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THE MOD H NIELSEN NUMBER

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Abstract. Let $f : X \to X$ be a self-map of a connected finite polyhedron X. In this short note, we say that the mod H Nielsen number $N_H(f)$ is well-defined without the algebraic condition $f_\pi(H) \subseteq H$ and that $N_H(f)$ is the same as the q-Nielsen number $N_q(f)$ in any case.

1. Introduction

Let $f : X \to X$ be a self-map of a connected finite polyhedron X. The Nielsen number $N(f)$ of f, by its homotopy invariance, provides a lower bound for the minimum number of fixed points over all maps homotopic to f. The mod H Nielsen number $N_H(f)$ is a lower bound for $N(f)$ that might be easier to compute and is essential to study for the fixed point of fiber preserving maps. See [2], [3] or [5] for more details.

In 1992, Woo and Kim [4] introduced the q-Nielsen number $N_q(f)$ that is a generalization of the original mod H Nielsen number $N_H(f)$. The q-Nielsen number has an advantage for geometric questions and is defined without the condition $f_\pi(H) \subseteq H$ that is a requirement in the definition of the original mod H Nielsen number.

The purpose of this short note is to say that we can define the mod H Nielsen fixed point class without the condition $f_\pi(H) \subseteq H$, which means that $N_H(f)$ is defined without the condition $f_\pi(H) \subseteq H$, and that the mod H Nielsen theory can share everything with the q-Nielsen theory.

2. Mod H Nielsen number

Let X be a connected finite polyhedron and $f : X \to X$ a self-map. Let Π be the group of covering transformations for the universal covering

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projection $p : \widetilde{X} \to X$. In what follows, we will fix a lifting $\tilde{f} : \widetilde{X} \to \widetilde{X}$ of f on the universal covering. Then we have a homomorphism $\varphi : \varPi \to \varPi$ defined by

$$\tilde{f} \circ \alpha = \varphi(\alpha) \circ \tilde{f}, \quad \alpha \in \varPi.$$

The homomorphism φ is the same as $\tau_\beta f_\pi$ for some $\beta \in \varPi$ where τ_β is a conjugation by β and $f_\pi : \pi_1(X) \to \pi_1(X)$ is the homomorphism induced by f. Take a normal subgroup H of $\varPi$ with $f_\pi(H) \subseteq H$, so $\varphi(H) \subseteq H$. Let $\overline{X}$ be the quotient space $\widetilde{X}/H$ of the universal covering space $\widetilde{X}$ and $\bar{p} : \overline{X} \to X$ a regular covering projection concerning to H. Since $\varphi(H) \subseteq H$, $\tilde{f}$ induces the lifting $\bar{f} : \overline{X} \to \overline{X}$ so that the following diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{X} \\ \downarrow & & \downarrow \\ \overline{X} & \xrightarrow{\bar{f}} & \overline{X} \\ \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle \bar{p}} \\ X & \xrightarrow{f} & X \end{array}$$

commutes.

For the lifting $\bar{f}$, we also have a homomorphism $\bar{\varphi} : \varPi/H \to \varPi/H$ defined by

$$\bar{f} \circ \bar{\alpha} = \bar{\varphi}(\bar{\alpha}) \circ \bar{f}, \quad \bar{\alpha} \in \varPi/H$$

so that the following diagram commutes:

$$\begin{array}{ccc} \varPi & \xrightarrow{\varphi} & \varPi \\ \downarrow & & \downarrow \\ \varPi/H & \xrightarrow{\bar{\varphi}} & \varPi/H \end{array}$$

The homomorphism $\bar{\varphi}$ induces the *Reidemeister action* of $\varPi/H$ on $\varPi/H$ as follows:

$$\varPi/H \times \varPi/H \to \varPi/H, \quad (\bar{\gamma}, \bar{\alpha}) \mapsto \bar{\gamma}\bar{\alpha}\bar{\varphi}(\bar{\gamma})^{-1}.$$

Denote the sets of Reidemeister classes of $\varPi/H$ determined by $\bar{\varphi}$ by $\mathcal{R}[\bar{\varphi}]$. Then the fixed point set $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$ splits into a disjoint union of *mod H fixed point classes*, that is,

$$\mathrm{Fix}(f) = \bigcup_{[\bar{\alpha}] \in \mathcal{R}[\bar{\varphi}]} \bar{p}(\mathrm{Fix}(\bar{\alpha} \circ \bar{f})).$$

The original *mod H Nielsen number* $N_H(f)$ is the number of mod H fixed point classes with nonzero fixed point index. (see [1] or [3] for the details)

THEOREM 2.1 ([3, Theorem 2.2]). *Two fixed points x_0 and x_1 of f belong to the same mod H fixed point class if and only if there is a path c from x_0 to x_1 such that $\langle c\,(f\circ c)^{-1}\rangle \in H$.*

In [4], Woo and Kim introduced another approach for the mod H Nielsen number. Let $q : X \to Y$ be a map. Two fixed point x_0 and x_1 of a map $f : X \to X$ are said to be in the same *q-fixed point class* if there is a path c from x_0 to x_1 such that

$$q\circ c \simeq q\circ f\circ c \quad \text{(rel. end points)}.$$

Each q-fixed point class is a union of fixed point classes of f and the *q-Nielsen number*, denoted by $N_q(f)$, is the number of q-fixed point classes with nonzero fixed point index. The q-Nielsen number $N_q(f)$ is a homotopy invariant lower bound for $N(f)$ and is a generalization of the original mod H Nielsen number.

THEOREM 2.2 ([4, Theorem 1.5]). *If there is a map $f_q : Y \to Y$ such that the following diagram*

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & X \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle q} \\ Y & \xrightarrow{\;f_q\;} & Y \end{array}$$

commutes, then $N_q(f) = N_H(f)$ where $H = \ker(q_\pi : \pi_1(X) \to \pi_1(Y))$.

The definition of q-fixed point class does not use any covering spaces and it works directly on X and Y. So, $N_q(f)$ is always defined without any restriction to a map f and it is more convenient in geometric problems.

We now consider the mod H fixed point class again. For $\alpha \in \Pi$, let $\bar\alpha \in \Pi/H$ be the quotient element of α. The mod H fixed point class $\bar p(\mathrm{Fix}(\bar\alpha\circ\bar f))$ splits into a union of fixed point classes as follows:

$$\bar p(\mathrm{Fix}(\bar\alpha\circ\bar f)) = \bigcup_{\alpha'\in[\alpha]_H} p(\mathrm{Fix}(\alpha'\circ\tilde f))$$

where $[\alpha]_H$ is the mod H conjugacy class of α which is defined by the equivalent relation $\alpha' \sim \alpha$ if $\alpha' = h\gamma\alpha\varphi(\gamma)^{-1}$ for some $h \in H$ and $\gamma \in \Pi$. (see [3]) If a map $f : X \to X$ does not satisfy $f_\pi(H) \subseteq H$, there does not exist any lifting of f on the regular covering space, so the mod H fixed point class $\bar p(\mathrm{Fix}(\bar\alpha\circ\bar f))$ is not defined. But, there is still no problem to get the mod H conjugacy class $[\alpha]_H$ and $p(\mathrm{Fix}(\alpha'\circ\tilde{f}))$ for each $\alpha'\in[\alpha]_H$ without the condition $f_\pi(H)\subseteq H$.

DEFINITION 2.3. For a map $f : X \to X$ and a normal subgroup H of $\mathit{\Pi}$, the set

$$F_H(\alpha\circ\tilde{f}) = \bigcup_{\alpha'\in[\alpha]_H} p(\mathrm{Fix}(\alpha'\circ\tilde{f}))$$

is called a *mod H fixed point class of f* labeled by the mod H conjugate class $[\alpha]_H$. The *mod H Nielsen number $N_H(f)$* is the number of mod H fixed point classes with nonzero fixed point index.

The algebraic condition $f_\pi(H)\subseteq H$ is not necessary in Definition 2.3 and neither is it in the following theorem.

THEOREM 2.4. *Two fixed points x_0 and x_1 of f belong to the same mod H fixed point class if and only if there is a path c from x_0 to x_1 such that $\langle c\,(f\circ c)^{-1}\rangle\in H$.*

Proof. Suppose that fixed points x_0 and x_1 are in the same mod H fixed point class $F_H(\alpha\circ\tilde{f})$. Then we may assume that $x_0\in p(\mathrm{Fix}(\alpha\circ\tilde{f}))$ and $x_1\in p(\mathrm{Fix}(h\gamma\alpha\varphi(\gamma)^{-1}\circ\tilde{f}))$ for some $h\in H$ and $\gamma\in\mathit{\Pi}$. Choose $\tilde{x}_0\in p^{-1}(x_0)$ and $\tilde{x}_1\in p^{-1}(x_1)$ such that $(\alpha\circ\tilde{f})(\tilde{x}_0)=\tilde{x}_0$ and $(h\gamma\alpha\varphi(\gamma)^{-1}\circ\tilde{f})(\tilde{x}_1)=\tilde{x}_1$. Take a path $\tilde{c}$ from $\tilde{x}_0$ to $\gamma^{-1}(\tilde{x}_1)$. Let $c=p\circ\tilde{c}$, then c is a path from x_0 to x_1. Let $k=\gamma^{-1}h\gamma\in H$. Then

$$\begin{aligned}(k\alpha\circ\tilde{f})(\gamma^{-1}(\tilde{x}_1)) &= (\gamma^{-1}h\gamma\alpha\circ\tilde{f}\circ\gamma^{-1})(\tilde{x}_1)\\ &= (\gamma^{-1}h\gamma\alpha\varphi(\gamma)^{-1}\circ\tilde{f})(\tilde{x}_1)\\ &= \gamma^{-1}((h\gamma\alpha\varphi(\gamma)^{-1}\circ\tilde{f})(\tilde{x}_1))\\ &= \gamma^{-1}(\tilde{x}_1).\end{aligned}$$

Therefore, since $\tilde{c}$ is a path from $\tilde{x}_0$ to $\gamma^{-1}(\tilde{x}_1)$, this implies that $k\alpha\circ\tilde{f}\circ\tilde{c}$ is a path from $k(\tilde{x}_0)$ to $\gamma^{-1}(\tilde{x}_1)$, hence $\tilde{c}\,(k\alpha\circ\tilde{f}\circ\tilde{c})^{-1}$ is a path from $\tilde{x}_0$ to $k(\tilde{x}_0)$. Since $\tilde{c}\,(k\alpha\circ\tilde{f}\circ\tilde{c})^{-1}$ is a lifting of $c\,(f\circ c)^{-1}$, we can conclude that $\langle c\,(f\circ c)^{-1}\rangle=k\in H$.

Conversely, suppose that there is a path c from x_0 to x_1 such that $\langle c\,(f\circ c)^{-1}\rangle\in H$. Choose $\tilde{x}_0\in p^{-1}(x_0)$ and let $\tilde{c}$ be the lifting of c with initial point $\tilde{x}_0$. Let $\tilde{x}_1$ be the terminal point of $\tilde{c}$. Let $\alpha\in\mathit{\Pi}$ be the covering transformation with $(\alpha\circ\tilde{f})(\tilde{x}_1)=\tilde{x}_1$. Then $\tilde{c}\,(\alpha\circ\tilde{f}\circ\tilde{c})^{-1}$ is a lifting of $c\,(f\circ c)^{-1}$ with initial point $\tilde{x}_0$. Since $\langle c\,(f\circ c)^{-1}\rangle\in H$, there exists $h\in H$ such that $h(\tilde{x}_0)$ is the terminal point of the lifting $\tilde{c}\,(\alpha\circ\tilde{f}\circ\tilde{c})^{-1}$. This implies that $h(\tilde{x}_0)$ is the initial point of the path

$\alpha \circ \tilde{f} \circ \tilde{c}$. Thus we have

$$\begin{aligned}(h^{-1}\alpha \circ \tilde{f})(\tilde{x}_0) &= h^{-1}(\alpha \circ \tilde{f}(\tilde{x}_0)) \\ &= h^{-1}(h(\tilde{x}_0)) \\ &= \tilde{x}_0.\end{aligned}$$

Therefore, we have $x_0 \in p(\mathrm{Fix}(h^{-1}\alpha \circ \tilde{f}))$. Since $(\alpha \circ \tilde{f})(\tilde{x}_1) = \tilde{x}_1$, we have $x_1 \in p(\mathrm{Fix}(\alpha \circ \tilde{f}))$. Consequently, x_0 and x_1 are in the same mod H fixed point class. □

THEOREM 2.5. *Let $H = \ker(q_\pi : \pi_1(X) \to \pi_1(Y))$. The mod H fixed point classes are the same as the q-fixed point classes.*

Proof. Let x_0 and x_1 be fixed points of f and c a path from x_0 to x_1. Then

$$\begin{aligned} & q \circ c \simeq q \circ f \circ c \text{ (rel. end points)} \\ \text{if and only if } & q(c(f \circ c)^{-1}) \simeq e \text{ (rel. base point } q(x_0)) \text{ where} \\ & e \text{ is the constant loop at } q(x_0) \\ \text{if and only if } & q_\pi(\langle c\,(f \circ c)^{-1}\rangle) = \langle e \rangle \\ \text{if and only if } & \langle c\,(f \circ c)^{-1}\rangle \in H. \end{aligned}$$

By Theorem 2.4, we obtain the desired result. □

COROLLARY 2.6. *For a self-map $f : X \to X$ and a map $q : X \to Y$, we have*

$$N_H(f) = N_q(f)$$

where $H = \ker(q_\pi)$.

EXAMPLE 2.7. Let X be a torus with three holes. Then the fundamental group $\pi_1(X)$ of X is a free group on four generators:

$$\pi_1(X) = \langle a, b, c, d \rangle.$$

Let $f : X \to X$ be a map such that

$$\begin{aligned} f_\pi(a) &= a^{10}ba^{-6}c^{-2}, \\ f_\pi(b) &= cbd, \\ f_\pi(c) &= ad, \\ f_\pi(d) &= c^5d^{-5}ad^{-1}a^2d^2. \end{aligned}$$

The Lefschetz number $L(f)$ of f is:

$$L(f) = 1 - ((10-6) + 1 + 0 + (-5 - 1 + 2)) = 0,$$

and it is very hard to calculate the Nielsen number $N(f)$. Thus we do not even know the existence of the fixed point of a map that is homotopic to f by $L(f)$ and $N(f)$. But, we can easily compute the mod H Nielsen number $N_H(f)$ for H which is the normal closure of the subgroup $\langle a, b, c\rangle$. Note that $f_\pi(H) \nsubseteq H$.

Now, consider a map $q : X \to S^1$ such that $\ker(q_\pi) = H$. Then, it is easy to check that there are at least five q-Nielsen fixed point classes such that exactly five of those classes have nonzero indices, so $N_H(f) = N_q(f) = 5$. Thus, all maps in the homotopy class of f have at least five fixed points.

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