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COMPETING CONSTANTS FOR THE SOBOLEV TRACE INEQUALITY

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Abstract. A comparison of constants is given to show that a better constant for the Sobolev trace inequality can be obtained from the conjectured extremal function.

1. Introduction

An extremal function for the Sobolev trace inequality

$$\left(\int_{\mathbb{R}^n} |f(x)|^q dx\right)^{1/q} \le A_{p,q} \left(\int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^p dxdy\right)^{1/p}, \frac{1}{q} = \frac{n+1}{np} - \frac{1}{n}$$

was conjectured in [8] as the function of the form

$$\phi(x,y) \equiv [(1+y)^2 + |x|^2]^{-\frac{n+1-p}{2(p-1)}}.$$

No proof has been given to show that this is an actual extremal function for this inequality. It is noted that the function is not radially symmetric. Actually it can be easily seen that no radial function can be extremal due to the boundary condition of the Euler-Lagrange equations associated with the inequality. It would be useful however to restrict this inequality to the space of radial functions. With this symmetry, the inequality becomes an inequality of one variable functions, and an extremal function can be found and the best constant can be computed. This gives a good bound for the best constant.

In this paper, it is given the comparison of constants arising from the conjectured extremal function ϕ and the radial extremal function. This comparison shows that a better constant for the Sobolev trace inequality

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can be obtained from the conjectured extremal function. The proof heavily depends on knowledge of the gamma functions, their derivatives and properties.

In this paper, we denote $q' = q/(q-1)$, $p' = p/(p-1)$ and $1/q = (n+1)/np - 1/n$.

2. Constants and comparison

2.1. Constant from radial functions

We will look at the Sobolev trace inequality for radial functions u:

$$(1) \qquad \left(\int_{\mathbb{R}^n} |u(x,0)|^q dx \right)^{1/q} \le \mathbf{C_R^{-1}} \left(\int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^p dxdy \right)^{1/p},$$

where $\mathbf{C_R}$ is the best constant for the trace inequality restricted on radial functions on $\mathbb{R}^{n+1}_+$. Without loss of generality, we may assume that functions are positive and decreasing. With those assumptions, we can rewrite (1) as follows: for $\sigma_k = 2\pi^{k/2}/\Gamma(k/2)$,

$$(2) \qquad \left(\int_0^\infty |u(t)|^q t^{n-1} dt \right)^{1/q} \le \mathbf{C_R^{-1}} \sigma_n^{-1/q} \sigma_{n+1}^{1/p} 2^{-1/p} \left(\int_0^\infty |u'(t)|^p t^n dt \right)^{1/p}.$$

By applying Bliss' argument [7] we find an extremal function u:

$$u(t) = (a+bt)^{-\frac{n+1-p}{p-1}},$$

where a and b are positive constants. Also, Bliss' argument can be exploited to show that the best constant for (2)

$$(3) \qquad n^{-1/p} \left(\frac{p-1}{n+1-p} \right)^{1/q'} \left(\frac{\Gamma(n)\Gamma(\frac{n}{p-1} - 1)}{\Gamma(\frac{np}{p-1})} \right)^{1/q - 1/p}$$

is attained at this function u. Combining (2) and (3) together, we get

$$(4) \qquad \mathbf{C_R} = \frac{\sqrt{\pi} n^{1/p}}{2^{1/q}} \left[\frac{n+1-p}{p-1} \right]^{1/p'} \frac{\Gamma(\frac{n}{2})^{1/q}}{\Gamma(\frac{n+1}{2})^{1/p}} \left[\frac{\Gamma(n)\Gamma(\frac{n}{p-1})}{\Gamma(n\frac{p}{p-1})} \right]^{1/np'}.$$

2.2. Constant from the conjectured extremal

We will compute the constant $\mathbf{C_E}$ from *Escobar's conjectured extremal function*

$$\phi(x,y) \equiv [(1+y)^2 + |x|^2]^{-\frac{n+1-p}{2(p-1)}}, \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^+,$$

that is,

$$\mathbf{C_E} \equiv \frac{\left(\int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^p dxdy\right)^{1/p}}{\left(\int_{\mathbb{R}^n} |u(x,0)|^q dx\right)^{1/q}}.$$

A routine computation with this function ϕ yields

$$\mathbf{C_E} = \sqrt{\pi}^{1/p'} \left[\frac{n+1-p}{p-1}\right]^{1/p'} \left[\frac{\Gamma(\frac{n}{2}\frac{1}{p-1})}{\Gamma(\frac{n}{2}\frac{p}{p-1})}\right]^{1/np'}.$$

2.3. Comparison of the constants

We shall compare the constants $\mathbf{C_R}$ and $\mathbf{C_E}$. For that, take the ratio of the two numbers to get

$$\frac{\mathbf{C_E}}{\mathbf{C_R}} = 2^{\frac{1}{q}+\frac{1}{p}(1-n)} n^{-\frac{1}{p}} \frac{\Gamma(n)^{\frac{1}{q}}}{\Gamma(\frac{n}{2})^{\frac{1}{q}+\frac{1}{p}}} \left[\frac{\Gamma(n\frac{p}{p-1})\Gamma(\frac{n}{2}\frac{1}{p-1})}{\Gamma(n\frac{1}{p-1})\Gamma(\frac{n}{2}\frac{p}{p-1})}\right]^{\frac{1}{np'}}.$$

We rewrite the ratio in terms of $t = np'$ with $t \geq n+1$ as follows:

$$(5) \qquad f(t) \equiv 2^{2-n+\frac{n^2-2n-1}{t}} n^{\frac{n}{t}-1} \frac{\Gamma(n)^{1-\frac{n+1}{t}}}{\Gamma(\frac{n}{2})^{2-\frac{2n+1}{t}}} \left[\frac{\Gamma(t)\Gamma(\frac{t}{2}-\frac{n}{2})}{\Gamma(t-n)\Gamma(\frac{t}{2})}\right]^{\frac{1}{t}}.$$

Theorem 2.1. $0 < f(t) < 1$ *for* $t \geq n+1$.

Proof. Since it is clear that $f(t) > 0$, it is enough to show $f(t) < 1$. Take the logarithms of both sides of (5) and differentiate with respect to t to get

$$(6) \qquad \frac{d}{dt}[\ln f(t)] = \frac{1}{t^2} L(n,t),$$

where for $\psi(t) = d \ln \Gamma(t)/dt$,

$$L(n,t) = \ln \frac{\Gamma(t-n)\Gamma(\frac{t}{2})}{\Gamma(t)\Gamma(\frac{t}{2}-\frac{n}{2})} + t\left[\psi(t) - \psi(t-n) + \frac{1}{2}\{\psi(\frac{t}{2}-\frac{n}{2}) - \psi(\frac{t}{2})\}\right]$$
$$+(-n^2+2n+1)\ln 2 - n\ln n + (n+1)\ln\Gamma(n) - (2n+1)\ln\Gamma(\frac{n}{2}).$$

To finish the proof of the theorem, the following lemma is needed.

Lemma 2.2. $\frac{d}{dt}[\ln f(t)] < 0$ *for any* $t > n+1$.

Proof. From (6), it suffices to show $L(n,t) < 0$ for $t \geq n+1$. First, differentiate $L(n,t)$ with respect to t to get

$$\frac{d}{dt}L(n,t) = t\left[\psi'(t) - \psi'(t-n) + \frac{1}{4}\{\psi'(\frac{t}{2}-\frac{n}{2}) - \psi'(\frac{t}{2})\}\right].$$

Using the following series representations for the function ψ

$$\psi(x-n) = \psi(x) - \sum_{k=1}^{n} \frac{1}{x-k}$$
$$\frac{1}{2}[\psi(\frac{x+1}{2}) - \psi(\frac{x}{2})] = \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k},$$

we obtain two expressions for $\frac{d}{dt}L(n,t)$ depending on n:

$$\frac{d}{dt}L(n,t) = \begin{cases} t\left[-\sum_{k=1}^{m} \frac{1}{(t-2k+1)^2}\right] & \text{if } n = 2m, \\ t\left[-\sum_{k=1}^{m-1} \frac{1}{(t-2k)^2} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(t+k)^2}\right] & \text{if } n = 2m-1. \end{cases}$$

In the case $n = 2m-1$, $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(t+k)^2}$ is absolutely convergent, and so we may rearrange the terms to estimate the series without changing the sum. In both cases, we see that $\frac{d}{dt}L(n,t) < 0$. □

Now we have that $L(n,t) < L(n,n+1)$ for $t \geq n+1$. We only need to show $L(n,n+1) < 0$ to complete the proof.

$$L(n,n+1) = (-n^2+2n+1)\ln 2 + \ln \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2})}{\Gamma(n)\Gamma(\frac{1}{2})} + (n+1)\ln \frac{\Gamma(n)}{n\Gamma(\frac{n}{2})^2}$$
$$+(n+1)\left[\psi(n+1) - \psi(1) + \frac{1}{2}\{\psi(\frac{1}{2}) - \psi(\frac{n+1}{2})\}\right].$$

Use the duplication formula for the gamma function to get

$$\frac{1}{(n+1)}L(n,n+1) = (2-n)\ln 2 + \ln \frac{\Gamma(n)}{n\Gamma(\frac{n}{2})^2}$$
$$+\psi(n+1) - \psi(1) + \frac{1}{2}\{\psi(\frac{1}{2}) - \psi(\frac{n+1}{2})\}.$$

Use $\psi(x+n) = \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k}$ to express

$$L(n,n+1) = (n+1)\left[(2-n)\ln 2 + \ln \frac{\Gamma(n)}{n\Gamma(\frac{n}{2})^2} + \Lambda(n)\right],$$

where

$$\Lambda(n) = \begin{cases} \sum_{k=1}^{m} \frac{1}{2k} & \text{if } n = 2m \\ \sum_{k=1}^{m} \frac{1}{2k-1} - \ln 2 & \text{if } n = 2m-1. \end{cases}$$

Apply the induction on $\frac{1}{(n+1)}L(n,n+1)$ with $m = 1,2,3,\cdots$ to show $L(n,n+1) < 0$ with the help of the fact that $\ln(\frac{x}{x+1}) + \frac{1}{x+1} < 0$ for $x \geq 1$. By Lemma 2.2, we have

$$\begin{aligned} &\ln f(t) < \ln f(n+1) \quad \text{for } t \geq n+1 \\ \text{or} \quad & f(t) < f(n+1) \qquad \text{for } t \geq n+1. \end{aligned}$$

With the duplication formula for the gamma function, we can see that

$$f(n+1)^{n+1} \;=\; \frac{1}{2^{n-1}}\frac{1}{n}\frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{n+1}{2})} \;=\; 1.$$

We conclude that $f(t) < f(n+1) = 1$ to finish the proof of theorem. □

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