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INTUITIONISTIC FUZZY SEMIPRIME IDEALS OF ORDERED SEMIGROUPS

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Abstract. In this paper, we introduce the notion of intuitionistic fuzzy semiprimality in an ordered semigroup, which is an extension of fuzzy semiprimality and investigate some properties of intuitionistic fuzzification of the concept of several ideals.

1. Introduction

After the introduction of fuzzy sets by L. A. Zadeh [6], several researches were conducted on the generalizations of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by K. T. Atanassov [1], as a generalization of the notion of fuzzy set. In [2], N. Kuroki gave some properties of fuzzy ideals and fuzzy semiprime ideals in semigroups. In this paper, we introduce the notion of intuitionistic fuzzy semiprimality in an ordered semigroup, which is an extension of fuzzy semiprimality and investigate some properties of intuitionistic fuzzification of the concept of several ideals.

2. Preliminaries

We include some elementary aspects of ordered semigroups that are necessary for this paper. In what follows, we use S to denote an ordered semigroup unless otherwise specified.

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By an *ordered semigroup* we mean an ordered set S at the same time a semigroup satisfying the following conditions:

$$(\forall a, b, x \in S)(a \leq b \Rightarrow xa \leq xb \text{ and } ax \leq bx)$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset U of S is called a *subsemigroup* of S if $U^2 \subseteq U$.

A non-empty subset A of an ordered semigroup S is called a *left* (resp. *right*) *ideal* of S if it satisfies:

- $SA \subseteq A$ (resp. $AS \subseteq A$),
- $(\forall a \in A)(\forall b \in S)(b \leq a \Rightarrow b \in A)$.

Both a left and a right ideal of S is said to be *ideal* of S.

Definition 2.1. Let $(S, \cdot, \leq)$ bean ordered semigroup and $\emptyset \neq T \subseteq S$. Then T is called *prime* if $xy \in T \Rightarrow x \in T$ or $y \in T$ for all $x, y \in S$. Let T be an ideal of S. If T is prime subset of S, then T is called *prime ideal.*

Definition 2.2. Let $(S, \cdot, \leq)$ bean ordered semigroup and $\emptyset \neq T \subseteq S$. Then T is called *semiprime* if $a^2 \in T \Rightarrow a \in T$ for all $a \in S$. Let T be an ideal of S. If T is semiprime subset of S, then T is called *semiprime ideal.*

A mapping $\mu : S \to [0, 1]$, where S is an arbitrary non-empty set, is called a *fuzzy set* in S. The *complement* of μ, denoted by $\tilde{\mu}$, is the fuzzy set in S given by $\tilde{\mu}(x) = 1 - \mu(x)$ for all $x \in S$.

Let $(S, \cdot, \leq)$ bean ordered semigroup. A fuzzy subset μ of S is called a *fuzzy ideal* of S, if the following axioms are satisfied:

- If $x \leq y$, then $\mu(x) \geq \mu(y)$,
- $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$, for all $x, y \in S$.

An intuitionistic fuzzy set (briefly, IFS) A in a nonempty set X is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$$

where the functions $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership and the degree of nonmembership, respectively, and

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$$

for all $x \in X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$.

In what follows, we use S to denote an ordered semigroup unless otherwise specified.

Definition 2.3. For an IFS $A = (\mu_A, \gamma_A)$ in S, consider the following axioms:
(IS_1) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$,
(IS_2) $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}, \forall x, y \in S$.
Then $A = (\mu_A, \gamma_A)$ is called an *intuitionistic fuzzy subsemigroup* (briefly, IFS) of S if it satisfies (IS_1) and (IS_2).

Definition 2.4. For an IFS $A = (\mu_A, \gamma_A)$ in S, consider the following axioms:
(IL_1) $x \leq y$ implies $\mu_A(x) \geq \mu_A(y)$ and $\mu_A(xy) \geq \mu_A(y)$,
(IL_2) $x \leq y$ implies $\gamma_A(x) \leq \gamma_A(y)$ and $\gamma_A(xy) \leq \gamma_A(y), \forall x, y \in S$.
Then $A = (\mu_A, \gamma_A)$ is called an *intuitionistic fuzzy left ideal* (briefly, $IFLI$) of S if it satisfies (IL_1) and (IL_2).

Definition 2.5. For an IFS $A = (\mu_A, \gamma_A)$ in S, consider the following axioms:
(IR_1) $x \leq y$ implies $\mu_A(x) \geq \mu_A(y)$ and $\mu_A(xy) \geq \mu_A(x)$,
(IR_2) $x \leq y$ implies $\gamma_A(x) \leq \gamma_A(y)$ and $\gamma_A(xy) \leq \gamma_A(x), \forall x, y \in S$.
Then $A = (\mu_A, \gamma_A)$ is called an *intuitionistic fuzzy right ideal* (briefly, $IFRI$) of S if it satisfies (IR_1) and (IR_2). Then $A = (\mu_A, \gamma_A)$ is called an *intuitionistic fuzzy ideal* (briefly, IFI) of S if it is a left ideal and a right ideal.

3. Main results

Definition 3.1. Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset μ of S is called *prime*, if

$$\mu(xy) = \max\{\mu(x), \mu(y)\}, \quad \forall x, y \in S.$$

A fuzzy ideal μ of S is called a *fuzzy prime ideal* of S if μ is a prime fuzzy subset of S.

Definition 3.2. For an IFS $A = (\mu_A, \gamma_A)$ in S, consider the following axioms:
(IP_1) $\mu_A(xy) = \max\{\mu_A(x), \mu_A(y)\}$,
(IP_2) $\gamma_A(xy) = \min\{\gamma_A(x), \gamma_A(y)\}, \forall x, y \in S$.
Then $A = (\mu_A, \gamma_A)$ is called a *first* (resp. *second*) *intuitionistic fuzzy prime* (briefly, IFP_1 (resp. IFP_2)) if it satisfies (IP_1) (resp. (IP_2)). Also, $A = (\mu_A, \gamma_A)$ is said to be an *intuitionistic fuzzy prime* (briefly, IFP) if it is both a first and a second intuitionistic fuzzy prime.

Let χ_U denote the characteristic function of a nonempty subset U of an ordered semigroup.

THEOREM 3.3. *If U is a prime ideal, then $\tilde{U} = (\chi_U, \tilde{\chi_U})$is an IFP of S.*

Proof. Let $x, y \in S$. If $xy \in U$, then $x \in U$ or $y \in U$. Thus $\chi_U(x) = 1$ or $\chi_U(y) = 1$. Thus we have

$$\chi_U(xy) = 1 = \max\{\chi_U(x), \chi_U(y)\}$$

and

$$\tilde{\chi}_U(xy) = 1 - \chi_U(xy) = 0 = \min\{\tilde{\chi}_U(x), \tilde{\chi}_U(y)\}.$$

If $xy \notin U$, then $x \notin U$ and $y \notin U$. Thus $\chi_U(x) = 0$ and $\chi_U(y) = 0$. Thus we have

$$\chi_U(xy) = 0 = \max\{\chi_U(x), \chi_U(y)\}$$

and

$$\tilde{\chi}_U(xy) = 1 - \chi_U(xy) = 1 = \min\{\tilde{\chi}_U(x), \tilde{\chi}_U(y)\}.$$

□

THEOREM 3.4. *Let U be a non-empty subset of S. If $\tilde{U} = (\chi_U, \tilde{\chi_U})$ is IFP_1 or IFP_2 of S, then U is prime.*

Proof. Suppose that $\tilde{U} = (\chi_U, \tilde{\chi_U})$ is an IFP_1 of S and $xy \in U$. In this case, $u = xy$ for some $u \in U$. It follows from (IP_1) that

$$1 = \chi_U(u) = \chi_U(xy) = \max\{\chi_U(x), \chi_U(y)\}.$$

Hence $\chi_U(x) = 1$, or $\chi_U(y) = 1$, i.e. $x \in U$ or $y \in U$. Thus U is prime. Now, assume that $\tilde{U} = (\chi_U, \tilde{\chi_U})$ is an IFP_2 of S and $x'y' \in U$. Then $u' = x'y'$ for some $u' \in U$. Using (IP_2), we get

$$\begin{aligned}
\tilde{\chi}_U(u') &= 1 - \chi_U(u') = 0 = \tilde{\chi}_U(x'y') \\
&= \min\{\tilde{\chi}_U(x'), \tilde{\chi}_U(y')\} \\
&= \min\{1 - \chi_U(x'), 1 - \chi_U(y')\},
\end{aligned}$$

and so $1 - \chi_U(x') = 0$ or $1 - \chi_U(y') = 0$. Therefore $\chi_U(x') = 1$ or $\chi_U(y') = 1$, i.e. $x' \in U$ or $y' \in U$. This completes the proof. □

DEFINITION 3.5. Let μ be a fuzzy subset of an ordered semigroup S. Then μ is called *semiprime* if $\mu(a) \geq \mu(a^2)$, for all $a \in S$. A fuzzy ideal μ of S is called a *fuzzy semiprime ideal* of S if μ is a fuzzy semiprime subset of S.

DEFINITION 3.6. For an IFS $A = (\mu_A, \gamma_A)$ in S, consider the following axioms:

(IS_1) $\mu_A(x) \geq \mu_A(x^2)$,

(IS_2) $\gamma_A(x) \leq \gamma_A(x^2), \forall x \in S$.

Then $A = (\mu_A, \gamma_A)$ is called a *first* (resp. *second*) *intuitionistic fuzzy semiprime* (briefly, $IFSP_1$ (resp. $IFSP_2$)) if it satisfies (IS_1) (resp. (IS_2)). Also, $A = (\mu_A, \gamma_A)$ is said to be *intuitionistic fuzzy semiprime* (briefly, $IFSP$) if it is both a first and a second intuitionistic fuzzy semiprime.

THEOREM 3.7. *If U is semiprime, then $\tilde{U} = (\chi_U, \tilde{\chi_U})$is an $IFSP$ of S.*

Proof. Let a be any element of S. If $a^2 \in U$, then since U is semiprime, we have $a \in U$. Thus

$$\chi_U(a) = 1 \geq \chi_U(a^2)$$

and

$$\bar{\chi}_U(a) = 1 - \chi_U(a) = 0 \leq \bar{\chi}_U(a^2).$$

If $a^2 \notin U$, then we have $\chi_U(a^2) = 0$. Therefore,

$$\chi_U(a) \geq 0 = \chi_U(a^2)$$

and

$$\bar{\chi}_U(a^2) = 1 - \chi_U(a^2) = 1 \geq \bar{\chi}_U(a).$$

This proves the theorem. □

THEOREM 3.8. *Let U be a non-empty subset of S. If $\tilde{U} = (\chi_U, \tilde{\chi_U})$ is $IFSP_1$ or $IFSP_2$ of S, then U is semiprime.*

Proof. Suppose that $\tilde{U} = (\chi_U, \tilde{\chi_U})$ is an $IFSP_1$ of S and $a^2 \in U$. In this case, $u = a^2$ for some $u \in U$. It follows from (IS_1) that

$$1 = \chi_U(u) = \chi_U(a^2) \leq \chi_U(a).$$

Hence $\chi_U(a) = 1$, i.e. $a \in U$. Thus U is semiprime. Now, assume that $\tilde{U} = (\chi_U, \tilde{\chi_U})$ is an $IFSP_2$ of S and $a_0^2 \in U$. Then $u_0 = a_0^2$ for some $u_0 \in U$. Using (IS_2), we get

$$\tilde{\chi}_U(a_0) \leq \tilde{\chi}_U(a_0^2) = 1 - \chi_U(a_0^2) = 1 - 1 = 0,$$

i.e. $\tilde{\chi}_U(a_0) = 1 - \chi_U(a_0) = 0$. Thus $\chi_U(a_0) = 1$, and so $a_0 \in U$. This completes the proof. □

THEOREM 3.9. *For any intuitionistic fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of S, if $A = (\mu_A, \gamma_A)$ is intuitionistic fuzzy semiprime, $A(a) = A(a^2)$ holds.*

Proof. Let a be an element of S. Then, since μ_A is a fuzzy subsemigroup of S, we have

$$\mu_A(a) \geq \mu_A(a^2) = \min\{\mu_A(a), \mu_A(a)\} = \mu_A(a),$$

and so we have $\mu_A(a) = \mu_A(a^2)$. Also, we have

$$\gamma_A(a) \leq \gamma_A(a^2) = \max\{\gamma_A(a), \gamma_A(a)\} = \gamma_A(a).$$

Thus $\gamma_A(a) = \gamma_A(a^2)$. This proves the theorem. □

An ordered semigroup S is called *left* (resp. *right*) *regular* if, for each element a of S, there exists an element x in S such that $a \leq xa^2$ (resp. $a \leq a^2x$).

THEOREM 3.10. *Let S be left regular. Then, for every intuitionistic fuzzy left ideal $A = (\mu_A, \gamma_A)$ of S, $A(a) = A(a^2)$ holds for all $a \in S$.*

Proof. Let a be any element of S. Since S is left regular, there exists an element x in S such that $a \leq xa^2$. Thus we have

$$\mu_A(a) \geq \mu_A(xa^2) \geq \mu_A(a^2) \geq \mu_A(a),$$

and so we have $\mu_A(a) = \mu_A(a^2)$. Also, we have

$$\gamma_A(a) \leq \gamma_A(xa^2) \leq \gamma_A(a^2) \leq \gamma_A(a).$$

Thus $\gamma_A(a) = \gamma_A(a^2)$. So, $A(a) = A(a^2)$. This proves the theorem. □

THEOREM 3.11. *Let S be left regular. Then, every intuitionistic fuzzy left ideal of S is intuitionistic fuzzy semiprime.*

Proof. Let IFS $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy left ideal of S and let $a \in S$. Then, there exists an element x in S such that $a \leq xa^2$ since S is left regular. So, we have $\mu_A(a) \geq \mu_A(xa^2) \geq \mu_A(a^2)$, and $\gamma_A(a) \leq \gamma_A(xa^2) \leq \gamma_A(a^2)$. This proves the theorem.

□

An ordered semigroup S is called *intra-regular* if, for each element a of S, there exist elements x and y in S such that $a \leq xa^2y$.

DEFINITION 3.12. For an IFS $A = (\mu_A, \gamma_A)$ in S, consider the following axioms:

(II_1) $x \leq y$ implies $\mu_A(x) \geq \mu_A(y)$ and $\mu_A(xsy) \geq \mu_A(s)$,

(II_2) $x \leq y$ implies $\gamma_A(x) \leq \gamma_A(y)$, and $\gamma_A(xsy) \leq \gamma_A(s), \forall x, y \in S$.

Then $A = (\mu_A, \gamma_A)$ is called a *intuitionistic fuzzy interior ideal* (briefly, $IFII$) of S if it satisfies (II_1) and (II_2).

THEOREM 3.13. *Let $A = (\mu_A, \gamma_A)$ be an IFS in an intra-regular ordered semigroup S. Then, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy interior ideal of S if and only if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of S.*

Proof. Let a, b be any elements of S, and let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy interior ideal of S. Then, since S is intra-regular, there exist elements x, y, u and v in S such that $a \leq xa^2y$ and $b \leq ub^2v$. Then, since μ_A is a fuzzy interior ideal of S, we have

$$\mu_A(ab) \geq \mu_A((xa^2y)b) = \mu_A((xa)a(yb)) \geq \mu_A(a)$$

and

$$\mu_A(ab) \geq \mu_A(a(ub^2v)) = \mu_A((au)b(bv)) \geq \mu_A(b).$$

Also, we have

$$\gamma_A(ab) \leq \gamma_A((xa^2y)b) = \gamma_A((xa)a(yb)) \leq \gamma_A(a)$$

and

$$\gamma_A(ab) \leq \gamma_A(a(ub^2v)) = \gamma_A((au)b(bv)) \leq \gamma_A(b).$$

On the other hand, let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of S. Then, since μ_A is a fuzzy ideal of S, we have

$$\mu_A(xay) = \mu_A(x(ay)) \geq \mu_A(ay) \geq \mu_A(a),$$

and

$$\gamma_A(xay) = \gamma_A(x(ay)) \leq \gamma_A(ay) \leq \gamma_A(a)$$

for all x, a and $y \in S$. This proves the theorem. □

THEOREM 3.14. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of S. If S is intra-regular, then $A = (\mu_A, \gamma_A)$ is intuitionistic fuzzy semiprime.*

Proof. Let a be any element of S. Then since S is intra-regular, there exist x and y in S such that $a \leq xa^2y$. So, we have

$$\mu_A(a) \geq \mu_A(xa^2y) \geq \mu_A(a^2y) \geq \mu_A(a^2),$$

and

$$\gamma_A(a) \leq \gamma_A(xa^2y) \leq \gamma_A(a^2y) \leq \gamma_A(a^2).$$

This proves the theorem. □

THEOREM 3.15. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy interior ideal of S. If S is an intra-regular, then $A = (\mu_A, \gamma_A)$ is intuitionistic fuzzy semiprime.*

Proof. Let a be any element of S. Then since S is intra-regular, there exist x and y in S such that $a \leq xa^2y$. So, we have

$$\mu_A(a) \geq \mu_A(xa^2y) \geq \mu_A(a^2),$$

and

$$\gamma_A(a) \leq \gamma_A(xa^2y) \leq \gamma_A(a^2).$$

This proves the theorem. □

THEOREM 3.16. *Let S be intra-regular. Then, for all intuitionistic fuzzy interior ideal $A = (\mu_A, \gamma_A)$ and for all $a \in S$, $A(a) = A(a^2)$ holds*

Proof. Let a be any element of S. Then since S is intra-regular, there exist x and y in S such that $a \leq xa^2y$. So, we have

$$\begin{aligned}\mu_A(a) &\geq \mu_A(xa^2y) \geq \mu_A(a^2)\\ &\geq \mu_A((xa^2y)(xa^2y))\\ &= \mu_A((xa)a(yxa^2y)) \geq \mu_A(a),\end{aligned}$$

and

$$\begin{aligned}\gamma_A(a) &\leq \gamma_A(xa^2y) \leq \gamma_A(a^2)\\ &\leq \gamma_A((xa^2y)(xa^2y))\\ &= \gamma_A((xa)a(yxa^2y)) \leq \gamma_A(a).\end{aligned}$$

So, we have $A(a) = A(a^2)$. This proves the theorem. □

THEOREM 3.17. *Let S be intra-regular. Then, for all intuitionistic fuzzy interior ideal $A = (\mu_A, \gamma_A)$ and for all $a, b \in S$, $A(ab) = A(ba)$ holds*

Proof. Let a be any element of S. Then since S is intra-regular, there exist x and y in S such that $a \leq xa^2y$. So, we have

$$\begin{aligned}\mu_A(ab) &= \mu_A((ab)^2) = \mu_A(a(ba)b) \geq \mu_A(ba)\\ &= \mu_A((ba)^2) = \mu_A(b(ab)a) \geq \mu_A(ab),\end{aligned}$$

and

$$\begin{aligned}\gamma_A(ab) &= \gamma_A((ab)^2) = \gamma_A(a(ba)b) \leq \gamma_A(ba)\\ &= \gamma_A((ba)^2) = \gamma_A(b(ab)a) \leq \gamma_A(ab).\end{aligned}$$

So, we have $A(ab) = A(ba)$. This proves the theorem. □

An ordered semigroup S is called *archimedean* if, for any elements a, b, there exists a positive integer n such that $a^n \in SbS$.

THEOREM 3.18. *Let S be an archimedean ordered semigroup. Then, every intuitionistic fuzzy semiprime fuzzy ideal of S is a constant function.*

Proof. Let $A = (\mu_A, \gamma_A)$ be any intuitionistic fuzzy semiprime fuzzy ideal of S and $a, b \in S$. Then since S is archimedean, there exist x and y in S such that $a^n = xby$ for some integer n. Then, we have

$$\mu_A(a) = \mu_A(a^n) = \mu_A(xby) \geq \mu_A(b),$$

and

$$\mu_A(b) = \mu_A(b^n) = \mu_A(xay) \geq \mu_A(a).$$

Thus, we have $\mu_A(a) = \mu_A(b)$. Also, we have

$$\gamma_A(a) = \gamma_A(a^n) = \gamma_A(xby) \leq \gamma_A(b),$$

and

$$\gamma_A(b) = \gamma_A(b^n) = \gamma_A(xay) \leq \gamma_A(a).$$

Therefore, we have $A(a) = A(b)$ for all $a, b \in S$. This proves the theorem. □

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