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ON THE FUZZY COMPLETE NORMED LINEAR SPACE

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Abstract. In this paper, we introduce the notion of the complete fuzzy norm on a linear space. And we consider some relations between the fuzzy completeness and ordinary completeness on a linear space.

1. Introduction

The notions of fuzzy vector spaces and fuzzy topological vector spaces were introduced in Katsaras and Liu [4]. These ideas were modified by Katsaras [2], and in [3] Katsaras defined the fuzzy norm on a vector space. In [5] Krishna and Sarma discussed the generation of a fuzzy vector topology from an ordinary vector topology on vector space. Also Krishna and Sarma [6] observed the convergence of sequence of fuzzy points. Rhie et al. [9] introduced the notion of fuzzy α-Cauchy sequence of fuzzy points and fuzzy completeness.

In this paper, we first observe a type of the convergence of sequences as an analogy of Bag and Samanta [1] in a fuzzy normed linear space. Secondly, we introduce the notion of a complete fuzzy norm using the convergence of sequences on a linear space. And we consider some relations between the fuzzy completeness and the ordinary completeness on a linear space.

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2. Preliminaries

Throughout this paper, X is a vector space over the field $K(R$ or $C)$. Fuzzy subsets of X are denoted by Greek letters in general. χ_A denotes the characteristic function of the set A.

DEFINITION 2.1. [4] For two fuzzy subset μ_1 and μ_2 of X, the fuzzy subset $\mu_1 + \mu_2$ is defined by

$$(\mu_1 + \mu_2)(x) = \vee\{\mu_1(x_1) \wedge \mu_2(x_2) \mid \ x = x_1 + x_2\}.$$

And for a scalar t of K and a fuzzy subset μ of X, the fuzzy subset $t\mu$ is defined by

$$(t\mu)(x) = \begin{cases} \mu(x/t) & \text{if} \quad t \neq 0 \\ 0 & \text{if} \quad t = 0 \ \text{ and } \ x \neq 0 \\ \vee\{\mu(y) \mid \ y \in X\} & \text{if} \quad t = 0 \ \text{ and } \ x = 0. \end{cases}$$

DEFINITION 2.2. [2] $\mu \in I^X$ is said to be

1. *convex* if $t\mu + (1-t)\mu \subseteq \mu$ for each $t \in [0,1]$
2. *balanced* if $t\mu \subseteq \mu$ for each $t \in K$ with $\mid t \mid \leq 1$
3. *absorbing* if $\vee\{t\mu(x) \mid t > 0\} = 1$ for all $x \in X$.

DEFINITION 2.3. [2] Let (X, τ) be a topological space and $\omega(\tau) = \{f : (X, \tau) \to [0,1] \mid f \ \textit{is lower semicontinuous}\}$. Then $\omega(\tau)$ is a fuzzy topology on X. This topology is called the fuzzy topology generated by τ on X. The fuzzy usual topology on K means the fuzzy topology generated by the usual topology of K.

DEFINITION 2.4. [2] A *fuzzy linear topology* on a vector space X over K is a fuzzy topology on X such that the two mappings

$$\begin{aligned} + \ &: \ X \times X \to X, \qquad && (x,y) \to x + y \\ \cdot \ &: \ K \times X \to X, \qquad && (t,x) \to tx \end{aligned}$$

are continuous when K has the fuzzy usual topology and $K \times X$ and $X \times X$ have the corresponding product fuzzy topologies. A linear space with a fuzzy linear topology is called a *fuzzy topological linear space* or a *fuzzy topological vector space*.

DEFINITION 2.5. [2] Let x be a point in a fuzzy topological space X. A family F of neighborhoods of x is called a base for the system of all neighborhoods of x if for each neighborhood μ of x and each $0 < \theta < \mu(x)$, there exists $\mu_1 \in F$ with $\mu_1 \leq \mu$ and $\mu_1(x) > \theta$.

Definition 2.6. [3] A *fuzzy seminorm* on X is a fuzzy set ρ in X which is convex ,balanced and absorbing. If in addition $\wedge\{(t\rho)(x) \mid t > 0\} = 0$ for $x \neq 0$, then ρ is called a *fuzzy norm.*

Theorem 2.7. [3] *If ρ is a fuzzy seminorm on X, then the family $B_\rho = \{\theta \wedge (t\rho) \mid 0 < \theta \leq 1,\ t > 0\}$ is a base at zero for a fuzzy linear topology τ_ρ. The fuzzy topology τ_ρ is called the fuzzy topology induced by the fuzzy seminorm ρ . And a linear space equipped with a fuzzy seminorm (resp. fuzzy norm) is called a fuzzy seminormed (resp. fuzzy normed) linear space.*

Definition 2.8. [5] Let ρ be a fuzzy seminorm on X. $P_\epsilon : X \to R_+$ is defined by

$$P_\epsilon(x) = \wedge\{t > 0 \mid t\rho(x) > \epsilon\}$$

for each $\epsilon \in (0, 1)$.

Theorem 2.9. [5] *The P_ϵ is a seminorm on X for each $\epsilon \in (0, 1)$. Further P_ϵ is a norm on X for each $\epsilon \in (0, 1)$ if and only if ρ is a fuzzy norm on X.*

3. Fuzzy convergence and fuzzy completeness

In this section, we introduce the notion of a complete fuzzy norm on a linear space. And we consider some relations between the fuzzy completeness and the ordinary completeness on a linear space.

Now, we define the convergent sequence and the Cauchy sequence in a fuzzy normed linear space as an analogy of Bag and Samanta [1, Definition 2.2, 2.3].

Definition 3.1. Let (X, ρ) be a fuzzy normed linear space. A sequence $\{x_n\} \subset X$ is said to be convergent to $x \in X$ if for every $t > 0$ and $\epsilon \in (0, 1)$, there exists a positive integer M such that $n \geq M$ implies $t\rho(x_n - x) > 1 - \epsilon$.

Theorem 3.2. *Let (X, ρ) be a fuzzy normed linear space. A sequence $\{x_n\} \subset X$ converges to $x \in X$ if and only if for every $t > 0$ and $\epsilon \in (0, 1)$, there exists a positive integer M such that $n \geq M$ implies $P_{1-\epsilon}(x_n - x) < t$.*

Proof. Let $t > 0$ and $\epsilon \in (0, 1)$ be given. Since $\{x_n\}$ converges to x, there exists a positive integer M such that

$$\begin{aligned} & n \geq M \quad \text{implies} \quad \tfrac{t}{2}\rho(x_n - x) > 1 - \epsilon \\ \Longrightarrow \quad & n \geq M \quad \text{implies} \quad P_{1-\epsilon}(x_n - x) \leq \tfrac{t}{2} < t. \end{aligned}$$

For the converse, let $t > 0$ and $\epsilon > 0$ be given. Then there exists a positive integer M such that

$$\begin{aligned} & n \geq M \quad \text{implies} \quad P_{1-\epsilon}(x_n - x) < t \\ \Longrightarrow \quad & n \geq M \quad \text{implies} \quad t'\rho(x_n - x) > 1 - \epsilon \\ & \qquad\qquad \text{for some} \quad t' \in (P_{1-\epsilon}(x_n - x), t) \\ \Longrightarrow \quad & n \geq M \quad \text{implies} \quad t\rho(x_n - x) \geq t'\rho(x_n - x) > 1 - \epsilon. \end{aligned}$$

This completes the proof. □

The next definition of a Cauchy sequence in a fuzzy normed linear space is an analogy of Bag and Samanta [1, Definition 2.3].

Definition 3.3. Let (X, ρ) be a fuzzy normed linear space. A sequence $\{x_n\} \subset X$ is a Cauchy sequence if and only if for every $t > 0$ and $\epsilon \in (0, 1)$, there exists a positive integer M such that $n, m \geq M$ implies $t\rho(x_n - x_m) > 1 - \epsilon$.

Theorem 3.4. *Let (X, ρ) be a fuzzy normed linear space. A sequence $\{x_n\} \subset X$ is a Cauchy sequence if and only if for every $t > 0$ and $\epsilon \in (0, 1)$, there exists a positive integer M such that $n, m \geq M$ implies $P_{1-\epsilon}(x_n - x_m) < t$.*

Proof. The proof is similar to that of Theorem 3.2. We omit it. □

The following theorem is easily verified with elementary skills from Theorem 3.2. and Theorem 3.4.

Theorem 3.5. *Every convergent sequence in a fuzzy normed linear space is a Cauchy sequence.*

Now, we introduce the complete fuzzy norm using the Cauchy sequence defined above.

Definition 3.6. A fuzzy norm ρ on a linear space X is said to be fuzzy complete if every Cauchy sequence in X converges to a point in X.

Lemma 3.7. *Let $(X, \|\cdot\|)$ be a normed linear space and B the closed unit ball of X. Then every Cauchy sequence in the fuzzy normed linear space (X, χ_B) is a Cauchy sequence with respect to the ordinary norm.*

Proof. Let $\{x_n\} \subset X$ be a Cauchy sequence and $\delta > 0$. Since $\{x_n\}$ is a Cauchy sequence, for this δ and for every $\epsilon \in (0, 1)$, there exists a positive integer M such that

$$\begin{array}{rlll} & n,m \geq M & \text{implies} & \frac{\delta}{2}\chi_B(x_n - x_m) > 1-\epsilon \\ \Longrightarrow & n,m \geq M & \text{implies} & \chi_B(\frac{2}{\delta}(x_n - x_m)) > 1-\epsilon \\ \Longrightarrow & n,m \geq M & \text{implies} & \chi_B(\frac{2}{\delta}(x_n - x_m)) = 1 \\ \Longrightarrow & n,m \geq M & \text{implies} & \| x_n - x_m \| \leq \frac{\delta}{2} < \delta. \end{array}$$

Therefore $\{x_n\}$ is Cauchy sequence in $(X, \| \cdot \|)$. This prove the lemma. □

THEOREM 3.8. *Let $(X, \| \cdot \|)$ be a Banach space. Then the fuzzy normed linear space (X, χ_B) is fuzzy complete where B is the closed unit ball of X.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X, χ_B). Then it is a Cauchy sequence with respect to the ordinary norm $\| \cdot \|$ by the above lemma. Since $(X, \| \cdot \|)$ is complete, there exists an $x \in X$ such that $\| x_n - x_m \| \to 0$. Now, we show that $\{x_n\}$ converges to this x in (X, χ_B). Let $t > 0$ and $\epsilon \in (0,1)$. Then there exists a positive integer M such that

$$\begin{array}{rlll} & n \geq M & \text{implies} & \| x_n - x \| < t \\ \Longrightarrow & n \geq M & \text{implies} & \| \frac{1}{t}(x_n - x) \| < 1 \\ \Longrightarrow & n \geq M & \text{implies} & \chi_B(\frac{1}{t}(x_n - x)) = 1 \\ \Longrightarrow & n \geq M & \text{implies} & t\chi_B(x_n - x) > 1-\epsilon. \end{array}$$

That is $\{x_n\}$ converges to x , therefore (X, χ_B) is fuzzy complete. This completes the proof. □

COROLLARY 3.9. *The field K(R or C) with the fuzzy topology generated by the usual topology on K is a fuzzy complete normed linear space.*

DEFINITION 3.10. [3] Two fuzzy seminorms ρ_1, ρ_2 on X are said to be *equivalent* iff $\tau_{\rho_1} = \tau_{\rho_2}$.

THEOREM 3.11. [9] *Let $(X, \| \cdot \|)$ be a normed linear space. If ρ is a lower semi-continuous fuzzy norm on X, and has the bounded support: $\{x \in X | \ \rho(x) > 0\}$ is bounded, then ρ is equivalent to the fuzzy norm χ_B where B is the closed unit ball of X.*

By Theorem 3.8. and the above theorem, we get the following theorem which is the main result of this paper.

THEOREM 3.12. *If X is a Banach space and ρ is a lower semicontinuous fuzzy norm on X having the bounded support, then the fuzzy normed linear space (X, ρ) is fuzzy complete.*

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