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POINCARÉ'S INEQUALITY ON A NEW FUNCTION SPACE $L_\alpha(X)$

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Abstract. We prove the homogeneous property of the norm of the new space $L_\alpha(X)$ which has been developed in [3]. We also present Poincaré's inequality that is fitted to the function space $L_\alpha(X)$ with an appropriate slope condition.

1. Introduction

A new function space has been developed in [3] which generalizes the classical Lebesgue space. The motivation of this research came from taking a close look at the L^p-norm: $\|f\|_{L^p} = \left(\int_X |f(x)|^p \, d\mu\right)^{1/p}$ of the classical Lebesgue spaces $L^p(X)$, $1 \le p < \infty$. It can be rewritten as

$$\|f\|_{L^p} := \alpha^{-1}\left(\int_X \alpha(|f(x)|)\, d\mu\right),$$

with

$$\alpha(x) := x^p.$$

Even though the positive real-variable function $\alpha(x) := x^p$ has very beautiful and convenient algebraic and geometric properties(Section 2.1 in [3]), it also has some practical limitations. One of such practical limitations which cannot be overlooked arises from some problems of partial differential equations which contain the *flux* term $\mathbf{J}$ such as the parabolic equations:

$$\frac{\partial}{\partial t}c(u) - \nabla \cdot \mathbf{J} = f.$$

The most common non-linear assumptions for the flux terms are $\mathbf{J} = |\nabla u|^{p-2}\nabla u$ which produce the p-Laplacian $\Delta_p\, u := \nabla \cdot |\nabla u|^{p-2}\nabla u$. A

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critical reason for this assumption is that the function spaces that have been used to deal with these problems are just the Lebesgue spaces $L^p(X)$. The new function space has been designed to handle solutions of nonlinear equations having more general flux term than p-Laplacian, and which could not otherwise be dealt with the well-known spaces[4].

This paper has two-folds: first, we discuss the homogeneity property of $\|\cdot\|_{L_\alpha}$. Indeed, in [3] it was shown the functional $\|\cdot\|_{L_\alpha}$ satisfies the triangle inequality(Minkowski's inequality) so that it can produce a metric space on $L_\alpha(X)$, but it was not known at that time the homogeneity of $\|\cdot\|_{L_\alpha}$: for $k \in \mathbb{R}$ and $f \in L_\alpha(X)$,

$$\|kf\|_{L_\alpha} = |k|\|f\|_{L_\alpha}.$$

In this paper, we prove the homogeneity of $\|\cdot\|_{L_\alpha}$. We introduce the dual space argument for it(Section 3). Second, we prove the L_α-version of Poincaré's inequality in Section 4 that is useful for the theory of partial differential equations and harmonic analysis.

Even though it may be criticized that the presenting space is similar to the classical Orlicz spaces, we emphasize that two concepts are basically different, and we present a new approach to discover new spaces which generalizes the space L^p.

2. Preliminary results on the space $L_\alpha(X)$

We briefly introduce some terminologies to review a new function space $L_\alpha(X)$ defined in [3]. In the following, $(X, \mathfrak{M}, \mu)$ always represents a given measurable space.

A *pre-Hölder's function* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$[1] is an *absolutely continuous bijective* function satisfying

$$[H1] \qquad \alpha(0) = 0, \quad \alpha(1) = 1.$$

If there exists a pre-Hölder's function β satisfying

$$[H2] \qquad \alpha^{-1}(x)\beta^{-1}(x) = x \qquad \text{for all } x \geq 0,$$

then β is called the *conjugate* (*pre-Hölder's*) *function* of α. In the relation [H2], the notation α^{-1} and β^{-1} are meant to be the inverse functions of α and β, respectively.

[1] $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$

Definition 2.1. *A pre-Hölder's function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ together with the conjugate function β is said to be a Hölder's function if for any positive constants $a, b > 0$, there exist positive constants θ_1, θ_2 (depending on a, b) such that*

$$\theta_1 + \theta_2 \leq 1$$

and that a comparable condition

$$[H3] \qquad \alpha^{-1}(x)\beta^{-1}(y) \leq \theta_1 \frac{ab}{\alpha(b)}\, x + \theta_2 \frac{ab}{\beta(a)}\, y$$

holds for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$.

We summarize some basic identities for a pre-Hölder's pair (α, β):

$$(1) \qquad x = \beta\left(\frac{x}{\alpha^{-1}(x)}\right) \quad \text{or} \quad \alpha(x) = \beta\left(\frac{\alpha(x)}{x}\right),$$

$$(2) \qquad x = \frac{\alpha(x)}{\beta^{-1}(\alpha(x))},$$

$$(3) \qquad \frac{\beta^{-1}(x)}{\alpha'(\alpha^{-1}(x))} + \frac{\alpha^{-1}(x)}{\beta'(\beta^{-1}(x))} = 1,$$

$$(4) \qquad \frac{y}{\alpha'(x)} + \frac{x}{\beta'(y)} = 1 \quad \text{for} \quad y := \frac{\alpha(x)}{x},$$

$$(5) \qquad \alpha'(x) = \frac{\alpha(x)}{x} + \frac{\alpha(x)}{\beta'\left(\frac{\alpha(x)}{x}\right) - x}.$$

We say that α obeys a *commutative condition* if

$$[\beta, \alpha](x) := \beta \circ \alpha \circ \beta^{-1} \circ \alpha^{-1}(x) = x \qquad [C]$$

for all $x \geq 0$. Then it can be readily checked that

$$(6) \qquad \beta \circ \alpha^{-1}\left(\frac{\alpha(a)}{a}\right) = a, \qquad a > 0,$$

and so it can be noted that

$$(7) \qquad \frac{\alpha(x)}{x} = \left(\frac{\beta(x)}{x}\right)^{-1} = \alpha \circ \beta^{-1}(x) = \beta^{-1} \circ \alpha(x),$$

$x > 0$, provided that the pre-Hölder's function α is commutative with its conjugate function β.

Hölder type inequality and Minkowski's inequality on the new space $L_\alpha(X)$ are presented as follows:

REMARK 2.1 (Hölder and Minkowski inequality). *Let α be a Hölder's function and β be the corresponding Hölder's conjugate function. Then for any $f \in L_\alpha(X)$ and any $g \in L_\beta(X)$, we have*

$$\left| \int_X f(x)g(x)\, d\mu \right| \leq \|f\|_{L_\alpha} \|g\|_{L_\beta},$$

and for any $f_1, f_2 \in L_\alpha(X)$, we have

$$\|f_1 + f_2\|_{L_\alpha} \leq \|f_1\|_{L_\alpha} + \|f_2\|_{L_\alpha}.$$

It was pointed out in [3] that $L_\alpha(X)$ is a *topological vector space* with *inhomogeneous* norm $\|\cdot\|_{L_\alpha}$. Furthermore, for $k \geq 0$

$$\lceil k^{-1} \rceil^{-1} \|f\|_{L_\alpha} \leq \|k\, f\|_{L_\alpha} \leq \lceil k \rceil\, \|f\|_{L_\alpha}, \tag{8}$$

where $\lceil k \rceil$ is the *ceiling* of k, the smallest integer that is not less than k.

REMARK 2.2. *The metric space $L_\alpha(X)$ is complete with respect to the metric:*

$$d(f, g) := \|f - g\|_{L_\alpha}, \quad \text{for } f, g \in L_\alpha(X).$$

3. Homogeneity of $\|\cdot\|_{L_\alpha}$: dual space argument

A natural question to arise on $\|\cdot\|_{L_\alpha}$ is that

" does the functional $\|\cdot\|_{L_\alpha}$ produce a *norm* on $L_\alpha(X)$? "

Unlike the classical Lebesgue spaces, this would not be a trivial question. In fact, it is not easy to verify the *homogeneity* required for norms:

$$\|kf\|_{L_\alpha} = |k|\; \|f\|_{L_\alpha}, \quad k \in \mathbb{R}, \;\; f \in L_\alpha(X).$$

We present a dual space argument in order to verify that the functional $\|\cdot\|_{L_\alpha}$ actually satisfies the homogeneity. To do that, we first note that by Hölder's inequality, each $g \in L_\beta(X)$ defines a bounded linear functional $\mathcal{F}_g$ on $L_\alpha(X)$ by

$$\mathcal{F}_g(f) := \int_X f(x)g(x)d\mu,$$

and the operator norm of $\mathcal{F}_g$ is at most $\|g\|_{L_\beta}$:

$$\|\mathcal{F}_g\|_{L_\alpha^*} := \sup \left\{ \frac{|\int_X fgd\mu|}{\|f\|_{L_\alpha}} : f \in L_\alpha(X), f \neq 0 \right\} \leq \|g\|_{L_\beta}. \tag{9}$$

Also, for $0 \neq g \in L_\beta(X)$, we put $f(x) := \frac{\beta(|g(x)|)\ \operatorname{sgn} g(x)\, \|g\|_{L_\beta}}{|g(x)|\beta(\|g\|_{L_\beta})}$ to have that $\|f\|_{L_\alpha} \leq 1$ (by (1), (8)) and

$$\|\mathcal{F}_g\|_{L_\alpha^*} = \sup \left\{ \frac{\left|\int_X fgd\mu\right|}{\|f\|_{L_\alpha}} : f \in L_\alpha(X), f \neq 0 \right\} \geq \frac{\left|\int_X fgd\mu\right|}{\|f\|_{L_\alpha}} = \|g\|_{L_\beta}.$$

This implies that the mapping $g \mapsto \mathcal{F}_g$ is an isometry from $L_\beta(X)$ into $L_\alpha(X)^*$. Furthermore, the linear transformation $\mathcal{F} : L_\beta(X) \to L_\alpha(X)^*$ is *onto*:

Proposition 3.1 (Dual of $L_\alpha(X)$). *Let α be a Hölder's function and β be the corresponding Hölder's conjugate function. Then the dual space $L_\alpha(X)^*$ is isometrically isomorphic to the conjugate space $L_\beta(X)$. Actually, for each $\varphi \in L_\alpha(X)^*$ there exists $g \in L_\beta(X)$ such that*

$$\varphi(f) = \int_X f(x)g(x)d\mu$$

for all $f \in L_\alpha(X)$.

Proof. This is an L_α-version of the Riesz representation theorem. So we put in Appendix the proof of the case when $(X, \mathfrak{M}, \mu)$ is a σ-finite measure space. Here we only explain the case when μ is arbitrary.

Now, any $\varphi \in L_\alpha(X)^*$ is given. In Appendix, we prove that for each σ-finite subset $E \subset X$ there is an essentially unique $g_E \in L_\beta(E)$ such that

$$\varphi(f) = \int_E f g_E d\mu$$

for all $f \in L_\alpha(E)$, and $\|g_E\|_{L_\beta} \leq \|\varphi\|_{L_\alpha^*}$. If F is σ-finite and $E \subset F$, then $g_F = g_E$ almost everywhere on E, so $\|g_F\|_{L_\beta} \geq \|g_E\|_{L_\beta}$. Let

$$M := \sup\left\{\|g_E\|_{L_\beta} :\ E \text{ is a } \sigma\text{-finite subset of } X\right\},$$

and we choose a sequence $\{E_n\}$ so that $\|g_{E_n}\|_{L_\beta} \to M$. Also, set $F := \cup E_n$, and we see that F is σ-finite and $\|g_F\|_{L_\beta} \geq \|g_{E_n}\|_{L_\beta}$ for all n, hence $\|g_F\|_{L_\beta} = M$. Now, if A is a σ-finite subset of X containing F, we have

$$\begin{aligned}\int_X \beta(|g_F|)\, d\mu + \int_X \beta(|g_{A-F}|)\, d\mu = \int_X \beta(|g_A|)\, d\mu &= \beta(\|g_A\|_{L_\beta}) \\ &\leq \beta(M) \\ &= \int_X \beta(|g_F|)\, d\mu\end{aligned}$$

and thus $g_A = 0$ almost everywhere on $A - F$. Hence for any $f \in L_\alpha(X)$, we have

$$\varphi(f) = \int_X f\, g_{A_f}\, d\mu = \int_X f\, g_F\, d\mu$$

by the fact that $A_f = \{x \in X : f(x) \neq 0\} \cup F$ is a σ-finite subset of X containing F. Thus we may take $g = g_F$, and the proof is complete. □

Corollary 3.1. *The above proposition implies the homogeneity of* $\|\cdot\|_{L_\alpha}$. *That is, for all* $k \geq 0$ *and* $f \in L_\alpha(X)$,

$$\|kf\|_{L_\alpha} = k\|f\|_{L_\alpha}.$$

4. Poincaré's inequality on $L_\alpha(\Omega)$

In this section we present the main theorem(Poincaré's inequality) on the new space $L_\alpha(\Omega)$.

We say that a pre-Hölder function β satisfies the *slope condition* if there exists some positive constant $c > 1$ for which it holds

$$\beta'(x) \geq c\, \frac{\beta(x)}{x} \tag{10}$$

for almost every $x > 0$.

Theorem 4.1. *Let* (α, β) *be a Hölder pair with the commutative condition and the slope condition (10) and let* Ω *be an open set in* $\mathbb{R}^n$ *which is bounded in some direction, that is, there is a vector* $v \in \mathbb{R}^n$ *such that*

$$\sup\{|x \cdot v| : x \in \Omega\} < \infty.$$

Then there is a constant $C > 0$ *such that for any* $f \in W^1_\alpha(\Omega)$ *with* $f(x) = 0$[2] *for* $x \in \partial\Omega$ *and* $x \cdot v \neq 0$,

$$\|f\|_{L_\alpha} \leq C\|v \cdot \nabla f\|_{L_\alpha}.$$

Proof. Without loss of generality, we may assume that $v = (1, 0, \cdots, 0)$. For $\phi \in C_c^\infty(\Omega)$, we have

$$\partial_{x_1}(x_1 \alpha(|\phi(x)|)) = \alpha(|\phi(x)|) + x_1 \alpha'(|\phi(x)|)(\mathrm{sgn}\phi(x))(\partial_{x_1}\phi(x)).$$

Integration of both sides yields

$$\begin{aligned} &\int_\Omega \partial_{x_1}(x_1 \alpha(|\phi(x)|))\, dx \\ &= \int_\Omega \alpha(|\phi(x)|)\, dx + \int_\Omega x_1 \alpha'(|\phi(x)|)(\mathrm{sgn}\phi(x))(\partial_{x_1}\phi(x))\, dx. \end{aligned} \tag{11}$$

[2] in the sense of trace map

The boundary condition makes the left-hand side be zero:

$$\int_\Omega \partial_{x_1}(x_1\alpha(|\phi(x)|))\,dx = 0.$$

Then by virtue of Hölder's inequality, identity (11) becomes

$$\begin{aligned}(12)\qquad &\int_\Omega \alpha(|\phi(x)|)dx \\ &\le k\beta^{-1}\Big(\int_\Omega \beta\big(\alpha'(|\phi(x)|)\big)dx\Big)\alpha^{-1}\Big(\int_\Omega \alpha(|\partial_{x_1}\phi(x)|)dx\Big),\end{aligned}$$

where $k = \sup\{|x\cdot v| : x \in \Omega\}$. Owing to identity (5), we have

$$\begin{aligned}&\beta^{-1}\left(\int_\Omega \beta\left(\alpha'(|\phi(x)|)\right)dx\right)\\ &= \beta^{-1}\left(\int_\Omega \beta\left(\frac{\alpha(|\phi(x)|)}{|\phi(x)|} + \frac{\alpha(|\phi(x)|)}{\beta'\left(\frac{\alpha(|\phi(x)|)}{|\phi(x)|}\right) - |\phi(x)|}\right)dx\right).\end{aligned}$$

Now the commutative condition (7) delivers

$$\frac{\alpha(|\phi(x)|)}{|\phi(x)|} = \left(\frac{\beta(|\phi(x)|)}{|\phi(x)|}\right)^{-1},$$

wherein the notation $(\,\cdot\,)^{-1}$ represents the inverse functions of $(\,\cdot\,)$. Then the slope condition (10) gives

$$\beta'\left(\frac{\alpha(|\phi(x)|)}{|\phi(x)|}\right) \ge c|\phi(x)|,$$

for some positive constant $c > 1$. Hence by virtue of identity (1), we have

$$\begin{aligned}\beta^{-1}\Big(\int_\Omega \beta\big(\alpha'(|\phi(x)|)\big)\,dx\Big) &\le \beta^{-1}\Big(\int_\Omega \beta\Big(\frac{\alpha(|\phi(x)|)}{|\phi(x)|} + \frac{\alpha(|\phi(x)|)}{(c-1)(|\phi(x)|)}\Big)\,dx\Big)\\ &= \beta^{-1}\left(\int_\Omega \beta\left(\left(1+\frac{1}{c-1}\right)\frac{\alpha(|\phi(x)|)}{|\phi(x)|}\right)dx\right)\\ &= \left(1+\frac{1}{c-1}\right)\beta^{-1}\left(\int_\Omega \beta\left(\frac{\alpha(|\phi(x)|)}{|\phi(x)|}\right)dx\right)\\ &= \left(1+\frac{1}{c-1}\right)\beta^{-1}\left(\int_\Omega \alpha(|\phi(x)|)dx\right).\end{aligned}$$

Hence from the estimate (12), we conclude that

$$\alpha(\|\phi\|_{L_\alpha}) \le C\beta^{-1}\circ\alpha(\|\phi\|_{L_\alpha})\|\partial_{x_1}\phi\|_{L_\alpha},$$

or equivalently,

$$\frac{\alpha(\|\phi\|_{L_\alpha})}{\beta^{-1}\circ\alpha(\|\phi\|_{L_\alpha})} \le C\|\partial_{x_1}\phi\|_{L_\alpha}.$$

Identity (2) leads to $\|\phi\|_{L_\alpha} \le C\|\partial_{x_1}\phi\|_{L_\alpha}$. The density argument gives the desired inequality. □

Appendix: Proof of Proposition 3.1

First, let us suppose that μ is finite so that all simple functions are in $L_\alpha(X)$. Choose a functional φ in $L_\alpha(X)^*$. We will show that φ is of the form $\varphi(f) = \int_X fg d\mu$ for all $f \in L_\alpha(X)$. To accomplish it, define $\nu(E) := \varphi(\chi_E)$, for any measurable set $E \in \mathfrak{M}$. Then ν is a measure on $(X, \mathfrak{M})$. In fact, for any sequence (E_j) of disjoint measurable sets satisfying $E := \cup_{j=1}^\infty E_j$, we have $\chi_E = \sum_{j=1}^\infty \chi_{E_j}$ where the series converges in the L_α-norm:

$$\begin{aligned}\left\|\chi_E - \sum_{j=1}^{n}\chi_{E_j}\right\|_{L_\alpha} = \left\|\sum_{j=n+1}^{\infty}\chi_{E_j}\right\|_{L_\alpha} &= \alpha^{-1}\left(\int_X \alpha\left(\sum_{j=n+1}^{\infty}\chi_{E_j}\right)\, d\mu\right)\\ &= \alpha^{-1}\left(\int_X \alpha\left(\chi_{\cup_{j=n+1}^\infty E_j}\right)\, d\mu\right)\\ &= \alpha^{-1}\left(\int_X \chi_{\cup_{j=n+1}^\infty E_j}\, d\mu\right)\\ &= \alpha^{-1}(\mu(\chi_{\cup_{j=n+1}^\infty E_j})) \to 0\end{aligned}$$

as $n \to \infty$. The linearity and continuity of φ imply that

$$\nu(E) = \varphi(\chi_E) = \sum_{j=1}^{\infty}\varphi(\chi_{E_j}) = \sum_{j=1}^{\infty}\nu(E_j).$$

And also, $\nu(\varnothing) = \varphi(\chi_\varnothing) = 0$, so that ν is a measure. Furthermore, if $\mu(E) = 0$, then $\chi_E = 0$ as an element of $L_\alpha(X)$, so $\nu(E) = 0$, which implies absolute continuity of the measure μ with respect to ν. Hence by virtue of Radon-Nikodym theorem, there exists a unique integrable function g satisfying

$$\varphi(\chi_E) = \nu(E) = \int_E g\, d\mu$$

for all $E \in \mathfrak{M}$. Therefore we have $\varphi(f) = \int_X fgd\mu$ for all simple functions f, and hence it does for all $f \in L_\alpha(X)$. Also, we notice that

$$\|\varphi\|_{L_\alpha^*} = \sup\left\{\frac{\left|\int_X fgd\mu\right|}{\|f\|_{L_\alpha}} : f \in L_\alpha(X), f \neq 0\right\} = \|g\|_{L_\beta}.$$

Now suppose that μ is σ-finite. Let $\{E_n\}$ be an increasing sequence of sets such that $0 < \mu(E_n) < \infty$ and $X = \bigcup_{n=1}^\infty E_n$, and let us agree to identify $L_\alpha(E_n)$ and $L_\beta(E_n)$ with the subspaces of $L_\alpha(X)$ and $L_\beta(X)$ consisting of functions which vanish outside E_n. The preceding argument shows that for each n, there exists $g_n \in L_\beta(E_n)$ such that

$$\varphi(f) = \int_X fg_n\, d\mu$$

for all $f \in L_\alpha(E_n)$, and

$$\|g_n\|_{L_\beta} = \|\left.\varphi\right|_{E_n}\|_{L_\alpha(E_n)^*} \leq \|\varphi\|_{L_\alpha(X)^*}.$$

Since g_n is unique modulo alterations on null sets, $g_n = g_m$ almost everywhere on E_n for $n < m$, and so we can define g on X by setting $g = g_n$ on E_n. Then from the fact that $\|g\|_{L_\beta} = \lim_{n\to\infty} \|g_n\|_{L_\beta} \leq \|\varphi\|_{L_\alpha(X)^*}$, we get $g \in L_\beta$. Moreover, if $f \in L_\alpha$, then $f\chi_{E_n} \to f$ in the L_α-norm by the dominated convergence theorem, so we conclude

$$\varphi(f) = \lim_{n\to\infty} \varphi\left(f\chi_{E_n}\right) = \lim_{n\to\infty} \int_{E_n} fg_n\, d\mu = \int_X fg\, d\mu.$$

□

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