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MULTIPLE SOLUTIONS FOR THE NONLINEAR PARABOLIC PROBLEM

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Abstract. We investigate the multiple solutions for the nonlinear parabolic boundary value problem with jumping nonlinearity crossing two eigenvalues. We show the existence of at least four nontrivial periodic solutions for the parabolic boundary value problem. We restrict ourselves to the real Hilbert space and obtain this result by the geometry of the mapping.

1. Introduction

Let Ω be a bounded, connected open subset of R^n with smooth boundary $\partial\Omega$ and let Δ be the Laplace operator. In this paper we consider the multiplicity of the solutions of the following parabolic boundary value problem

$$
\begin{aligned}
D_t u = \Delta u + bu^+ - au^- - s\phi_1 \qquad &\text{in } \ \Omega \times R, \qquad (1.1)\\
u(x,t) = 0, \qquad &x \in \partial\Omega,\ t \in R,\\
u(x,t) = u(x,t+2\pi), \qquad &\text{in } \ \Omega \times R.
\end{aligned}
$$

The physical model for this kind of the jumping nonlinearity problem can be furnished by traveling waves in suspension bridges. The nonlinear equations with jumping nonlinearity have been extensively studied by McKenna and Walter [8], Tarantello [14], Micheletti and Pistoia [10,11] and many the other authors. Tarantello, Micheletti and Pistoia dealt with the biharmonic equations with jumping nonlinearity and proved the existence of nontrivial solutions by degree theory and critical points theory. Lazer and McKenna [7] dealt with the one dimensional elliptic

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equation with jumping nonlinearity for the existence of nontrivial solutions by the global bifurcation method. For the multiplicity results of the solutions of the nonlinear parabolic problem we refer to [6, 9].

The purpose of this paper is to find the number of weak solutions of (1.1)

The steady-state case of (1.1) is the elliptic problem

$$\Delta w + bw^+ - aw^- - s\phi_1 = 0 \qquad \text{in } \ \Omega, \tag{1.2}$$

$$w = 0 \qquad \text{on } \ \partial\Omega.$$

For the multiplicity results for the solutions of (1.2) we refer to [9].

We observe that $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \to \infty$ are the eigenvalues of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, $u|_{\partial\Omega} = 0$ and ϕ_k is the eigenfunction corresponding to the eigenvalue λ_k for each k. We note that the first eigenfunction $\phi_1(x) > 0$.

The main results are the following:

THEOREM 1.1. *Assume that $a < \lambda_1 < \lambda_2 < b < \lambda_3$ and $s > 0$. Then (1.1) has at least four periodic solutions.*

Generally we have the following result:

THEOREM 1.2. *Assume that $\lambda_n < a < \lambda_{n+1} < \lambda_{n+2} < b < \lambda_{n+3}$, $n \geq 0$, and $s > 0$. Then (1.1) has at least four periodic solutions.*

For the proof of Theorem 1.1 and Theoem 1.2 we use the variational reduction method. The organization of this paper is the folllowing: In section 2 we introduce the Hilbert space H whose elements are expressed by the square integrable Fourier series expansions on $\Omega \times (0, 2\pi)$, consider the parabolic problem (1.2) on H and obtain some results on the operator $D_t - \Delta$. In section 3 we prove Theorem 1.1 and Theorem 1.2.

2. Parabolic problem on the Hilbert space H

Let Q be the space $\Omega \times (0, 2\pi)$. The space $L_2(\Omega \times (0, 2\pi))$ is a Hilbert space equipped with the usual inner product

$$< v, w >= \int_0^{2\pi} \int_\Omega v(x,t)\bar{w}(x,t)dxdt$$

and a norm

$$\|v\|_{L_2(Q)} = \sqrt{< v, v >}.$$

We shall work first in the complex space $L_2(\Omega \times (0, 2\pi))$ but shall later switch to the real space. The functions

$$\Phi_{jk}(x,t) = \phi_k \frac{e^{ijt}}{\sqrt{2\pi}}, \qquad j = 0, \pm 1, \pm 2, \ldots, \; k = 1, 2, 3, \ldots$$

form a complete orthonormal basis in $L_2(\Omega \times (0, 2\pi))$. Every elements $v \in L_2(\Omega \times (0, 2\pi))$ has a Fourier expansion

$$v = \sum_{jk} v_{jk} \Phi_{jk}$$

with $\sum |v_{jk}|^2 < \infty$ and $v_{jk} =< v, \Phi_{jk} >$. Let us define a subspace H of $L_2(\Omega \times (0, 2\pi))$ as

$$H = \{u \in L_2(\Omega \times (0, 2\pi)) | \sum_{jk} (j^2 + \lambda_k^2)^{\frac{1}{2}} u_{jk}^2 < \infty\}. \tag{2.1}$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum_{jk} (j^2 + \lambda_k^2)^{\frac{1}{2}} u_{jk}^2]^{\frac{1}{2}}.$$

A weak solution of problem (1.1) is of the form $u = \sum u_{jk} \Phi_{jk}$ satisfying $\sum |u_{jk}|^2 (j^2 + \lambda_k^2)^{\frac{1}{2}} < \infty$, which implies $u \in H$. Thus we have that if u is a weak solution of (1.1), then $u_t = D_t u = \sum_{j\ k} ij u_{jk} \Phi_{jk}$ belong to H and $-\Delta u = \sum \lambda_k u_{jk} \Phi_{jk}$ belong to H.

We have some properties on $\| \cdot \|$ and $D_t - \Delta$. Since $|ij + \lambda_k| \geq 1$ for all j, k, we have that:

LEMMA 2.1. *(i)* $\|u\| \geq \|u(x,0)\| \geq \|u(x,0)\|_{L_2(\Omega)}$.
(ii) $\|u\|_{L_2(Q)} = 0$ *if and only if* $\|u\| = 0$.
(iii) $u_t - \Delta u \in H$ *implies* $u \in H$.

Proof. (i) Let $u = \sum_{j\ k} u_{jk} \Phi_{jk}$. Then

$$\|u\|^2 = \sum (j^2 + \lambda_k^2)^{\frac{1}{2}} u_{jk}^2 \geq \sum \lambda_k^2 u_{jk}^2 (x.0) = \|u(x.0)\|^2$$

$$\geq \sum u_{jk}^2 (x,0) = \|u(x,0)\|_{L_2(\Omega)}^2.$$

(ii) Let $u = \sum_{j\ k} u_{jk} \Phi_{jk}$.

$$\|u\| = 0 \Leftrightarrow \sum_{j\ k} (j^2 + \lambda_k^2)^{\frac{1}{2}} u_{jk}^2 = 0 \Leftrightarrow \sum_{j\ k} u_{jk}^2 = 0 \Leftrightarrow \|u\|_{L_2(Q)} = 0.$$

(iii) Let $u_t - \Delta u = f \in H$. Then f can be expressed by

$$f = \sum f_{jk}\Phi_{jk}, \qquad \sum_{j\ k}(j^2+\lambda_k^2)^{\frac{1}{2}} f_{jk}^2 < \infty.$$

Then we have

$$\|(D_t-\Delta)^{-1}f\|^2 = \sum_{j\ k} \frac{(j^2+\lambda_k^2)^{\frac{1}{2}}}{j^2+\lambda_k^2} f_{jk}^2 < C\sum_{j\ k} f_{jk}^2 < \infty$$

for some $C > 0$. □

Lemma 2.2. *For any real $\alpha \neq \lambda_k$, the operator $(D_t - \Delta - \alpha)^{-1}$ is linear, self-adjoint, and a compact operator from $L_2(\Omega \times (0, 2\pi))$ to H with the operator norm $\frac{1}{|\alpha-\lambda_k|}$, where λ_k is an eigenvalue of $-\Delta$ closest to α.*

Proof. Suppose that $\alpha \neq \lambda_k$. Since $\lambda_k \to +\infty$, the number of elements in the set $\{\lambda_k | \ \lambda_k < \alpha\}$ is finite, where λ_k is an eigenvalue of $-\Delta$. Let $h = \sum_{j\ k} h_{jk}\Phi_{jk}$, where $\Phi_{jk} = \phi_k \frac{e^{ijt}}{\sqrt{2\pi}}$. Then

$$(D_t - \Delta - \alpha)^{-1}h = \sum_{j\ k} \frac{1}{im + \lambda_n - \alpha} h_{jk}\Phi_{jk}.$$

Hence

$$\|(D_t - \Delta - \alpha)^{-1}h\|^2 = \sum_{j\ k} \frac{1}{j^2 + (\lambda_k-\alpha)^2}(j^2+(\lambda_k-\alpha)^2)^{\frac{1}{2}} h_{jk}^2$$

$$\leq \sum_{j\ k} C h_{jk}^2 < \infty$$

for some $C > 0$. Thus $(D_t - \Delta - \alpha)^{-1}$ is a bounded operator from $L_2(\Omega \times (0, 2\pi))$ to H and also send bounded subset of $L_2(\Omega \times (0, 2\pi))$ to a compact subset of H, hence $(D_t - \Delta - \alpha)^{-1}$ is a compact operator. □

From Lemma 2.2 we obtain the following lemma:

Lemma 2.3. *Let $F(x, t, u) \in L_2(\Omega\times(0, 2\pi))$. Then all the solutions of*

$$u_t - \Delta u = F(x, t, u) \qquad \text{in } L_2(\Omega \times (0, 2\pi))$$

belong to H.

With the aid of Lemma 2.3 it is enough to investigate the existence of solutions of (1.1) in the subspace H of $L_2(\Omega \times (0, 2\pi))$, namely

$$D_t u = \Delta u + bu^+ - au^- - s\phi_1 \qquad \text{in } H. \tag{2.2}$$

From now on we restrict ourselves to the real L_2-space and observe that this is an invariant space for R. So $L_2(\Omega \times (0, 2\pi))$ denotes the real square-integrable functions on $\Omega \times (0, 2\pi)$ and H the subspace of $L_2(\Omega \times (0, 2\pi))$ satisfying (2.1).

3. Proof of Theorem 1.1 and Theorem 1.2

Assume that $a < \lambda_1 < \lambda_2 < b < \lambda_3$ and $s > 0$. We shall use the contraction mapping theorem to reduce the problem from an infinite dimensional one in $L_2(Q)$ to a finite dimensional one.

Let V be the two dimensional subspace of H spanned by $\Phi_{01}(x)$ and $\Phi_{02}(x)$ and W the subspace spanned by Φ_{0n}, $n \geq 3$ and Φ^c_{mn}, Φ^s_{mn}, $m \geq 1$. Then W is the orthogonal complement of V in H.

From now on we restrict ourselves to the real L_2-space and observe that this is an invariant space for R. So $L_2(\Omega \times (0, 2\pi))$ denotes the real square-integrable functions on $\Omega \times (0, 2\pi)$ and H the subspace of $L_2(\Omega \times (0, 2\pi))$ satisfying (2.1). Let P be an orthogonal projection from H onto V. Then for all $u \in H$, $u = v + w$, where $v = Pu$, $w = (I - P)u$.

Therefore (2.2) is equivalent to

$$\begin{aligned} &(a) \qquad w = (D_t - \Delta)^{-1}(I - P)(b(v + w)^+ - a(v + w)^{-1}), \\ &(b) \qquad D_t v = \Delta v + P(b(v + w)^+ - a(v + w)^- - s\phi_1), \end{aligned} \tag{3.1}$$

where $D_t = \frac{\partial}{\partial t}$.

Let us show that for fixed v, (3.1.a) has a unique solution $w = \theta(v)$ and that $\theta(v)$ is Lipschitz continuous in terms of v. Let σ be the spectrum of $D_t - \Delta$. Then $\sigma = \{\lambda_n \pm im |\, n \geq 1,\ m \geq 0\}$. Let $\alpha = \frac{1}{2}(\lambda_1 + \lambda_2)$. We rewrite (3.1.a) as

$$(D_t - \Delta - \alpha)w = (I - P)(b(v + w)^+ - a(v + w)^{-1} - \alpha(v + w))$$

or

$$w = (D_t - \Delta - \alpha)^{-1}(I - P)g_v(w) \tag{3.2}$$

where

$$g_v(w) = b(v + w)^+ - a(v + w)^{-1} - \alpha(v + w).$$

Since

$$\begin{aligned} |g_v(w_1) - g_v(w_2)| &\leq \max\{|b - \alpha|, |a - \alpha|\}|w_2 - w_1|, \\ \||g_v(w_1) - g_v(w_2)\|| &\leq \max\{|b - \alpha|, |a - \alpha|\}\||w_2 - w_1\||, \end{aligned}$$

where $\|\cdot\|$ is the norm in H. Since the operator $(D_t - \alpha)^{-1}(I-P)$ is a self-adjoint, compact linear map from $(I-P)H$ onto itself, it follows that

$$\|(D_t - \Delta - \alpha I)^{-1}(I-P)\| = \mathrm{dist}(\alpha, \{(\lambda_n \pm im - \alpha)^{-1} |\ m \geq 0,\ n \geq 2\}).$$

Therefore for fixed $v \in V$, the right hand side of (3.2) defines a Lipschitz mapping $(I-P)H$ into itself with Lipschitz constant $\gamma < 1$. Therefore by the contraction mapping principle, for given $v \in V$, there exists a unique $w = \theta(v) \in W$ which satisfies (3.2). it follows that, by the standard argument principle, $\theta(v)$ is Lipschitz continuous in terms of v.

Thus we have a reduced equation (2.2) to the equivalent equation

$$D_t v = \Delta v + P(b(v+\theta(v))^+ - a(v+\theta(v))^- - s\phi_1) \tag{3.3}$$

defined on the two dimensional subspace PH spanned by $\{\Phi_{01}(x), \Phi_{02}(x)\}$.

We note that if $v \geq 0$ or $v \leq 0$, then $\theta(v) = 0$. If we put $v \geq 0$ ($v \leq 0$) and $\theta(v) = 0$ in (3.1.a), equation (3.1.a) is satisfied, respectively. Since $v = c_1\Phi_{01} + c_2\Phi_{02}$, there exists a cone C_1 defined by $c_1 \geq 0$, $|c_2| \leq \epsilon_0 c_1$ so that $v \geq 0$ for all $v \in C_1$ and a cone C_2, $c \leq 0$, $|c_2| \leq \epsilon_0|c_1|$ so that $v \leq 0$ for all $v \in C_2$. We know that $w = \theta(v) = 0$ for $v \in C_1 \cup C_2$, but we do not know $\theta(v)$ for all $v \in PH$. We consider the map

$$v \mapsto T(v) = -D_t v + \Delta v + P(b(v+\theta(v))^+ - a(v+\theta(v))^-).$$

First we consider the image of the cone C_1. If $v = c_1\Phi_{01} + c_2\Phi_{02}$, we have that

$$\begin{aligned} T(v) &= -\lambda_1 c_1 \Phi_{01} - \lambda_2 C_2 \Phi_{02} + b(c_1\Phi_{01} + c_2\Phi_{02}) \\ &= (\lambda_1 - b)c_1\Phi_{01} + (\lambda_2 - b)c_2\Phi_{02}. \end{aligned}$$

Thus the image of the rays $c_1\Phi_{01} \pm \epsilon_0 c_1 \Phi_{02}$ are

$$(\lambda_1 - b)c_1\Phi_{01} + (\lambda_2 - b)\epsilon_0 c_1 \Phi_{02}$$

or the rays

$$d_1\Phi_{01} \pm \epsilon_0(\frac{\lambda_2 - b}{\lambda_1 - b})d_1\Phi_{02}.$$

Thus T maps C_1 into the cone

$$D_1 = \{d_1\Phi_{01} + d_2\Phi_{02} | d_1 \geq 0,\ |d_2| \leq \epsilon_0(\frac{b-\lambda_2}{b-\lambda_1})\}.$$

Similary for C_2 we can calculate the image under T. If $c_1 \leq 0$,

$$T(c_1\Phi_{01} \pm \epsilon_0 c_1 \Phi_{02}) = (a - \lambda_1)c_1\Phi_{01} \pm (a - \lambda_2)\epsilon_0 c_1 \Phi_{02}.$$

Thus $T(v) = s\phi_1$ has one solution in each of the cones C_1, C_2, namely $\frac{s\Phi_{01}}{b-\lambda_1}$, $\frac{s\Phi_{01}}{a-\lambda_1}$. Now we need a lemma.

LEMMA 3.1. *There exists $d > 0$ so that*

$$(T(c_1\Phi_{01} + c_2\Phi_{02}), \Phi_{01}) \geq d|c_2|.$$

Proof. By the definition of $T(v)$,

$$\begin{aligned} T(c_1\Phi_{01} + c_2\Phi_{02}) = &(-D_t + \Delta)(c_1\Phi_{01} + c_2\Phi_{02}) \\ &+P(b(c_1\Phi_{01} + c_2\Phi_{02} + \theta(c_1\Phi_{01} + c_2\Phi_{02}))^+ \\ &-a(c_1+\Phi_{01}+c_2\Phi_{02}+\theta(c_1\Phi_{01}+c_2\Phi_{02}))^-). \end{aligned}$$

So if $u = c_1\Phi_{01} + c_2\Phi_{02} + \theta(c_1\Phi_{01} + c_2\Phi_{02})$, then

$$\begin{aligned} (T(c_1\Phi_{01} + c_2\Phi_{02}), \Phi_{01}) = &((-D_t + \Delta + \lambda_1)(c_1\Phi_{01} + c_2\Phi_{02}), \Phi_{01}) \\ &+(bu^+ - au^- - \lambda_1 u, \Phi_{01}). \end{aligned}$$

The first term is zero because $(-D_t + \Delta + \lambda_1)\Phi_{01} = 0$ and $-D_t + \Delta$ is self-adjoint. The second term satisfies $bu^+ - au^- - \lambda_1 u \geq \gamma|u|$, where $\gamma = \min\{b - \lambda_1, \lambda_1 - a\} > 0$. Therefore $(T(c_1\Phi_{01} + c_2\Phi_{02}), \Phi_{01}) \geq \gamma \int |u|\Phi_{01}$. Now there exists $d > 0$ so that $\gamma\Phi_{01} \geq d|\Phi_{02}|$ and therefore

$$\gamma \int |u|\Phi_{01} \geq d \int |u||\Phi_{02}| \geq d|\int u\Phi_{02}| = d|(u, \Phi_{02})|.$$

Thus we prove the lemma. □

We shall describe the behavior of T in the complement of the two cases C_1 and C_2. Let us consider the image under T of $c_1\Phi_{01} + c_2\Phi_{02}$ with $c_2 \geq \epsilon|c_1|$, $c_2 = l$ for some $l > 0$. By Lemma 3.1, the image $T(L)$ of $c_2 = l$, $|c_1| \leq \frac{1}{\epsilon}l$ must lie to the right of the line $c_1 = dl$ and must cross the positive Φ_{01} axis in the image space. Thus we have shown that if $u = c_1\Phi_{01} + l\Phi_{02} + \theta(c_1\Phi_{01} + l\Phi_{02})$, $l > 0$, $|c_1| \leq \frac{l}{\epsilon}$. Then u satisfies, for some c_1, $-D_t u + \Delta u + bu^+ - au^- = s\phi_1$ for some $s > dl$ and $l > 0$. Letting $\tilde{u} = \frac{t}{s}u$, we see that $\tilde{u}$ satisfies

$$(-D_t + \Delta)\tilde{u} + b\tilde{u} - a\tilde{u} = t\phi_1.$$

Similarly we can show the existence of another solution $\check{u}$ satisfying

$$-D_t\check{u} + \Delta\check{u} + b\check{u}^+ - a\check{u}^- = t\phi_1$$

with $(\check{u}, \Phi_{02}) < 0$. Thus we have four solutions, one in each of the four cones, where C_1, C_2 divide the Φ_{01}, Φ_{02} plane into. We prove Theorem 1.1. For the proof of Theorem 1.2 we set V be the two dimensional subspace of H spanned by $\Phi_{0n+1}(x)$ and $\Phi_{0n+2}(x)$ and W the subspace spanned by Φ_{0n}, Φ_{0n+3}, $n \geq 1$ and Φ^c_{mn}, Φ^s_{mn}, $m \geq 1$. Then W is the orthogonal complement of V in H. The other parts of the proof of Theorem 1.2 have the similar process to that of Theorem 1.1. □

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