JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 22, No. 2, June 2009

SPECTRA OF ASYMPTOTICALLY QUASISIMILAR SUBDECOMPOSABLE OPERATORS

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Abstract. In this paper, we prove that asymptotically quasisimilar subdecomposable operators have equal spectra and quasisimilar decomposable operators have equal spectra. Moreover, every subscalar operator is admissible.

1. Introduction

Let X be a Banach space over the complex plane $\mathbb{C}$. Let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on X. For a given $T \in \mathcal{L}(X)$, $\sigma(T)$ denotes the spectrum of T and $\ker(T)$ denotes the kernel of T. The *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$ is defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to X$ which satisfies $(T - \lambda)f(\lambda) = x$ for all $\lambda \in U$. The *local spectrum* $\sigma_T(x)$ of T at x is then defined as $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. An operator $T \in \mathcal{L}(X)$ is said to have the *single-valued extension property*, abbreviated SVEP, if for every open set $U \subseteq \mathbb{C}$, the only analytic solution $f : U \to X$ of the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on U.

For every closed subset F of $\mathbb{C}$, let $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$ denote the corresponding *analytic spectral subspace* of T. It is easy to see that $X_T(F)$ is a invariant subspace for T. An operator $T \in \mathcal{L}(X)$ is said to have *Dunford's property (C)* if $X_T(F)$ is closed for every closed $F \subseteq \mathbb{C}$. This condition plays an important role in the theory of spectral operators. It is well known that Dunford's property (C) implies the single-valued extension property [7].

Received April 22, 2009; Accepted May 26, 2009.
2000 *Mathematics Subject Classifications*: Primary 47A11; Secondary 47B40.
Key words and phrases: algebraic spectral subspace, analytic spectral subspace, subdecomposable operator.
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Let $\sigma_{sur}(T)$ denote the *surjectivity spectrum* of an operator T. That is, $\sigma_{sur}(T) = \{\lambda \in \mathbb{C} \ : \ (T - \lambda)X \neq X\}$. It is well known that [7]

$$\sigma_{sur}(T) = \bigcup_{x \in X} \sigma_T(x) \subseteq \sigma(T).$$

Let $A \subseteq \mathbb{C}$. Then the *algebraic spectral subspace* $E_T(A)$ is the largest subspaces of X on which all the restrictions of $T-\lambda$, $\lambda \in \mathbb{C}\setminus A$, are surjective. That is,

$$E_T(F) = \mathrm{span}\{Y \in \mathrm{lat}(T) \ : \ \sigma_{sur}(T|Y) \subseteq F\},$$

where $\mathrm{lat}(T)$ denotes the collection of $T-$invariant subspaces of X, and $T|Y$ denotes the restriction of T on Y. Thus $E_T(A)$ is the largest subspace of X with this surjectivity property but this space need not be closed in general. An operator $T \in \mathcal{L}(X)$ is said to be *admissible* if for each closed $F \subseteq \mathbb{C}$ the algebraic spectral subspace $E_T(F)$ is closed. Laursen proved in [4] that if $E_T(F)$ is closed then $E_T(F) = X_T(F)$. Thus the admissible operators allow us to combine the analytic tools associated with the space $X_T(F)$ with the algebraic tools associated with the space $E_T(F)$.

2. Local spectral properties of subdecomposable operators

The following Proposition is found in [4].

PROPOSITION 2.1. *Let T be a bounded linear operator on a Banach space X. Then we have*
(1) $E_{T-\lambda}(\mathbb{C} \setminus \{0\}) = E_T(\mathbb{C} \setminus \{\lambda\})$ for every $\lambda \in \mathbb{C}$.
(2) $E_T(\cap_\alpha F_\alpha) = \cap_\alpha E_T(F_\alpha)$ for any family $\{F_\alpha\}$ of subsets of $\mathbb{C}$.
(3) $E_T(F) = \cap_{\lambda \notin F} E_T(\mathbb{C} \setminus \{\lambda\})$.
(4) $\ker(T - \lambda) \subseteq E_T(\{\lambda\})$ for every $\lambda \in \mathbb{C}$.

We say that the operator T has *finite descent* if for every $\lambda \in \mathbb{C}$ there is an integer $n \in \mathbb{N}$ such that $(T - \lambda)^n X = (T - \lambda)^{n+1} X$.

THEOREM 2.2. *Let T be a bounded linear operator on a Banach space X. If T has finite descent then there exists a positive integer $p \in \mathbb{N}$ for* *which*

$$E_T(F) = \bigcap_{\lambda \notin F} (T-\lambda)^p X$$

for any subset F of $\mathbb{C}$. In particular, $E_T(\phi) = \bigcap_{\lambda\in\mathbb{C}}(T-\lambda)^p X$.

Proof. Let $p \in \mathbb{N}$ such that $(T-\lambda)^p X = (T-\lambda)^{p+1}X$. Then by Proposition 2.1, we have,

$$E_T(F) = \bigcap_{\lambda \notin F} E_T(\mathbb{C}\setminus\{\lambda\}) = \bigcap_{\lambda \notin F} E_{T-\lambda}(\mathbb{C}\setminus\{0\}).$$

Thus it suffices to show that $E_{T-\lambda}(\mathbb{C}\setminus\{0\}) = (T-\lambda)^p X$ for each $\lambda \in \mathbb{C}$. It is easily show that

$$\begin{aligned} E_{T-\lambda}(\mathbb{C}\setminus\{0\}) &\subseteq (T-\lambda)^p X \\ &\subseteq E_{(T-\lambda)^p}(\mathbb{C}\setminus\{0\}) \\ &\subseteq E_{T-\lambda}(\mathbb{C}\setminus\{0\}). \end{aligned}$$

Hence $(T-\lambda)^p X = E_{T-\lambda}(\mathbb{C}\setminus\{0\})$. Therefore we have,

$$E_T(F) = \bigcap_{\lambda \notin F} (T-\lambda)^p X.$$

This completes the proof. □

We denote by $C^\infty(\mathbb{C})$ the Fréchet algebra of all infinitely differentiable complex valued functions $\varphi(z)$, $z = x_1 + ix_2$, x_1, $x_2 \in \mathbb{R}$, defined on the complex plane $\mathbb{C}$ with the topology of uniform convergence of every derivative on each compact subset of $\mathbb{C}$. That is, with the topology generated by the family of pseudo-norm

$$|\varphi|_{K,m} = \max_{|p|\le m} \sup_{z\in K} |D^p\varphi(z)|,$$

where K is an arbitrary compact subset of $\mathbb{C}$, m a non-negative integer, $p = (p_1, p_2)$, p_1, $p_2 \in \mathbb{N}$, $|p| = p_1 + p_2$ and

$$D^p\varphi = \frac{\partial^{|p|}\varphi}{\partial {x_1}^{p_1} \partial {x_2}^{p_2}}, \quad z = x_1 + ix_2.$$

An operator $T \in \mathcal{L}(X)$ on a complex Banach space X is called a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : C^\infty(\mathbb{C}) \to \mathcal{L}(X)$ satisfying $\Phi(1) = I$ and $\Phi(z) = T$, where I is the identity operator on X and z denotes the identity function on $\mathbb{C}$. Such a continuous function Φ is in fact an operator valued distribution and it is called a *spectral distribution* for T. The class of generalized scalar operators were introduced by Colojoară and Foiaş [1]. An operator $T \in \mathcal{L}(X)$ on a complex Banach space X is said to be *subscalar* if T is similar to the restriction of a generalized scalar operator to one of its closed invariant subspaces.

An operator $T \in \mathcal{L}(X)$ is called *decomposable* if for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$, there exist Y, $Z \in \mathrm{Lat}(T)$ the collection of all closed T-invariant linear subspaces of X such that

$$\sigma(T|Y) \subset U, \ \ \sigma(T|Z) \subset V \ \text{ and } \ Y + Z = X.$$

An important subclass of the decomposable operators is formed by the generalized scalar operators. It is well known that decomposable operator T on a Banach space has Dunford's property (C) [7]. An operator $T \in \mathcal{L}(X)$ is called *subdecomposable* if T is similar to the restriction of a decomposable operator to one of its closed invariant subspaces.

Proposition 2.3. *If $T \in \mathcal{L}(X)$ is subscalar then T is admissible.*

Proof. Let $F \subseteq \mathbb{C}$ be a closed, and let $S \in \mathcal{L}(Y)$ be a generalized scalar extension of T, that is, $T \in \mathcal{L}(X)$ is similar to the restriction of a decomposable operator $S \in \mathcal{L}(Y)$ to one of its closed invariant subspace. Clearly T inherits Dunford's property (C) from the decomposability of S. Thus T has Dunford's property (C). Since S is a generalized scalar, $E_S(F) = Y_S(F)$ for all closed F subset of $\mathbb{C}$. Hence we have,

$$\begin{aligned} \overline{E_T(F)} &\subseteq E_S(F) \cap X \\ &= Y_S(F) \cap X \\ &\subseteq Y_S(F) \end{aligned}$$

If $\lambda \in \mathbb{C} \setminus F$, then $T-\lambda$ is bijective on $\overline{E_T(F)}$. Thus $\lambda \in \rho(T|\overline{E_T(F)})$ and hence $\sigma(T|\overline{E_T(F)}) \subseteq F$. Therefore we have,

$$E_T(F) \subseteq \overline{E_T(F)} \subseteq X_T(F) \subseteq E_T(F).$$

This completes the proof. □

Let X and Y be Banach spaces over the complex field $\mathbb{C}$ and let $\mathcal{L}(X,Y)$ denote the space of all bounded linear operators from X to Y. For given operators $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$, we consider the corresponding commutator $C(S,T) : \mathcal{L}(X,Y) \longrightarrow \mathcal{L}(X,Y)$ defined by

$$C(S,T)A = SA - AT \quad \text{for all} \quad A \in \mathcal{L}(X,Y).$$

It is easy to see that for each $n \in \mathbb{N}$,

$$\begin{aligned} C(S,T)^n A &= C(S,T)^{n-1}(SA-AT) \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k S^{n-k} A T^k. \end{aligned}$$

LEMMA 2.4. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$. Suppose that $A \in \mathcal{L}(X,Y)$ with $C(S,T)A = 0$. Then $AE_T(F) \subseteq E_S(F)$ for all $F \subseteq \mathbb{C}$. Furthermore, if $\ker(A) \subseteq E_T(F)$ and $E_S(F) \subseteq AX$ then $AE_T(F) = E_S(F)$.*

Proof. Since $AE_T(F) = A(T-\lambda)E_T(F) = (S-\lambda)AE_T(F)$ for all $\lambda \in \mathbb{C} \setminus F$, $AE_T(F) \subseteq E_S(F)$ by the maximality of $E_S(F)$. Suppose that $\ker(A) \subseteq E_T(F)$ and $E_S(F) \subseteq AX$. We will show that $A^{-1}E_S(F) \subseteq E_T(F)$ for all $F \subseteq \mathbb{C}$. Let $Z = A^{-1}E_S(F)$. Then Z is invariant with respect to T. Let $x \in Z$. Then,

$$Ax \in E_S(F) = (S-\lambda)E_S(F) \subseteq (S-\lambda)AX.$$

Thus there exists an element $y \in X$ such that $Ay \in E_S(F)$ and $Ax = (S-\lambda)Ay = A(T-\lambda)y$. Therefore we have,

$$x - (T-\lambda)y \in \ker(A) \subseteq E_T(F) \subseteq (T-\lambda)E_T(F).$$

Hence there exists an element $z \in E_T(F)$ for which $x-(T-\lambda)y = (T-\lambda)z$. Thus we have,

$$A(x+y) \in E_S(F) + AE_T(F) = E_S(F).$$

Hence $y+z \in Z$. Therefore we have,

$$x = (T-\lambda)(y+z) \in (T-\lambda)Z = (T-\lambda)A^{-1}E_S(F) \subseteq E_T(F).$$

This implies that $A^{-1}E_S(F) \subseteq E_T(F)$ for all $F \subseteq \mathbb{C}$. It follows from $E_S(F) \subseteq AX$ that

$$E_S(F) \subseteq AA^{-1}E_S(F) \subseteq AE_T(F).$$

Hence $AE_T(F) = E_S(F)$. □

Two bounded linear operators $R \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are called *quasi-similar* if there exist $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$, each injective and with dense range, so that $AT = SA$ and $TB = BS$.

An operator $A \in \mathcal{L}(X,Y)$ is said to *intertwines* S *and* T *asymptotically* if $\|C(S,T)^nA\|^{\frac{1}{n}} \longrightarrow 0$ as $n \to \infty$. This condition has been investigated by Colojoară and C. Foiaş [1]. Two operators $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are said to be *asymptotically quasisimilar* if there are $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$, both injective and with dense range, for which $\|C(S,T)^nA\|^{\frac{1}{n}} \to 0$ and $\|C(T,S)^nB\|^{\frac{1}{n}} \to 0$ as $n \to \infty$.

Proposition 2.5. *Let $T \in \mathcal{L}(X)$ and let $S \in \mathcal{L}(Y)$. If $A \in \mathcal{L}(X,Y)$ intertwines S and T asymptotically. Then the local spectrum $\sigma_T(Ax)$ of T at Ax is contained in $\sigma_S(x)$. In particular, if T and S are asymptotically quasisimilar then $\sigma_T(x) = \sigma_S(x)$ for every $x \in X$.*

Proof. Let $x \in X$ and let $\mu \notin \sigma_T(x)$. Then there exists an open neighborhood U of μ in $\mathbb{C}$ and an analytic function $f : U \longrightarrow X$ on an open sunset U of $\mathbb{C}$ such that $(T-\lambda)f(\lambda) = x$ for all $\lambda \in U$. Define the map $g : U \longrightarrow Y$ by

$$g(\lambda) = \sum_{n=0}^{\infty}(-1)^n C(S,T)^n(A)\frac{f^n(\lambda)}{n!} \quad \text{for all} \quad \lambda \in U.$$

Then it is easily see that the infinite series converges locally uniformly on U, and hence $g : U \to Y$ is well defined analytic function such that $(S-\lambda)g(\lambda) = Ax$ for all $\lambda \in U$. Thus we have $\mu \notin \sigma_S(Ax)$ and hence $\sigma_S(Ax) \subseteq \sigma_T(x)$. Suppose that T and S are asymptotically quasisimilar. Then by the above result and symmetry we conclude that $\sigma_T(x) = \sigma_S(x)$ for every $x \in X$. □

THEOREM 2.6. *Let $T \in \mathcal{L}(X)$ and let $S \in \mathcal{L}(Y)$ be a subdecomposable operator. Suppose that $A : X \longrightarrow Y$ is a bounded linear operator with dense range such that $C(S,T)A = 0$. Then $\sigma(S) \subseteq \sigma(T)$.*

Proof. It is clear that $\sigma_{sur}(T)$ is closed and equal to the union of the local spectra $\sigma_T(x)$ for all $x \in X$. This implies that $X = X_T(\sigma_{sur}(T))$. It follows from proposition 2.5 that

$$Y = (AX)^- = (AX_T(\sigma_{sur}(T)))^- \subseteq Y_S(\sigma_{sur}(T))^- = Y_S(\sigma_{sur}(T)),$$

since S has property (C). Also we have,

$$\sigma_{sur}(S) = \sigma_{sur}(S|Y_S(\sigma_{sur}(T))) \subseteq \sigma_{sur}(T).$$

Hence we conclude that

$$\sigma(S) = \sigma_{sur}(S) \subseteq \sigma_{sur}(T) \subseteq \sigma(T).$$

□

THEOREM 2.7. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be two subdecomposable operators. Suppose that T and S are asymptotically quasisimilar. Then $\sigma(T) = \sigma(S)$.*

Proof. Assume that T and S are asymptotically quasisimilar. Then there are $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$, both injective and with dense range, for which $\|C(S,T)^n A\|^{\frac{1}{n}} \to 0$ and $\|C(T,S)^n B\|^{\frac{1}{n}} \to 0$ as $n \to \infty$. By Proposition 2.5, we have,

$$AX_T(F) \subseteq Y_S(F) \quad \text{and} \quad BY_S(F) \subseteq X_T(F)$$

hold for all closed $F \subseteq \mathbb{C}$. Let $y \in Y$. Then $\sigma_S(y) \subseteq \sigma_{sur}(S)$ and so $y \in Y_S(\sigma_{sur}(S))$. This implies that $Y = Y_S(\sigma_{sur}(S))$. Since T has property (C) we have,

$$X = (BY)^- = (BY_S(\sigma_{sur}(S)))^- \subseteq X_T(\sigma_{sur}(S))^- = X_T(\sigma_{sur}(S)).$$

Therefore we have,

$$\sigma(T) = \sigma_{sur}(T) \subseteq \sigma_{sur}(S) \subseteq \sigma(S).$$

Similarly, we have $\sigma(S) \subseteq \sigma(T)$. □

Corollary 2.8. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be two subdecomposable operators. Suppose that T and S are quasisimilar. Then $\sigma(T) = \sigma(S)$.*

Corollary 2.9. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be two decomposable operators. Suppose that T and S are quasisimilar. Then $\sigma(T) = \sigma(S)$.*

Let T be a bounded linear operator on a Hilbert space over the complex plane $\mathbb{C}$. Then T is said to be an *M-hyponormal operator* if there is a constant $M > 0$ so that $\|T^*x\| \leq M\|Tx\|$ for every $x \in H$. Eschmeier and Putinar proved that every M-hyponormal operator is similar to a subdecomposable operator [3].

Corollary 2.10. *Let H and K be Hilbert spaces. And let $T \in \mathcal{L}(H)$ and $S \in \mathcal{L}(K)$ be two M-hyponormal operators. Suppose that T and S are quasisimilar. Then $\sigma(T) = \sigma(S)$.*

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