Proposition 3.2 and [2, Theorem 1.5]. □

COROLLARY 3.7. *If β is a $\{0,1\}$-valued powerful integral fractionary fuzzy ideal of R, then $\sqrt{\beta}$ is a prime fractionary fuzzy ideal of R.*

Proof. This follows from Proposition 3.2, [2, Proposition 1.9], and the fact that $(\sqrt{\beta})_* = \sqrt{\beta_*}$ ([10, Corollary 3.6.6]). □

The following result is the fuzzification of [2, Lemma 1.2].

PROPOSITION 3.8. *Let β be a $\{0,1\}$-valued integral fractionary fuzzy ideal of R. Then the following statements are equivalent:*

(1) *β is a powerful fuzzy ideal of R.*
(2) *$(\frac{1}{x})_1 \circ \beta \subseteq \chi_R^{(0)}$ for each $x \in K \setminus R$.*
(3) *$(\frac{1}{x})_1 \circ \beta \subseteq \chi_R^{(0)}$ for each $x \in K \setminus \beta_*$.*

Proof. (1) ⇒ (2) Suppose that β is a powerful fuzzy ideal of R and let $x \in K \setminus R$. If $w \in R$, then $((\frac{1}{x})_1 \circ \beta)(w) \leq 1 = \chi_R^{(0)}(w)$. If $w \notin R$ and $\beta(xw) = 1$, then $< x_1 > \circ < w_1 > \subseteq \beta$. Since β is powerful, we have $< x_1 > \subseteq \chi_R^{(0)}$ or $< w_1 > \subseteq \chi_R^{(0)}$. Then $x \in R$ or $w \in R$, which is a contradiction. Thus $((\frac{1}{x})_1 \circ \beta)(w) = \beta(xw) = 0$. Hence $(\frac{1}{x})_1 \circ \beta \subseteq \chi_R^{(0)}$.

(2) ⇒ (3) This is trivial.

(3) $\Rightarrow$ (1) Suppose that $x, y \in K$ and $< x_1 > \circ < y_1 > \subseteq \beta$, but $< x_1 > \nsubseteq \chi_R^{(0)}$. Then $x \notin R$. By hypothesis, $(\frac{1}{x})_1 \circ \beta \subseteq \chi_R^{(0)}$. Thus $< y_1 > = < (\frac{1}{x})_1 > \circ < x_1 > \circ < y_1 > \subseteq (\frac{1}{x})_1 \circ \beta \subseteq \chi_R^{(0)}$. Hence β is a powerful fuzzy ideal of R. □

Corollary 3.9. *If β is a $\{0,1\}$-valued powerful integral fractionary fuzzy ideal of R, then β^3 is a powerful fuzzy ideal of R.*

Proof. Let $x \in K \setminus R$. Since β is a powerful fuzzy ideal of R, $(\frac{1}{x})_1 \circ \beta \subseteq \chi_R^{(0)}$. Then $(\frac{1}{x})_1 \circ \beta^3 \subseteq \chi_R^{(0)} \circ \beta^2 \subseteq \chi_R^{(0)}$. In fact, if $w \in K \setminus R$ and $(\chi_R^{(0)} \circ \beta^2)(w) = 1$, then there exist $u, v \in K$ such that $w = uv$ and $\chi_R^{(0)}(u) = 1, \beta^2(v) = 1$. Also, since $\beta^2(v) = 1$, there exist $a, b \in K$ such that $v = ab$ and $\beta(a) = 1 = \beta(b) = 1$. Hence $u, a, b \in R$, and so $w = uv \in R$, which is a contradiction. Thus $(\chi_R^{(0)} \circ \beta^2)(w) = 0$ for each $w \in K \setminus R$. Therefore β^3 is a powerful fuzzy ideal of R. □

Proposition 3.10. *Let β be a $\{0,1\}$-valued powerful integral fractionary fuzzy ideal of R. Then the following statements are equivalent:*

(1) *β is a powerful fuzzy ideal of R.*
(2) *If $x, y \in K$ and $< x_1 > \circ < y_1 > \subseteq \beta$, then $< x_1 > \subseteq \beta$ or $< y_1 > \subseteq \chi_R^{(0)}$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $x, y \in K$ and $< x_1 > \circ < y_1 > \subseteq \beta$. If $< x_1 > \nsubseteq \beta$, then $< (\frac{1}{x})_1 > \circ \beta \subseteq \chi_R^{(0)}$ by Proposition 3.8. Hence $< y_1 > = < (\frac{1}{x})_1 > \circ < (xy)_1 > \subseteq < (\frac{1}{x})_1 > \circ \beta \subseteq \chi_R^{(0)}$.

(2) $\Rightarrow$ (1) Let $x \in K \setminus \beta_*$ and let $w \in K \setminus R$. If $((\frac{1}{x})_1 \circ \beta)(w) = \beta(xw) = 1$, then $xw \in \beta_*$, and so $< x_1 > \circ < w_1 > \subseteq \beta$. By hypothesis $< x_1 > \subseteq \beta$ or $< w_1 > \subseteq \chi_R^{(0)}$. Since $x \notin \beta_*$, we have $< w_1 > \subseteq \chi_R^{(0)}$. Then $w \in R$, which is a contradiction. Thus $((\frac{1}{x})_1 \circ \beta)(w) = \beta(xw) = 0$. Hence $(\frac{1}{x})_1 \circ \beta \subseteq \chi_R^{(0)}$. Therefore β is a powerful fuzzy ideal of R. □

Proposition 3.11. *The finite sum of a $\{0,1\}$-valued powerful integral fractionary fuzzy ideals of R is a $\{0,1\}$-valued powerful fuzzy ideal of R.*

Proof. Let $\beta_1, \cdots, \beta_n$ be $\{0,1\}$-valued powerful integral fractionary fuzzy ideals of R and let $x \in K \setminus R$. Then $(\frac{1}{x})_1 \circ \beta_i \subseteq \chi_R^{(0)}$, where $i = 1, \cdots, n$. Thus $(\frac{1}{x})_1 \circ (\beta_1 + \cdots + \beta_n) = (\frac{1}{x})_1 \circ \beta_1 + \cdots + (\frac{1}{x})_1 \circ \beta_n \subseteq \chi_R^{(0)} + \cdots + \chi_R^{(0)} = \chi_R^{(0)}$. Hence $\beta_1 + \cdots + \beta_n$ is a powerful fuzzy ideal of R. □

Proposition 3.12. *An integral domain R is a pseudo-valuation domain if and only if some $\{0,1\}$-valued maximal integral fractionary fuzzy ideal of R is a powerful fuzzy ideal of R.*

Proof. Suppose that (R, M) is a pseudo-valuation domain. Then every $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R is a powerful fuzzy ideal of R by Theorem 2.5 and Corollary 3.4. Thus $\chi_M^{(0)}$ is a powerful fuzzy ideal of R. Conversely, let β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R and α be a $\{0,1\}$-valued maximal integral fractionary fuzzy ideal of R with α powerful. Then $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$ by Corollary 3.6. From the fact that α is a maximal and powerful fuzzy ideal, it follows that β is a strongly prime fuzzy ideal of R. Thus R is a pseudo-valuation domain. $\square$

Proposition 3.13. *If β is a $\{0,1\}$-valued powerful integral fractionary fuzzy ideal of R with $\beta_* \neq R$, then $\mu = \bigcap\{ \beta^k \mid k = 1, 2, \cdots \}$ is a strongly prime fuzzy ideal of R.*

Proof. Since β is finite-valued, we have $\mu_* = (\bigcap\{ \beta^k \mid k = 1, 2, \cdots \})_* = \bigcap\{ (\beta^k)_* \mid k = 1, 2, \cdots \} = \bigcap\{ (\beta_*)^k \mid k = 1, 2, \cdots \}$. Since β_* is a proper powerful ideal of R, μ_* is a strongly prime ideal of R. By Theorem 2.3, $\mu = \cap \beta^k$ is a strongly prime fuzzy ideal of R. $\square$

Proposition 3.14. *Let β be a $\{0,1\}$-valued powerful integral fractionary fuzzy ideal of R.*

(1) *If $x, y \in K$ and $< x_1 > \circ < y_1 > \subseteq \sqrt{\beta}$, then $< (x^n)_1 > \subseteq \beta$ or $< (y^n)_1 > \subseteq \beta$ for some $n \in \mathbb{N}$.*
(2) *If $x \in K$ and $< (x^n)_1 > \subseteq \beta$ for some $n \in \mathbb{N}$, then $< (x^{n+k})_1 > \subseteq \chi_R^{(0)}$ for each $k > 0$.*

Proof. (1) Since $xy \in (\sqrt{\beta})_* = \sqrt{\beta_*}$ and β_* is a powerful ideal of R, by [2, Proposition 1.9] either $x^n \in \beta_*$ or $y^n \in \beta_*$ for some $n \in \mathbb{N}$. Thus $< (x^n)_1 > \subseteq \beta$ or $< (y^n)_1 > \subseteq \beta$.
(2) Straightforward. $\square$

Recall from [2] that an integral domain R is said to be *seminormal* if $x \in K$ and $x^2, x^3 \in R$ imply $x \in R$.

Proposition 3.15. *Let R be a seminormal integral domain. If β is a $\{0,1\}$-valued powerful integral fractionary fuzzy ideal of R, then $\sqrt{\beta}$ is a powerful fuzzy ideal of R.*

Proof. Suppose that $x, y \in K$ and $< x_1 > \circ < y_1 > \subseteq \sqrt{\beta}$. By Proposition 3.14, $< (x^n)_1 > \subseteq \beta$ or $< (y^n)_1 > \subseteq \beta$ for some $n \in \mathbb{N}$.

Then $<(x^{n+k})_1> \subseteq \chi_R^{(0)}$ or $<(y^{n+k})_1> \subseteq \chi_R^{(0)}$ for each $k > 0$. Hence $x^{n+k} \in R$ or $y^{n+k} \in R$ for each $k > 0$. Since R is seminormal, we have $x \in R$ or $y \in R$. Then $< x_1 > \subseteq \chi_R^{(0)}$ or $< y_1 > \subseteq \chi_R^{(0)}$. Thus $\sqrt{\beta}$ is a powerful fuzzy ideal of R. □

Proposition 3.16. *Let β be a powerful integral fractionary fuzzy ideal of R. If $\alpha =< (x_1)_1, \cdots, (x_n)_1 >, x_1, \cdots, x_n \in R$ is a finitely generated prime integral fractionary fuzzy ideal of R with $\alpha \subseteq \beta$, then α is a maximal fractionary fuzzy ideal of R. Hence R is a pseudo-valuation domain.*

Proof. Since β is powerful, α is a $\{0, 1\}$-valued powerful fuzzy ideal of R. Since α_* is a finitely generated prime ideal of R and β_* is a powerful ideal of R, α_* is a maximal ideal of R by [2, Proposition 1.14]. Then α is a maximal integral fractionary fuzzy ideal of R. Hence R is a pseudo-valuation domain by Proposition 3.12. □

Lemma 3.17. *Let R be a valuation domain. If β is a $\{0, 1\}$-valued integral fractionary fuzzy ideal of R with $\beta_* \neq R$, then β is a powerful fuzzy ideal of R.*

Proof. Let $x \in K \setminus R$. Then $x^{-1} \in R$, and so $((\frac{1}{x})_1 \circ \beta)_* = x^{-1}\beta_* \subseteq \beta_* \subseteq R = (\chi_R^{(0)})_*$. Hence $(\frac{1}{x})_1 \circ \beta \subseteq \chi_R^{(0)}$. Thus β is a powerful fuzzy ideal of R. □

Theorem 3.18. *The following statements are equivalent:*

(1) *R is a valuation domain.*
(2) *Every $\{0, 1\}$-valued integral fractionary fuzzy ideal β of R with $\beta_* \neq R$ is a powerful fuzzy ideal of R.*

Proof. (1) $\Rightarrow$ (2) This follows from Lemma 3.17.

(2) $\Rightarrow$ (1) Let I be any proper ideal of R and let $\beta = \chi_I$. Then $\beta_* = (\chi_I)_* = I \neq R$. By hypothesis, β is a powerful fuzzy ideal of R. By Proposition 3.8, $(\frac{1}{x})_1 \circ \beta \subseteq \chi_R^{(0)}$ for each $x \in K \setminus \beta_*$. Then $x^{-1}\beta_* \subseteq R$, i.e., $x^{-1}I \subseteq R$. Hence R is a valuation domain by [11, Theorem 37]. □

We recall from [11] that an ideal I of R is called *divided* if $x^{-1}I \subseteq R$ for each $x \in R \setminus I$ and that an ideal I of R is said to be *super divided* if $x^{-1}I \subseteq R$ for each $x \in K \setminus I$.

Definition 3.19. An integral fractionary fuzzy ideal β of R is called a *divided fuzzy ideal of* R if $(\frac{1}{x})_1 \circ \beta \subseteq \beta$ for each $x \in K \setminus \beta_*$.

PROPOSITION 3.20. *Let β be a $\{0,1\}$-valued integral fractionary fuzzy ideal of R. Then β is a divided fuzzy ideal of R if and only if β_* is a super divided ideal of R.*

Proof. Let $x \in K \setminus \beta_*$. Since β is divided, we have $(\frac{1}{x})_1 \circ \beta \subseteq \beta$. Then $x^{-1}\beta_* \subseteq \beta_* \subseteq R$. Thus β_* is a super divided ideal of R. Conversely, let $x \in K \setminus \beta_*$ and $w \in K$. If $\beta(w) = 0$ and $((\frac{1}{x})_1 \circ \beta)(w) = \beta(xw) = 1$, then $xw \in \beta_*$, and so $w \in x^{-1}\beta_* \subseteq \beta_*$. Thus $\beta(w) = 1$, which is a contradiction. Hence $(\frac{1}{x})_1 \circ \beta \subseteq \beta$. Therefore β is a divided fuzzy ideal of R. □

PROPOSITION 3.21. *Let β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R. Then β is a divided fuzzy ideal of R if and only if β is a strongly prime fuzzy ideal of R.*

Proof. Let $x \in K \setminus R$. Then $x \in K \setminus \beta_*$. Since β is a divided fuzzy ideal of R, we have $(\frac{1}{x})_1 \circ \beta \subseteq \beta$. Thus β is a strongly prime fuzzy ideal of R by Proposition 2.7. Conversely, let $x \in K \setminus \beta_*$. Then $x \in R$ or $x \notin R$. If $x \notin R$, then $(\frac{1}{x})_1 \circ \beta \subseteq \beta$ since β is a strongly prime fuzzy ideal of R. Now suppose that $x \in R$ and $w \in K$. If $\beta(w) = 0$ and $((\frac{1}{x})_1 \circ \beta)(w) = \beta(xw) = 1$, then $xw \in \beta_*$. Since β_* is a strongly prime ideal of R, we have $x \in \beta_*$ or $w \in \beta_*$. Then $\beta(x) = 1$ or $\beta(w) = 1$, which is a contradiction. Thus $(\frac{1}{x})_1 \circ \beta \subseteq \beta$. Hence β is a divided fuzzy ideal of R. □

PROPOSITION 3.22. *The finite sum of $\{0,1\}$-valued divided integral fractionary fuzzy ideals of R is a $\{0,1\}$-valued divided fuzzy ideal of R.*

Proof. Let $\beta_1, \cdots, \beta_n$ be $\{0,1\}$-valued divided integral fractionary fuzzy ideals of R and let $x \in K \setminus (\beta_1 + \cdots + \beta_n)_*$. Then $x \notin (\beta_i)_*$ for each $i = 1, \cdots, n$. Since each β_i is divided, we have $(\frac{1}{x})_1 \circ \beta_i \subseteq \beta_i$, where $i = 1, \cdots, n$. Thus $(\frac{1}{x})_1 \circ (\beta_1 + \cdots + \beta_n) = (\frac{1}{x})_1 \circ \beta_1 + \cdots + (\frac{1}{x})_1 \circ \beta_n \subseteq \beta_1 + \cdots + \beta_n$. Hence $\beta_1 + \cdots + \beta_n$ is a divided fuzzy ideal of R. □

4. Strongly primary fuzzy ideals

We recall from [10] that a fuzzy ideal β of R is called *primary* if β is nonconstant and for any fuzzy ideals μ, ν of R, $\mu \circ \nu \subseteq \beta$ implies that $\mu \subseteq \beta$ or $\nu \subseteq \sqrt{\beta}$. A fuzzy ideal β of R is a primary if and only if $\beta(0) = 1$, β_* is a primary ideal of R and $|Im(\beta)| = 2$.

We recall from [2] that an ideal I of R is called a *strongly primary ideal of* R if $x, y \in K$ and $xy \in I$ imply that $x \in I$ or $y^n \in I$ for some

$n \in \mathbb{N}$ and an integral domain R is called an *almost pseudo-valuation domain* if every prime ideal of R is strongly primary.

In this section, we introduce the concept of strongly primary fuzzy ideals and characterize almost pseudo-valuation domains using this concept.

DEFINITION 4.1. Let β be an integral fractionary fuzzy ideal of R. Then β is called a *strongly primary fuzzy ideal of* R if for any fractionary fuzzy ideals μ, ν of R, $\mu \circ \nu \subseteq \beta$ implies that $\mu \subseteq \beta$ or $\nu \subseteq \sqrt{\beta}$.

It is clear that a strongly prime fuzzy ideal of R is a strongly primary fuzzy ideal of R.

THEOREM 4.2. *Let β be a $\{0,1\}$-valued integral fractionary fuzzy ideal of R. Then the following statements are equivalent:*

(1) *β is a strongly primary fuzzy ideal of R.*
(2) *β_* is a strongly primary ideal of R.*

Proof. (1) $\Rightarrow$ (2) Suppose that β is a strongly primary fuzzy ideal of R and let $x, y \in K$ and $xy \in \beta_*$. Then $< x_1 > \circ < y_1 > \subseteq \beta$. Since β is a strongly primary fuzzy ideal of R, $< x_1 > \subseteq \beta$ or $< y_1 > \subseteq \sqrt{\beta}$. Then $x \in \beta_*$ or $y \in (\sqrt{\beta})_* = \sqrt{\beta_*}$. Thus β_* is a strongly primary ideal of R.

(2) $\Rightarrow$ (1) Suppose that there exist fractionary fuzzy ideals μ, ν of R such that $\mu \circ \nu \subseteq \beta$, but $\mu \nsubseteq \beta$ and $\nu \nsubseteq \sqrt{\beta}$. Then there exist $x, y \in K$ such that $\mu(x) > \beta(x)$ and $\nu(y) > (\sqrt{\beta})(y)$. Since β is $\{0,1\}$-valued, we have $\beta(x) = 0$ and $(\sqrt{\beta})(y) = 0$. Then $x \notin \beta_*$ and $y \notin (\sqrt{\beta})_* = \sqrt{\beta_*}$. By hypothesis, $xy \notin \beta_*$, and so $\beta(xy) = 0$. But $0 = \beta(xy) \geq (\mu \circ \nu)(xy) \geq \mu(x) \wedge \nu(y) > 0$, which is a contradiction. Thus $\mu \subseteq \beta$ or $\nu \subseteq \sqrt{\beta}$. Therefore β is a strongly primary fuzzy ideal of R. □

COROLLARY 4.3. *Let β be a $\{0,1\}$-valued integral fractionary fuzzy ideal of R. If β is a strongly primary fuzzy ideal of R, then $\beta|_R$ is a primary fuzzy ideal of R.*

Proof. By Theorem 4.2, $\beta_* = (\beta|_R)_*$ is a strongly primary ideal of R. Then $\beta|_R$ is a primary fuzzy ideal of R by [10, Theorem 3.7.10]. □

COROLLARY 4.4. *Let R be a valuation domain and let β be a $\{0,1\}$-valued integral fractionary fuzzy ideal of R. If $\beta|_R$ is a primary fuzzy ideal of R, then β is a strongly primary fuzzy ideal of R.*

Proof. By [10, Theorem 3.7.10], $\beta_* = (\beta|_R)_*$ is a primary ideal of R. Since R is a valuation domain, β_* is a strongly primary ideal of R. Then β is a strongly primary fuzzy ideal of R by Theorem 4.2. □

Corollary 4.5. *Let R be a seminormal domain. If β is a $\{0,1\}$-valued strongly primary integral fractionary fuzzy ideal of R with $\beta_* \neq R$, then β is a powerful fuzzy ideal of R and $\sqrt{\beta}$ is a strongly prime fuzzy ideal of R. In particular, a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R is strongly prime if and only if it is strongly primary.*

Proof. Since β_* is a strongly primary ideal of R, β_* is a powerful ideal of R. Then β is a powerful fuzzy ideal of R by Proposition 3.2. It follows from Corollary 3.7 and Proposition 3.15 that $\sqrt{\beta}$ is a prime and powerful fuzzy ideal of R. Hence $\sqrt{\beta}$ is a strongly prime fuzzy ideal of R by Corollary 3.4. □

Proposition 4.6. *If β is a $\{0,1\}$-valued strongly primary integral fractionary fuzzy ideal of R, then $\mu = \bigcap_{n=1}^{\infty} \beta^n$ is a strongly prime fuzzy ideal of R.*

Proof. Since β_* is a strongly primary ideal of R, $\mu_* = (\bigcap_{n=1}^{\infty} \beta^n)_* = \bigcap_{n=1}^{\infty}(\beta^n)_* = \bigcap_{n=1}^{\infty}(\beta_*)^n$ is a strongly prime ideal of R. Thus $\mu = \bigcap_{n=1}^{\infty} \beta^n$ is a strongly prime fuzzy ideal of R by Theorem 2.3. □

We recall that for a subset S of R, $E(S) = \{x \in K \mid x^n \notin S$ for each $n \geq 1\}$. The following result is the fuzzification of [2, Lemma 2.3].

Proposition 4.7. *Let β be a $\{0,1\}$-valued integral fractionary fuzzy ideal of R. Then β is a strongly primary fuzzy ideal of R if and only if $(\frac{1}{x})_1 \circ \beta \subseteq \beta$ for each $x \in E(\beta_*)$.*

Proof. Let $x \in E(\beta_*)$. Since β_* is a strongly primary fuzzy ideal of R , $x^{-1}\beta_* \subseteq \beta_*$ by [2, Lemma 2.3]. Then $((\frac{1}{x})_1 \circ \beta)_* \subseteq \beta_*$. Thus $(\frac{1}{x})_1 \circ \beta \subseteq \beta$. Conversely, let $x, y \in K$ and $xy \in \beta_*$, $y \in E(\beta_*)$. Then $y \notin \beta_*$ and $(\frac{1}{y})_1 \circ \beta \subseteq \beta$. Thus $< x_1 >=< (\frac{1}{y})_1 > \circ < (xy)_1 >\subseteq (\frac{1}{y})_1 \circ \beta \subseteq \beta$. Hence $x \in \beta_*$, and so β_* is a strongly primary ideal of R. Therefore β is a strongly primary fuzzy ideal of R. □

Corollary 4.8. *Let β be a $\{0,1\}$-valued integral fractionary fuzzy ideal of R. If β is a strongly primary fuzzy ideal of R, then $(\frac{1}{x})_1 \circ \beta \subseteq \chi_R^{(0)}$ for each $x \in K \setminus \sqrt{\beta_*}$.*

Proof. Let $x \in K \setminus \sqrt{\beta_*}$. Then $x \in E(\beta_*)$. By Proposition 4.7, we have $(\frac{1}{x})_1 \circ \beta \subseteq \beta$. Then $\beta_* \subseteq x\beta_* \subseteq xR = (x_1 \circ \chi_R^{(0)})_*$. Hence $\beta \subseteq x_1 \circ \chi_R^{(0)}$, and so $(\frac{1}{x})_1 \circ \beta \subseteq \chi_R^{(0)}$. □

The power of a fuzzy ideal β can be defined recursively as follows: $\beta^1 = \beta$ and $\beta^n = \beta^1 \circ \beta^{n-1}$ for all $n \geq 2$.

PROPOSITION 4.9. *Let β be a $\{0,1\}$-valued strongly primary integral fractionary fuzzy ideal of R. If α is a $\{0,1\}$-valued integral fractionary fuzzy ideal of R with $\sqrt{\alpha_*} = \beta_*$, then $\alpha \circ \beta$ is a strongly primary fuzzy ideal of R. In particular, β^n is a strongly primary fuzzy ideal of R for $n \geq 1$.*

Proof. Let $x \in E((\alpha \circ \beta)_*) = E(\alpha_* \beta_*)$. Since $\sqrt{\alpha_* \beta_*} = \sqrt{\alpha_*} \cap \sqrt{\beta_*} = \beta_* \cap \sqrt{\beta_*} = \beta_*$, we have $x^n \notin \beta_*$ for all $n \geq 1$. Then $x \in E(\beta_*)$. Since β is a strongly primary fuzzy ideal of R, $(\frac{1}{x})_1 \circ \beta \subseteq \beta$. Hence $(\frac{1}{x})_1 \circ \alpha \circ \beta \subseteq \alpha \circ \beta$. Thus $\alpha \circ \beta$ is a strongly primary fuzzy ideal of R. $\square$

THEOREM 4.10. *The following statements are equivalent for an integral domain R:*

(1) *R is an almost pseudo-valuation domain.*
(2) *Every $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R is a strongly primary fuzzy ideal of R.*

Proof. (1) $\Rightarrow$ (2) Let β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R. Then $\beta_* = (\beta|_R)_*$ is a prime ideal of R. Since R is an almost pseudo-valuation domain, we have β_* is a strongly primary ideal of R. Hence β is a strongly primary fuzzy ideal of R by Theorem 4.2.

(2)$\Rightarrow$ (1) Let P be a prime ideal of R and assume that $x, y \in K$ and $xy \in P$. Then $< x_1 > \circ < y_1 > \subseteq \chi_P^{(0)}$. By hypothesis, $< x_1 > \subseteq \chi_P^{(0)}$ or $< y_1 > \subseteq \sqrt{\chi_P^{(0)}}$. Thus $x \in P$ or $y \in (\sqrt{\chi_P^{(0)}})_* = \sqrt{(\chi_P^{(0)})_*} = \sqrt{P} = P$. Hence P is a strongly primary ideal of R. Therefore R is an almost pseudo-valuation domain. $\square$

PROPOSITION 4.11. *Let R be an almost pseudo-valuation domain. If β is a $\{0,1\}$-valued non-maximal prime integral fractionary fuzzy ideal of R, then β is a strongly prime fuzzy ideal of R.*

Proof. Since $\beta_* = (\beta|_R)_*$ is a non-maximal prime ideal of R, β_* is a strongly prime ideal of R. Hence β is a strongly prime fuzzy ideal of R by Theorem 2.3. $\square$

5. Pseudo-strongly prime fuzzy ideals

We recall from [1] that a prime ideal P of R is called a *pseudo-strongly prime ideal* if whenever $x,\ y \in K$ and $xyP \subseteq P$, there exist $m \in \mathbb{N}$ such that $x^m \in R$ or $y^m P \subseteq P$ and an integral domain R is called a *pseudo-almost valuation domain* if every prime ideal is a pseudo-strongly prime ideal.

In this section, we introduce the concept of pseudo-strongly prime fuzzy ideals and characterize pseudo-almost valuation domain using this concept.

DEFINITION 5.1. Let β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R. Then β is called a *pseudo-strongly prime fuzzy ideal* of R if $x,\ y \in K$ and $(xy)_1 \circ \beta \subseteq \beta$ imply that there exists $m \in \mathbb{N}$ such that $(x^m)_1 \subseteq \chi_R^{(0)}$ or $(y^m)_1 \circ \beta \subseteq \beta$.

PROPOSITION 5.2. *Let β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R. Then β is a pseudo-strongly prime fuzzy ideal of R if and only if β_* is a pseudo-strongly prime ideal of R.*

Proof. Suppose that β is a pseudo-strongly prime fuzzy ideal of R and that $x, y \in K$ and $xy\beta_* \subseteq \beta_*$. Then there exist $m \in \mathbb{N}$ such that $(x^m)_1 \subseteq \chi_R^{(0)}$ or $(y^m)_1 \circ \beta \subseteq \beta$. Hence $x^m \in R$ or $y^m \beta_* \subseteq \beta_*$. Thus β_* is a pseudo-strongly prime ideal of R. Conversely, assume that β_* is a pseudo-strongly prime ideal of R and that $x,\ y \in K$ and $(xy)_1 \circ \beta \subseteq \beta$. Then $xy\beta_* \subseteq \beta_*$. By hypothesis, there exists $m \in \mathbb{N}$ such that $x^m \in R$ or $y^m P \subseteq P$. Hence $< (x^m)_1 > \circ \chi_R^{(0)}$ or $< (y^m)_1 > \circ \beta \subseteq \beta$. Thus β is a pseudo-strongly prime fuzzy ideal of R. $\square$

THEOREM 5.3. *The following statements are equivalent for an integral domain R:*

(1) *R is a pseudo-almost valuation domain.*
(2) *Every $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R is a pseudo-strongly prime fuzzy ideal of R.*

Proof. (1) $\Rightarrow$ (2) Let β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R. Then $\beta_* = (\beta|_R)_*$ is a prime ideal of R. Since R is a pseudo-almost valuation domain, we have β_* is a pseudo-strongly prime ideal of R. Hence β is a pseudo-strongly prime fuzzy ideal of R by Proposition 5.2.

(2)$\Rightarrow$ (1) Let P be a prime ideal of R. Since $\chi_P^{(0)}$ is a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R, $\chi_P^{(0)}$ is a pseudo-strongly prime fuzzy ideal of R. Then P is a pseudo-strongly prime ideal of R by Proposition 5.2. Hence R is a pseudo-almost valuation domain. $\square$

The following result is the fuzzification of [1, Lemma 2.1].

Proposition 5.4. *Let β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R. Then β is a pseudo-strongly prime fuzzy ideal of R if and only if for each $x \in E(R)$, $(x^{-n})_1 \circ \beta \subseteq \beta$ for some $n \in \mathbb{N}$.*

Proof. Suppose that β is a pseudo-strongly prime fuzzy ideal of R. Let $x \in E(R)$. Since $x_1 \circ (\frac{1}{x})_1 \circ \beta = (1)_1 \circ \beta \subseteq \beta$, we have $< (x^n)_1 > \subseteq \chi_R^{(0)}$ or $(x^{-n})_1 \circ \beta \subseteq \beta$ for some $n \in \mathbb{N}$. Since $x \in E(R)$, we have $(x^{-n})_1 \circ \beta \subseteq \beta$. Conversely, assume that $x,\ y \in K$ and $(xy)_1 \circ \beta \subseteq \beta$. If $x \in E(R)$, then by hypothesis, $(x^{-n})_1 \circ \beta \subseteq \beta$ for some $n \in \mathbb{N}$ and $(x^n)_1 \nsubseteq \chi_R^{(0)}$. Since $(x^n y^n)_1 \circ \beta \subseteq (xy)_1 \circ \beta \subseteq \beta$, we have $(y^n)_1 \circ \beta = (x^{-n})_1 \circ (x^n y^n)_1 \circ \beta \subseteq (x^{-n})_1 \circ \beta \subseteq \beta$. Thus β is a pseudo-strongly prime fuzzy ideal of R. □

Proposition 5.5. *If R is a pseudo-almost valuation domain, then every $\{0,1\}$-valued prime integral fractionary fuzzy ideals of R are linearly ordered.*

Proof. Let $\alpha,\ \beta$ be a $\{0,1\}$-valued prime integral fractionary fuzzy ideals of R and assume that $\alpha \nsubseteq \beta$ and $\beta \nsubseteq \alpha$. Then there exist $x,\ y \in K$ such that $\alpha(x) \gneq \beta(x)$ and $\beta(y) \gneq \alpha(y)$. Since α and β are $\{0,1\}$-valued, we have $\alpha(x) = 1$, $\beta(x) = 0$, $\alpha(y) = 0$, and $\beta(y) = 1$. Set $w = \frac{y}{x}$. If $w \notin E(R)$, then $w^n = \frac{y^n}{x^n} \in R$ for some $n \in \mathbb{N}$. Then $y^n \in x^n R \subseteq \alpha_*$. Hence $y \in \alpha_*$, which is a contradiction. So we conclude that $w \in E(R)$. Since R is a pseudo-almost valuation domain, α is a pseudo-strongly prime fuzzy ideal of R. By Proposition 5.4, $(w^{-n})_1 \circ \alpha \subseteq \alpha$ for some $n \in \mathbb{N}$. Then $\alpha(\frac{x^{n+1}}{y^n}) \geq ((\frac{x^n}{y^n})_1 \circ \alpha)(\frac{x^{n+1}}{y^n}) \geq \alpha(x) = 1$, and so $\alpha(\frac{x^{n+1}}{y^n}) = 1$. Then $\frac{x^{n+1}}{y^n} \in \alpha_*$ and $x^{n+1} \in \beta_*$. Since β_* is a prime ideal of R, we have $x \in \beta_*$. This is a contradiction. Therefore either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. □

Theorem 5.6. *An integral domain R is a pseudo-almost valuation domain if and only if some $\{0,1\}$-valued maximal integral fractionary fuzzy ideal is a pseudo-strongly prime fuzzy ideal of R.*

Proof. Suppose that some $\{0,1\}$-valued maximal integral fractionary fuzzy ideal α is a pseudo-strongly prime fuzzy ideal of R. Then α_* is a maximal pseudo-strongly prime ideal of R by Proposition 5.2. That R is a pseudo-almost valuation domain by [1, Theorem 2.5]. The converse is clear. □

In light of Theorem 5.6, we have the following corollary.

Corollary 5.7. *Let α be a pseudo-strongly prime fuzzy ideal of R. If β is a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R and $\beta \subseteq \alpha$, then β is a pseudo-strongly prime fuzzy ideal of R.*

Proof. Let α be a pseudo-strongly prime fuzzy ideal of R. Let β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideal of R such that $\beta \subseteq \alpha$. Let $x \in E(R)$. Since α is a pseudo-strongly prime fuzzy ideal of R, we have $(x^{-n})_1 \circ \alpha \subseteq \alpha$ for some $n \geq 1$. By Proposition 5.4, we need only show that $(x^{-l})_1 \circ \beta \subseteq \beta$ for some $l \geq 1$. Suppose that $(x^{-s})_1 \circ \beta \nsubseteq \beta$ for each $s \geq 1$. Then there exist $y \in K$ such that $((x^{-n})_1 \circ \beta)(y) > \beta(y)$. Hence $\beta(x^n y) = 1$ and $\beta(y) = 0$. Since $1 = ((x^{-n})_1 \circ \beta)(y) \leq ((x^{-n})_1 \circ \alpha)(y) \leq \alpha(y)$, we have $\alpha(y) = \alpha(x^n y) = 1$. Thus $x^n y \in \beta_* \subseteq \alpha_*$ and $y \in \alpha_* \setminus \beta_*$. Since α_* is a pseudo-strongly prime ideal of R, $x^{mn} = \frac{x^{mn}y^m}{y^m} \in \beta_* \subseteq R$ for some $m \geq 1$ by [1, Proposition 2.4]. Hence $x \notin E(R)$, which is a contradiction. Thus β is a pseudo-strongly prime fuzzy ideal of R. $\square$

The following two results are well-known ([1, Proposition 2.11 and Theorem 2.13]). Here we will give their proofs in terms of fractional fuzzy ideals.

Proposition 5.8. *If R is an almost pseudo-valuation domain, then R is a pseudo-almost valuation domain.*

Proof. Let β be a $\{0,1\}$-valued prime integral fractionary fuzzy ideals of R and let $x \in E(R)$. Then $x \in E(\beta_*)$. Since R is an almost pseudo-valuation domain, β is a strongly primary fuzzy ideal of R. Hence $(\frac{1}{x})_1 \circ \beta \subseteq \beta$. Thus β is a pseudo-strongly prime fuzzy ideal of R. Therefore R is a pseudo-almost valuation domain. $\square$

We recall that an integral domain R is *root closed* if, whenever $x \in K$ and $x^n \in R$ for some $n \geq 1$, then $x \in R$.

Theorem 5.9. *If R is a root closed pseudo-almost valuation domain, then R is a pseudo-valuation domain.*

Proof. Let M be the maximal ideal of R and let $x \in K \setminus R$. Then $x \in E(R)$. Since R is a pseudo-almost valuation domain, $\chi_M^{(0)}$ is a pseudo-strongly prime fuzzy ideal of R. Then $\chi_M^{(0)} \subseteq (x^n)_1 \circ \chi_M^{(0)}$. Let $m \in M$. Then $1 = \chi_M^{(0)}(m^n) \leq ((x^n)_1 \circ \chi_M^{(0)})(m^n) = \chi_M^{(0)}(\frac{m^n}{x^n})$. Hence $\chi_M^{(0)}(\frac{m^n}{x^n}) = 1$, and so $(\frac{m}{x})^n \in M$. Since R is root closed, we have $\frac{m}{x} \in M$. Then $(x_1 \circ \chi_M^{(0)})(m) = \chi_M^{(0)}(\frac{m}{x}) = 1$. Hence $\chi_M^{(0)} \subseteq x_1 \circ \chi_M^{(0)}$. Thus $\chi_M^{(0)}$ is a strongly prime fuzzy ideal of R. Therefore R is a pseudo-valuation domain. $\square$

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