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SOME APPLICATIONS OF EXTREMAL LENGTH TO CONFORMAL IMBEDDINGS

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Abstract. Let G be a Denjoy domain and let G' a Denjoy proper subdomain of G and homeomorphic to G. We consider conformal re-imbeddings of G' into G. Let G and G' are N-connected. We know that if $N = 2$, there is a re-imbedding f of G' into G such that $G - cl(f(G'))$ has an interior point. In this note, we obtain the following theorem.

If $N \geq 3$, G has a Denjoy proper subdomain G' such that, for any re-imbeddings f of G' into G, $G - cl(f(G'))$ has no interior point.

1. Introduction

Throughout this paper, $\widehat{\mathbb{C}}$ will denote extended complex plane, G is a Denjoy domain in $\widehat{\mathbb{C}}$, and $cl(G)$ is a closure of G.

Let $[a_N, b_N]$, $(N = 1, 2, 3, \cdots)$ be closed intervals such that $a_1 < b_1 < \cdots < a_N < b_N$ $(N \geq 2)$. We denote by

$$\mathcal{D}([a_1, b_1], [a_2, b_2], \cdots, [a_N, b_N])$$

the Denjoy domain $\widehat{\mathbb{C}} - \cup_{j=1}^{N}[a_j, b_j]$ in $\widehat{\mathbb{C}}$.

We note that each doubly or triply connected plane region is conformally equivalent to a Denjoy domain (see [2]).

Let

$$G' = \mathcal{D}([a'_1, b'_1], [a'_2, b'_2], \cdots, [a'_N, b'_N])$$

be a Denjoy proper subdomain of

$$G = \mathcal{D}([a_1, b_1], [a_2, b_2], \cdots, [a_N, b_N])$$

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such that

$$[a_j, b_j] \subset [a'_j, b'_j]$$

for each j $(1 \leq j \leq N)$. We consider re-imbeddings of G' into G, that is, conformal mappings f from G' into G with the following two conditions (1) and (2).

(1) Let denote by E_j the components of $\widehat{\mathbb{C}} - f(G')$, then $E_j \supset [a_j, b_j]$ $(1 \leq j \leq N)$
(2) $f(z) \to E_j$ if and only if $z \to [a'_j, b'_j]$ $(1 \leq j \leq N)$

2. Extremal length

Let T be a plane region and Γ be a family of curves in T. Let $\rho(z)$ be a non-negative Borel measurable function defined on T. Define

$$\begin{aligned} L(\rho, \Gamma) &= \inf \left\{ \int_\gamma \rho(z)|dz|; \gamma \in \Gamma \right\}, \\ A(\rho, T) &= \iint_T \rho(z)^2 dxdy \neq 0, \infty. \end{aligned}$$

Definition 2.1. ([7]) The quantity

$$\lambda(\Gamma) = \sup_\rho \frac{L(\rho, \Gamma)^2}{A(\rho, T)}$$

is called the extremal length of Γ.

We know that if $N = 2$, there is a re-imbedding f of G' into G such that $G - cl(f(G'))$ has an interior point, where the closure is taken in $\widehat{\mathbb{C}}$.

In fact, each doubly connected plane region with non-degenerate boundary components is mapped conformally onto an annulus, and the ratio of the radius of the annulus is uniquely determined. Moreover, two doubly connected plane regions are conformally equivalent if and only if these ratios of the radii of the annuli coinside.

We give an example.

Example 2.2. ([6]) *Let $\Delta(r, R) = \{z \,|\, r < |z| < R\}$, $(0 < r < R, R \neq \infty)$ on T. Then $\Delta_1(r_1, R_1)$ and $\Delta_2(r_2, R_2)$ are conformally equivalent if and only if*

$$R_1/r_1 = R_2/r_2 \tag{3}$$

For our proof we will need the following.

Proposition 2.3. ([8]) *Let Δ be the annulus $\Delta = \{z \mid a < |z| < b\}$ on T. Let Γ be the family of arcs in Δ which join the two contours. Then*

$$\lambda(\Gamma) = (1/2\pi) \log (b/a)$$

Proof. (Proof of example 2.1) (Method of extremal length) Since the proof of sufficient conditions is trivial, we discuss the proof of necessary conditions. Let Γ_Δ be the family of arcs in $\Delta(r, R)$ which join the two contours. Then by Proposition 2.1,

$$\lambda(\Gamma_\Delta) = (1/2\pi) \log(R/r). \tag{4}$$

Suppose that $\Delta_1(r_1, R_1)$ and $\Delta_2(r_2, R_2)$ are conformally equivalent and let f be a 1-1 conformal mapping on $\Delta_1(r_1, R_1)$ upon $\Delta_2(r_2, R_2)$. Then by the conformal invariance of extremal length,

$$\lambda(\Gamma_{\Delta_1}) = \lambda[f(\Gamma_{\Delta_1})] = \lambda(\Gamma_{\Delta_2}). \tag{5}$$

Hence by (4), (5), we obtain (3). □

Hereafter we assume that $N \geq 3$. Our proof is an elementary one using the method of extremal length.

Lemma 2.4. *Let $G = \mathcal{D}([a_1, b_1], [a_2, b_2], \cdots, [a_N, b_N])$, and let*

$$\Gamma = \Gamma(G; [a_1, b_1], \cup_{j=2}^{N} [a_j, b_j])$$

be the family of Jordan closed curves in G separating $[a_1, b_1]$ from $\cup_{j=2}^{N} [a_j, b_j]$. Let

$$\Gamma_0 = \Gamma(\widehat{\mathbb{C}} - [a_1, b_1] - [a_2, b_N]; [a_1, b_1], [a_2, b_N])$$

be the family of Jordan closed curves in $\widehat{\mathbb{C}} - [a_1, b_1] - [a_2, b_N]$ separating $[a_1, b_1]$ from $[a_2, b_N]$. Then

$$\lambda(\Gamma) = \lambda(\Gamma_0).$$

Proof. Since $\Gamma_0 \subset \Gamma$, it holds that $\lambda(\Gamma) \leq \lambda(\Gamma_0)$. Let $\omega = \varphi(z)$ be the conformal mapping from $\{\text{Im } z > 0\}$ onto the rectangle $\{0 < \text{Re } w < 1, 0 < \text{Im } w < \kappa\}$ such that $\varphi(a_1) = 0, \varphi(b_1) = 1, \varphi(a_2) = 1 + i\kappa$ and $\varphi(b_N) = i\kappa$. Then φ is analytic on $G \cap \{\text{Im } z = 0\}$.

Set

$$c_\tau = \{0 \leq \text{Re } w \leq 1, \text{Im } w = \tau\}$$

for $0 < \tau < \kappa$. Then

$$\gamma_\tau = \varphi^{-1}(c_\tau) \cup cl(\varphi^{-1}(c_\tau)) \in \Gamma,$$

where

$$cl(\varphi^{-1}(c_\tau)) = \{\bar{w}; w \in \varphi^{-1}(c_\tau)\}.$$

Denote by $\lambda(\{\gamma_\tau\})$ the extremal length of $\{\gamma_\tau\}_{0<\tau<1}$. Then

$$\frac{2}{\kappa} = \lambda(\{\gamma_\tau\}) \geq \lambda(\Gamma).$$

Consider the non-negative Borel measurable function $\rho(z)$ on G defined by

$$\rho(z) = \begin{cases} |\varphi'(z)|/2, & z \in G \cap \{\text{Im } z \geq 0\} \\ |\varphi'(\bar{z})|/2, & z \in G \cap \{\text{Im } z < 0\}. \end{cases}$$

Then

$$A(\rho, G) = \iint_G \rho^2 dxdy = \frac{\kappa}{2}.$$

Since $\int_c \rho(z)|dz| \geq 1$ for any $c \in \Gamma$, we have

$$\lambda(\Gamma) \geq A(\rho, G)^{-1} = \frac{2}{\kappa}.$$

By the same way, we can conclude

$$\lambda(\{\gamma_\tau\}) = \lambda(\Gamma_0).$$

Thus, $\lambda(\Gamma) = \lambda(\Gamma_0)$. □

Here we note the following fact. That is, let u be the harmonic measure of $[a_2, b_N]$ with respect to $\widehat{\mathbb{C}} - [a_1, b_1] - [a_2, b_N]$. That is, u is the bounded harmonic function on $\widehat{\mathbb{C}} - [a_1, b_1] - [a_2, b_N]$ such that $u = 0$ on $[a_1, b_1]$ and $u = 1$ on $[a_2, b_N]$. Then the Dirichlet integral $D(u)$ of u is finite.

We will need the following lemma.

LEMMA 2.5. ([3],[5]) *Let $\rho(z) = |grad\ u|$, then*

$$D(u) = \frac{L(\rho, \Gamma_0)^2}{A(\rho, G)} = \lambda(\Gamma_0).$$

3. Extremal length and conformal imbeddings

In contrast to the case $N = 2$, we can prove the following Theorem.

THEOREM 3.1. *If $N \geq 3$, any Denjoy domain G has a Denjoy proper subdomain G' such that, for any re-imbeddings f of G' into G, $G - cl(f(G'))$ has no interior point.*

Proof. Let

$$G = \mathcal{D}([a_1, b_1], [a_2, b_2], \cdots, [a_N, b_N])$$

and

$$G' = \mathcal{D}([a'_1, b'_1], [a'_2, b'_2], \cdots, [a'_N, b'_N]).$$

Assume that

$$[a_1, b_1] = [a'_1, b'_1], \quad a_2 = a'_2, \quad b_N = b'_N$$

and that G' be a proper subregion of G, that is,

$$[a_j, b_j] \subset [a'_j, b'_j], \quad (3 \leq j \leq N-1).$$

Let f be a re-imbedding of G' into G. We prove that the area measure of $G - cl(f(G'))$ is 0.

Contrary to the assertion, assume that the area measure of $G - cl(f(G'))$ is positive. Let

$$\Gamma = \Gamma(G; [a_1, b_1], \cup_{j=2}^{N}[a_j, b_j])$$

be the family of Jordan closed curves in G separating $[a_1, b_1]$ from $\cup_{j=2}^{N}[a_j, b_j]$. Let

$$\Gamma' = \Gamma(G'; [a_1, b_1], [a_2, b'_2] \cup \cdots \cup [a'_N, b_N])$$

be the family of Jordan closed curves in G' separating $[a_1, b_1]$ from $[a_2, b'_2] \cup \cdots \cup [a'_N, b_N]$. Set

$$f(\Gamma') = \{f(\gamma); \gamma \in \Gamma'\}.$$

By Lemma 2.2,

$$\lambda(\Gamma) = \lambda(\Gamma').$$

Since G' and $f(G')$ are conformally equivalent,

$$\lambda(\Gamma') = \lambda(f(\Gamma')).$$

In the meanwhile, we can prove that $\lambda(f(\Gamma')) > \lambda(\Gamma)$ to obtain a contradiction.

In fact, let u be the harmonic measure of $[a_2, b_N]$ with respect to $\hat{\mathbb{C}} - [a_1, b_1] - [a_2, b_N]$. Consider the the non-negative Borel measurable function $\rho(z) = |\text{grad } u|$ on G. By Lemma 2.3,

$$D(u) = \lambda(\Gamma) = \lambda(\Gamma').$$

Since $\Gamma \supset f(\Gamma')$,

$$L(\rho, \Gamma) = \inf\{\int_\gamma \rho|dz|; \gamma \in \Gamma\} \leq \inf\{\int_\gamma \rho|dz|; \gamma \in f(\Gamma')\} = L(\rho, f(\Gamma')).$$

Since $G - cl(f(G'))$ has a positive measure

$$A(\rho, f(G')) = \iint_{f(G')} \rho^2 dxdy < \iint_G \rho^2 dxdy = A(\rho, G).$$

Thus

$$\lambda(f(\Gamma')) \geq \frac{L^2(\rho, f(\Gamma'))}{A(\rho, f(G'))} > \frac{L^2(\rho, \Gamma)}{A(\rho, G)} = D(u) = \lambda(\Gamma).$$

This completes the proof of the theorem. □

References

[1] H. Ishida and F. Maitani, *Conformal imbeddings of Denjoy domains II* (in preparation).

[2] S. Nishimura and H. Ishida, *Extremal length and the moduli space of triply connected plane regions*, Acta Human. Sci. Univ. Sangio Kyotiensis, **25** (1995), 58-68.

[3] M. Ohtsuka, Dirichlet problem, extremal length and prime ends, Van Nostland, 1970.

[4] R. C. Penner, *Extremal lengths on Denjoy domains*, Proc. Amer. math. Soc. **102** (1988), 641-645.

[5] L. Sario, and K. Oikawa, *Capacity functions*, Springer-Verlag, 1969.

[6] B. H. Chung, *Some applications of extremal length to analytic functions*, Commun. Korean Math. Soc. **21** (2006), no. 1, 135-143.

[7] B. H. Chung, *Extremal length and geometric inequalities*, J. Chungcheong Math. Soc., **20** (2007), 147-156.

[8] B. H. Chung, and W. S. Jung, *Some results related to Extremal length*, J. Chungcheong Math. Soc. **16** (2003), 49-60.

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