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A NOTE ON THE GENERALIZED VARIATIONAL INEQUALITY WITH OPERATOR SOLUTIONS

Sangho Kum*

Abstract. In a series of papers [3, 4, 5], the author developed the generalized vector variational inequality with operator solutions (in short, GOVVI) by exploiting variational inequalities with operator solutions (in short, OVVI) due to Domokos and Kolumbán [2]. In this note, we give an extension of the previous work [4] in the setting of Hausdorff locally convex spaces. To be more specific, we present an existence of solutions of (GVVI) under the weak pseudomonotonicity introduced in Yu and Yao [7] within the framework of (GOVVI).

1. Introduction

In a series of papers [3, 4, 5], the author developed the generalized vector variational inequality with operator solutions (in short, GOVVI) by exploiting variational inequalities with operator solutions (in short, OVVI) due to Domokos and Kolumbán [2]. They designed (OVVI) to provide a unified approach to several kinds of (VI) and (VVI) problems in Banach spaces, and successfully described those problems in a wider context of (OVVI). Actually, motivated by the work of Domokos and Kolumbán [2], in a former paper [3], the author proposed (GOVVI) which extends (OVVI) into a multi-valued case under a standard pseudomonotonicity of the given operator. In a recent work [4], a more general pseudomonotone operator was treated in a normed space. As a continuation of works, in this note, we give an extension of the previous result [4, Theorem 3.2] in the setting of Hausdorff locally convex space. To be more specific, we present an existence of solutions of (GVVI) under

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the weak pseudomonotonicity introduced in Yu and Yao [7] within the framework of (GOVVI).

2. Preliminaries

Let E, F be Hausdorff t.v.s., and let X be a nonempty convex subset of E. Let $C_1 : X \to F$ be a multifunction such that for each $x \in X, C_1(x)$ is a convex cone in F with $\text{int}\, C_1(x) \neq \emptyset$ and $C_1(x) \neq F$. Let $L(E, F)$ be the space of all continuous linear operators from E to F and $T_1 : X \to L(E, F)$ a multifunction. From now on, unless otherwise specified, we work under the following settings:

Let X' be a nonempty convex subset of $L(E, F)$ and $T : X' \to E$ be a multifunction. Let $C : X' \to F$ be a multifunction such that for each $f \in X'$, $C(f)$ is a convex cone in F with $0 \notin C(f)$. Then the generalized variational inequalities with operator solutions (GOVVI) is defined as follows:

$$\text{Find } f_0 \in X' \text{ such that } \forall f \in X', \exists x \in T(f_0) \text{ with } \langle f - f_0, x\rangle \notin C(f_0).$$

Consider the multifunction $T_1 : X \to L(E, F)$. Then T_1 is said to be
(1) *weakly C_1-pseudomonotone* if $\forall x, y \in X$ and $\forall s \in T_1(x)$, we have

$$\langle s, y-x\rangle \notin -\text{int}C_1(x) \text{ implies } \langle t, y-x\rangle \notin -\text{int}C_1(x) \text{ for some } t \in T_1(y);$$

(2) *generalized hemicontinuous* if for any $x, y \in X$, the multifunction

$$\alpha \mapsto \langle T_1(x + \alpha(y - x)), y - x\rangle, \quad \forall \alpha \in [0, 1]$$

is upper semicontinuous at 0^+, where

$$\langle T_1(x + \alpha(y - x)), y - x\rangle = \{\langle s, y - x\rangle \mid s \in T_1(x + \alpha(y - x))\}.$$

In regard to monotonicity and continuity of T, two analogous definitions to those of T_1 in the above are necessary; $T : X' \to E$ is said to be
$(1)'$ *weakly C-pseudomonotone* if for any $f, g \in X'$ and for any $s \in T(f)$,

$$\langle g - f, s\rangle \notin C(f) \text{ implies } \langle g - f, t\rangle \notin C(f) \text{ for some } t \in T(g); \text{ and}$$

$(2)'$ *generalized hemicontinuous* if for any $f, g \in X'$, the multifunction

$$\alpha \mapsto \langle g - f, T(f + \alpha(g - f))\rangle, \quad \forall \alpha \in [0, 1]$$

is upper semicontinuous at 0^+, where

$$\langle g - f, T(f + \alpha(g - f))\rangle = \{\langle g - f, s\rangle \mid s \in T(f + \alpha(g - f))\}.$$

Recall that a locally convex space (in short, l.c.s.) E is said to be *bornological* if every circled, convex subset $A \subset E$ which absorbs every bounded set in E is a neighborhood of 0. Equivalently, a bornological space is a l.c.s. on which each seminorm that is bounded on bounded sets, is continuous. Now, we introduce a fixed-point theorem [6], originally established in [1], which plays the role of a basic tool to derive our main result.

LEMMA 2.1. *Let X be a nonempty convex subset of a locally convex space E. Let S, $V : X \to X$ be two multifunctions. Suppose that*

(i) for each $x \in X$, $S(x) \neq \emptyset$;
(ii) for each $x \in X$, $\mathrm{co}S(x) \subset V(x)$ where $\mathrm{co}S(x)$ stands for the convex hull of $S(x)$;
(iii) $X = \bigcup\{\mathrm{int}_X S^{-1}(z) \mid z \in X\}$;
(iv) the image $V(X)$ of the map V is contained in a compact subset D of X.
Then V has a fixed point $x_0 \in X$; that is, $x_0 \in V(x_0)$.

3. Main result

We begin with the following lemma in [4, Lemma 3.1] without proof.

LEMMA 3.1. *Let $T : X' \to E$ be a weakly C-pseudomonotone and generalized hemicontinuous multifunction with $T(f) \neq \emptyset$ for all $f \in X'$. Let $W : X' \to F$ be defined by $W(f) = F \setminus C(f)$ such that the graph $Gr(W)$ of W is closed in $X' \times F$ where $L(E,F)$ is endowed with either the topology of pointwise convergence or the topology of bounded convergence. Then the following two problems are equivalent:*
(i) Find $f \in X'$ such that $\forall g \in X'$, $\exists x \in T(f)$ with $\langle g-f, x\rangle \notin C(f)$.
(ii) Find $f \in X'$ such that $\forall g \in X'$, $\exists x \in T(g)$ with $\langle g-f, x\rangle \notin C(f)$.

THEOREM 3.2. *Let X' be a nonempty convex subset of $L(E,F)$ endowed with the topology of bounded convergence. Let $T : X' \to E$ be a weakly C-pseudomonotone and generalized hemicontinuous multifunction such that $T(f)$ is nonempty and compact for all $f \in X'$. Let $W : X' \to F$ be defined by $W(f) = F \setminus C(f)$ such that the graph $Gr(W)$ of W is closed in $X' \times F$. Assume that there exists a compact subset D of X' satisfying*

$$\{g \in X' \mid \exists f \in X' \text{ such that } \forall x \in T(f),\ \langle g-f, x\rangle \in C(f)\} \subset D. \quad (1)$$

Then (GOVVI) is solvable.

Proof. First note that $L(E,F)$ equipped with the topology of bounded convergence is a locally convex space. We define two multifunctions $S,\ V: X' \to X'$ to be

$$\begin{aligned} S(f) : &= \{g \in X' \mid \forall x \in T(g),\ \langle g-f, x\rangle \in C(f)\}, \\ V(f) : &= \{g \in X' \mid \forall x \in T(f),\ \langle g-f, x\rangle \in C(f)\}. \end{aligned}$$

The proof is organized in the following parts.
(i) It is clear that for each $f \in X'$, $V(f)$ is convex.
(ii) Since T is weakly C-pseudomonotone, we have $S(f) \subset V(f)$. By (i), we have $\text{co}S(f) \subset V(f)$ for all $f \in X'$.
(iii) V has no fixed point because $0 \notin C(f)$ for all $f \in X'$.
(iv) For each $g \in X'$, $S^{-1}(g)$ is open in X'. In fact, let $\{f_\lambda\}$ be a net in $(S^{-1}(g))^c$ convergent to $f \in X'$. Then $g \notin S(f_\lambda)$ and hence for some $x_\lambda \in T(g)$,

$$\langle g - f_\lambda, x_\lambda\rangle \notin C(f_\lambda).$$

Thus $\langle g - f_\lambda, x_\lambda\rangle \in W(f_\lambda)$. As $T(g)$ is compact, we may assume that $x_\lambda \to x$ for some $x \in T(g)$. Since $L(E,F)$ is endowed with the topology of bounded convergence and $T(g)$ is compact, $\langle g - f_\lambda, x_\lambda\rangle \to \langle g-f, x\rangle$. By virtue of the closedness of $Gr(W)$, we have $(f, \langle g-f, x\rangle) \in Gr(W)$, that is, $\langle g-f, x\rangle \notin C(f)$ for the particular $x \in T(g)$. Hence $g \notin S(f)$, so $f \in (S^{-1}(g))^c$. This shows that $(S^{-1}(g))^c$ is closed, i.e., $S^{-1}(g)$ is open in X'. Thus $X' = \bigcup\{\text{int}_{X'} S^{-1}(g) \mid g \in X'\}$.
(v) By (1), we have $V(X') \subset D$.
(vi) From (i)-(v), we see, by Lemma 2.1, there must be an $f_0 \in X'$ such that $S(f_0) = \emptyset$, namely,

$$\forall g \in X',\ \exists x \in T(g) \text{ such that } \langle g - f_0, x\rangle \notin C(f_0).$$

It follows from Lemma 3.1 that f_0 is a solution of (GOVVI). This completes the proof. □

As a direct consequence of Theorem 3.2, the following generalized VVI in a locally convex space is derived, which is a generalization of the corresponding Theorem 3.2 in [4].

THEOREM 3.3. *Let Y be a bornological l.c.s. and let Z be a Hausdorff l.c.s. Let X be a nonempty convex subset of Y and $C_1 : X \to Z$ be a multifunction such that for each $x \in X$, $C_1(x)$ is a convex cone in Z with int$C_1(x) \neq \emptyset$ and $C_1(x) \neq Z$. Let $T_1 : X \to L(Y,Z)$ be a weakly C_1-pseudomonotone and generalized hemicontinuous multifunction with nonempty compact values where $L(Y,Z)$ is the Hausdorff l.c.s. equipped with the topology of bounded convergence. Let $W_1 : X \to Z$ be defined* *by* $W_1(x) = Z \setminus -intC_1(x)$ *such that the graph* $Gr(W_1)$ *of* W_1 *is closed in* $X \times Z$. *Assume that there exists a compact subset* D *of* X *satisfying*

$$\{x \in X \mid \exists y \in X \text{ such that } \forall t \in T_1(y),\ \langle t, x - y\rangle \in C_1(y)\} \subset D. \quad (2)$$

Then there exists $x_0 \in X$ *such that*

$$\forall x \in X,\ \exists t \in T_1(x_0) \text{ with } \langle t, x - x_0\rangle \notin -intC_1(x_0).$$

Proof. We consider $E = L(Y, Z)$ as the Hausdorff l.c.s. of the continuous linear operators between Y and Z equipped with the topology of bounded convergence, and $F = Z$. Define a mapping $\phi : Y \to L(E, F)$ by $\phi(x) = f_x$ where $f_x(l) = \langle l, x\rangle$ for all $l \in E$. This ϕ is linear and injective. Indeed, assume that $l_i \to l$ in E. This implies that $\forall x \in Y$, $\langle l_i, x\rangle \to \langle l, x\rangle$ in $F = Z$. Thus $f_x(l_i) \to f_x(l)$ in F, so $f_x \in L(E, F)$. The linearity of ϕ is obvious. To show the injectivity of ϕ, it suffices to check that for each nonzero $x \in Y$, there exists an $l \in E$ such that $\langle l, x\rangle \neq 0$. By the separation theorem, we can find a $g \in Y^*$ with $g(x) = 1$. Define a linear operator $l : Y \to Z$ by

$$\langle l, y\rangle = g(y)z_0 \text{ for some } z_0 \neq 0 \text{ in } Z.$$

Clearly $l \in L(Y, Z)$ and $\langle l, x\rangle = g(x)z_0 = z_0 \neq 0$. Now let $X' = \phi(X)$ and $D' = \phi(D)$. Suppose that $L(E, F)$ is equipped with the topology of bounded convergence. Then $\phi : Y \to \phi(Y)$ is a homeomorphism by the proof of Theorem 3.4 in [5].

Now we define $T : X' \to E$, $C : X' \to F$ and $W : X' \to F$ as follows:

$$T(f_x) = T_1(x),\ C(f_x) = -\text{int}C_1(x),\ W(f_x) = W_1(x).$$

Then $0 \notin C(f_x)$ because $\text{int}C_1(x)$ is a proper convex cone of Z. The proof is organized in the following parts.

(i) The weak C_1-pseudomonotonicity of T_1 implies the weak C-pseudomonotonicity of T. In fact, for any $f_x,\ f_y \in X'$ and $s \in T(f_x) = T_1(x)$,

$$\begin{aligned}\langle f_y - f_x, s\rangle \notin C(f_x) \quad &\Rightarrow \quad \langle s, y - x\rangle \notin -\text{int}C_1(x)\\ &\Rightarrow \quad \langle t, y - x\rangle \notin -\text{int}C_1(x) \ \text{ for some } t \in T_1(y)\\ &\Rightarrow \quad \langle f_y - f_x, t\rangle \notin C(f_x) \ \text{ for some } t \in T(f_y).\end{aligned}$$

(ii) The generalized hemicontinuity of T_1 amounts to that of T. Actually, for any $f_x,\ f_y \in X'$ and $\alpha \in [0, 1]$,

$$\alpha \mapsto \langle f_y - f_x, T(f_x + \alpha(f_y - f_x))\rangle = \langle T_1(x + \alpha(y - x)), y - x\rangle$$

is upper semicontinuous at 0^+.

(iii) By the hypothesis, $T(f_x) = T_1(x)$ is nonempty and compact.

(iv) The graph $Gr(W)$ of W is closed in $X' \times F$. Indeed, let $\{f_{x_i}\}$ be a sequence in X' convergent to $f_x \in X'$. Let $w_i \in W(f_{x_i}) = W_1(x_i)$ such that $w_i \to w$ in F. Since ϕ is a homeomorphism, $\phi^{-1}(f_{x_i}) = x_i \to x = \phi^{-1}(f_x)$. Because the graph $Gr(W_1)$ of W_1 is closed in $X \times Z$, we have $w \in W_1(x) = W(f_x)$. This implies that $Gr(W)$ is closed in $X' \times F$.
(v) By (2), we see that

$$\{f_x \in X' \mid \exists f_y \in X' \text{ s.t. } \forall t \in T(f_y),\ \langle f_x - f_y, t\rangle \in C(f_y)\} \subset D' = \phi(D).$$

It follows from Theorem 3.1 that there exists $f_{x_0} \in X'$ such that for each $f_x \in X'$, there is $t \in T(f_{x_0})$ with $\langle f_x - f_{x_0}, t\rangle \notin C(f_{x_0})$. Therefore, there exists $x_0 \in X$ such that

$$\forall x \in X,\ \exists t \in T_1(x_0) \text{ with } \langle t, x - x_0\rangle \notin -\text{int}C_1(x_0).$$

This completes the proof. □

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Department of Mathematics Education
Chungbuk National University
Cheongju 361-763, Republic of Korea
E-mail: shkum@cbnu.ac.kr