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DIMENSIONALLY INVARIANT SPACES

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Abstract. We consider a code function from the unit interval which has a generalized dyadic expansion into a coding space which has an associated ultra metric. The code function is not a bi-Lipschitz map but a dimension-preserving map in the sense that the Hausdorff and packing dimensions of any subset in the unit interval and its image under the code function coincide respectively.

1. Introduction

Each point which has a generalized dyadic expansion in the unit interval has its own code in a coding space. We provide the coding space with a generalized ultra metric. Considering a code function from the unit interval with the Euclidean metric into the coding space with an ultra metric, we([5]) showed the unit interval and the ultra metric space are dimensionally equivalent in the sense that the Hausdorff and packing dimensions of the corresponding distribution sets in the two spaces $[0, 1)$ and the ultra metric space coincide. In [5], we conjectured that the code function is a dimension-preserving map even though it is not a bi-Lipschitz map. In this paper, we show that the code function f is a dimension preserving map in the sense that $\dim(E) = \dim(f(E))$ and $\mathrm{Dim}(E) = \mathrm{Dim}(f(E))$ for $E \subset [0, 1)$ denoting $\dim(E)$ by the Hausdorff dimension of E and $\mathrm{Dim}(E)$ by the packing dimension of E([9]). In this sense, we say that the unit interval and the ultra metric space are dimensionally invariant spaces. For this, we compare the definitions of Hausdorff measure and packing measure in the Euclidean space and those in the ultra metric space. For this comparison, we use the definitions of g-Hausdorff measure and g-packing measure in the Euclidean space where g is a complete bounded Vitali covering system([10]) instead

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of those of Hausdorff measure and packing measure in the Euclidean space(cf. [7]). We also show that the definitions of g-Hausdorff measure and g-packing measure in the ultra metric space where g is a complete bounded Vitali covering system([10]) coincide with those of Hausdorff measure and packing measure in the ultra metric space respectively.

2. Preliminaries

We([4]) recall F_a *the unit interval* $[0, 1)$ *having a generalized dyadic expansion with a base* a where $0 < a < 1$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}$ be the set of real numbers. We define a fundamental interval $J_{i_1 \cdots i_k} = f_{i_1} \circ \cdots \circ f_{i_k}(J)$ where $f_0(x) = ax$ and $f_1(x) = (1-a)x + a$ on $J = [0, 1)$, $i_j \in \{0, 1\}$ and $1 \le j \le k$. If $x \in F_a = [0, 1)$, then there is a unique code $\sigma \in \{0, 1\}^{\mathbb{N}}$ such that $\bigcap_{k=1}^{\infty} J_{\sigma|k} = \{x\}$ (Here $\sigma|k = i_1 i_2 \cdots i_k$ where $\sigma = i_1 i_2 \cdots i_k i_{k+1} \cdots$). We call a code $\sigma \in \{0, 1\}^{\mathbb{N}}$ where $\bigcap_{k=1}^{\infty} J_{\sigma|k} = \{x\}$ *a generalized dyadic expansion with a base* a of x. We note that the fundamental intervals $J_{i_1 \cdots i_k}$ where $x \in J_{i_1 \cdots i_k}$ is a k-cylinder $c_k(x)$([6]).

Let

$$f : F_a \longrightarrow \{0, 1\}^{\mathbb{N}}$$

be a function such that $f(x) = \sigma$ with $\{x\} = \bigcap_{k=0}^{\infty} J_{\sigma|k}$ where $\sigma \in \{0, 1\}^{\mathbb{N}}$. Then f is a code function in Introduction. Further $f(F_a) \subset \{0, 1\}^{\mathbb{N}}$(that is, $\{0, 1\}^{\mathbb{N}} - f(F_a) \neq \phi$).

We denote $n_0(\sigma|k)$ the number of times the digit 0 occurs in the first k places of σ(cf. [1]). Now, for $q \in [0, 1]$, we define the lower(upper) distribution set $\underline{F}(q)(\overline{F}(q))$ in $F_a = [0, 1)$ containing the digit 0 in proportion q by

$$\underline{F}(q) = \{x \in F_a : \liminf_{k \to \infty} \frac{n_0(f(x)|k)}{k} = q\},$$

$$\overline{F}(q) = \{x \in F_a : \limsup_{k \to \infty} \frac{n_0(f(x)|k)}{k} = q\}.$$

We write $\underline{F}(q) \cap \overline{F}(q) = F(q)$ and call it the distribution set in $[0, 1)$ containing the digit 0 in proportion q.

Similarly, for $q \in [0, 1]$, we define the lower(upper) distribution set $\underline{\Im}(q)(\overline{\Im}(q))$ in $\{0, 1\}^{\mathbb{N}}$ containing the digit 0 in proportion q by

$$\underline{\Im}(q) = \{\sigma \in \{0, 1\}^{\mathbb{N}} : \liminf_{k \to \infty} \frac{n_0(\sigma|k)}{k} = q\},$$

$$\overline{\Im}(q) = \{\sigma \in \{0,1\}^{\mathbb{N}} : \limsup_{k\to\infty} \frac{n_0(\sigma|k)}{k} = q\}.$$

We write $\underline{\Im}(q) \cap \overline{\Im}(q) = \Im(q)$ and call it the distribution set in $\{0,1\}^{\mathbb{N}}$ containing the digit 0 in proportion q.

We recall a coding space $\{0,1\}^{\mathbb{N}}$ with a generalized ultra metric $\rho_{x,y}$([2]) such that for $(x,y) \in \{(x,y)|0 < x,y < 1\}$, $\rho_{x,y}(\sigma,\sigma) = 0$ and if $\sigma \neq \tau$ then $\rho_{x,y}(\sigma,\tau) = x^{n_0(x|k)}y^{k-n_0(x|k)}$ where $\sigma = i_1 i_2 \cdots i_k i_{k+1} \cdots$ and $\tau = i_1 i_2 \cdots i_k j_{k+1} \cdots$ where $i_{k+1} \neq j_{k+1}$ for some $k = 0,1,2\cdots$.

Before going into our main results, we need some useful lemma. In this paper, the domain on which a function will be defined is a subset of $\mathbb{R}$ with the usual metric. We note that covering family used for the definition of Hausdorff measure on $\mathbb{R}$ is the family of the intervals on $\mathbb{R}$. In [2], it was shown that a covering family g of some intervals can be used instead of the covering family of the intervals on $\mathbb{R}$ giving $\dim_g$ which is the same value as the Hausdorff dimension dim. The following lemma gives g-Hausdorff dimension $\dim_g$ for a complete bounded Vitali covering([6]) g of $[0,1)$ is the same as Hausdorff dimension dim. Similarly it holds for the packing dimension Dim and the g-packing dimension Dim_g for a bounded Vitali covering([10]) g of $[0,1)$.

Lemma 2.1. *Let g be the family of the fundamental intervals $J_{i_1\cdots i_k}$ where $i_j \in \{0,1\}$ and $1 \le j \le k$ with $k \in \mathbb{N}$. Then* $\dim_g(E) = \dim(E)$ *and* $\mathrm{Dim}_g(E) = \mathrm{Dim}(E)$ *for all $E \subset [0,1)$.*

Proof. It is not difficult to show that the family of the fundamental intervals $J_{i_1\cdots i_k}$ where $i_j \in \{0,1\}$ and $1 \le j \le k$ is a complete bounded Vitali covering of $[0,1)$. It follows from the theorem 2.2 in [6] and the theorem 3.1 in [10]. □

3. Main results

From now on, let Γ be the family of the fundamental intervals $I_{i_1\cdots i_k} = i_1 \cdots i_k \times \{0,1\}^{\mathbb{N}} \subset \{0,1\}^{\mathbb{N}}$ where $i_j \in \{0,1\}$ and $1 \le j \le k$ with $k \in \mathbb{N}$. We denote H^s and p^s by s-dimensional Hausdorff measure and s-dimensional packing measure respectively on the ultra metric space $\{0,1\}^{\mathbb{N}}$, and H^s_Γ and p^s_Γ by s-dimensional Γ-Hausdorff measure and s-dimensional Γ-packing measure respectively on the ultra metric space $\{0,1\}^{\mathbb{N}}$. We also denote P^s and P^s_Γ by s-dimensional pre-packing measure and s-dimensional Γ-pre-packing measure respectively on the ultra metric space $\{0,1\}^{\mathbb{N}}$.

THEOREM 3.1. *$H^s_\Gamma(D) = H^s(D)$ and $p^s_\Gamma(D) = p^s(D)$ for all $D \subset \{0,1\}^{\mathbb{N}}$.*

Proof. Let $D \subset \{0,1\}^{\mathbb{N}}$. It is clear that $H^s_\Gamma(D) \geq H^s(D)$ from $\Gamma \subset 2^{\{0,1\}^{\mathbb{N}}}$. For each $B \subset \{0,1\}^{\mathbb{N}}$, there is $C_B \in \Gamma$ such that $B \subset C_B$ and $|B| = |C_B|$, which gives $H^s_\Gamma(D) \leq H^s(D)$.
If $\{B_i\}_{i=1}^\infty$ is disjoint open balls with centers in D, then $\{B_i\}_{i=1}^\infty$ is disjoint fundamental intervals with $D \cap B_i \neq \phi$, which gives $P^s(D) = P^s_\Gamma(D)$. This also gives $p^s(D) = p^s_\Gamma(D)$. □

REMARK 3.2. It is helpful for understanding the s-dimensional Γ-Hausdorff measure and the s-dimensional Γ-packing measure above to refer the one-dimensional Hausdorff measure and one-dimensional packing measure defined in a coding space with an ultra metric in [8] and their comparison with a net measure.

The above Theorem gives the following Theorem. Similarly in Preliminaries, we can consider Hausdorff dimension, Γ-Hausdorff dimension $\dim, \dim_\Gamma$ and packing dimension, Γ-packing dimension $\mathrm{Dim}, \mathrm{Dim}_\Gamma$ in $\{0,1\}^{\mathbb{N}}$.

THEOREM 3.3. $\dim_\Gamma(D) = \dim(D)$ *and* $\mathrm{Dim}_\Gamma(D) = \mathrm{Dim}(D)$ *for all* $D \subset \{0,1\}^{\mathbb{N}}$.

Proof. It is immediate from the above Theorem. □

From now on, we consider g as the family of the fundamental intervals $J_{i_1 \cdots i_k}$ where $i_j \in \{0,1\}$ and $1 \leq j \leq k$ with $k \in \mathbb{N}$. We fix a code function f

$$f : F_a \longrightarrow \{0,1\}^{\mathbb{N}}$$

where $\{0,1\}^{\mathbb{N}}$ is a coding space with a generalized ultra metric $\rho_{a,1-a}$.

THEOREM 3.4. $f(J_{i_1 \cdots i_k}) = \{\sigma \in \{0,1\}^{\mathbb{N}} : \sigma|k = i_1 \cdots i_k\}$
$-\{i_1 \cdots i_k 11 \cdots\}$ *where* $\{i_1 \cdots i_k 11 \cdots\} = i_1 \cdots i_k \times \{1\}^{\mathbb{N}}$ *and*

$$|f(J_{i_1 \cdots i_k})| = |J_{i_1 \cdots i_k}|.$$

Proof. We note that $\bigcap_{k=1}^\infty J_{\sigma|k} = \phi$ for $\sigma = i_1 \cdots i_k 11 \cdots$. It follows from the definition of the ultra metric and the fact that every $x \in J_{i_1 \cdots i_k}$ has its code $f(x)$ and $f(x)|k = i_1 \cdots i_k$. □

The end points of the fundamental intervals are at most countable, which gives the following theorem.

THEOREM 3.5. $H^s_g(E) = H^s_\Gamma(f(E))$ and $p^s_g(E) = p^s_\Gamma(f(E))$.

Proof. We note that $|f(G)| = |\overline{f(G)}|$ and the closure $\overline{f(G)}$ of $f(G)$ is in Γ where $G \in g$. It follows from the above Theorem with that

$$E \subset \cup G_n \Leftrightarrow f(E) \subset \cup \overline{f(G_n)}$$

where $G_n \in g$. □

COROLLARY 3.6. $\dim_g(E) = \dim_\Gamma(f(E))$ *and* $\mathrm{Dim}_g(E) = \mathrm{Dim}_\Gamma(f(E))$ *for all* $E \subset [0,1)$.

Proof. It is immediate from the above Theorem. □

The following corollary is our main result.

COROLLARY 3.7. $\dim(E) = \dim(f(E))$ *and* $\mathrm{Dim}(E) = \mathrm{Dim}(f(E))$ *for all* $E \subset [0,1)$.

Proof. It is immediate from Lemma 1 and Theorem 2 and the above Corollary. □

REMARK 3.8. We note that the code function f is not a bi-Lipschitz map but a dimension-preserving map.

REMARK 3.9.

$$\dim(\underline{F}(q)) = \dim(\underline{\Im}(q))$$

and

$$\dim(\overline{F}(q)) = \dim(\overline{\Im}(q)),$$

and

$$\mathrm{Dim}(\underline{F}(q)) = \mathrm{Dim}(\underline{\Im}(q))$$

and

$$\mathrm{Dim}(\overline{F}(q)) = \mathrm{Dim}(\overline{\Im}(q))$$

([5],cf.[3, 4]) since $\underline{\Im}(q) = f(\underline{F}(q))$ and $\overline{\Im}(q) = f(\overline{F}(q)))$ for each $q \in (0,1)$ and $\underline{\Im}(q) - f(\underline{F}(q))(\neq \phi)$ or $\overline{\Im}(q) - f(\overline{F}(q)))(\neq \phi)$ is a countable subset in $\{0,1\}^{\mathbb{N}}$ for $q = 0$ or 1(cf. [11]).

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