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ON THE HYERS-ULAM-RASSIAS STABILITY OF THE GENERALIZED POLYNOMIAL FUNCTION OF DEGREE 2

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Abstract. In this paper, we prove the stability of the functional equation

$$\sum_{i=0}^{3} {}_3C_i(-1)^{3-i} f(ix+y) = 0$$

in the sense of Th.M.Rassias on the punctured domain. Also, we investigate the superstability of the functional equation.

1. Introduction

Throughout this paper, let X be a normed space and Y a Banach space. For a given mapping $f : X \to Y$, define a mapping $E_3 f : X \times X \to Y$ by

$$E_3 f(x, y) := \sum_{i=0}^{3} {}_3C_i(-1)^{3-i} f(ix + y)$$

for all $x, y \in X$, where ${}_3C_i = \frac{3!}{i!(3-i)!}$. A mapping $f : X \to Y$ is called a generalized polynomial function of degree 2 if f satisfies the functional equation $E_3 f(x, y) = 0$. The functional equation $E_3 f(x, y) = 0$ is called a generalized polynomial functional equation of degree 2. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sum_{i=0}^{2} a_i x^i$ satisfies the functional equation $E_3 f = 0$, where a_i are real constants and $\mathbb{R}$ is the set of real numbers.

If we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality must be close to the solutions of the given equation? If the answer is affirmative, we would say that a given functional equation is stable.

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In 1941, D.H.Hyers [5] proved the stability of Cauchy equation $f(x+y)-f(x)-f(y)=0$ and in 1978, Th.M.Rassias [7] gave a significant generalization of the Hyers' result. Th.M.Rassias [8] during the 27th International Symposium on Functional Equations, that took place in Bielsko-Biala, Poland, in 1990, asked the question whether such a theorem can also be proved for a more general setting. Z.Gadja [3] following Th.M.Rassias's approach [7] gave an affirmative solution to the question. Recently, P.Găvruta [4] obtained a further generalization of Rassias' theorem, the so-called generalized Hyers-Ulam-Rassias stability.

A stability problem for the quadratic functional equation $f(x+y)+f(x-y)-2f(x)-2f(y)=0$ was proved by F.Skof [9] for a function $f:X\to Y$. P.W. Cholewa [1] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. S.Czerwik [2] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation.

In this paper, we prove the Hyers-Ulam-Rassias stability(if $p\geq 0$) and the superstability(if $p<0$) of the functional equation $E_3f=0$ in the sense of Th. M. Rassias.

2. Stability of the functional equation $E_3f=0$

Lemma 2.1. *Suppose that the odd function $f:X\to Y$ satisfies*

$$E_3f(x,y)=0 \tag{2.1}$$

for all $x,y\in X\setminus\{0\}$ and

$$f(2x)=2f(x)$$

for all $x\in X$. Then f is an additive function.

Proof. Since f is odd and $f(2x)=2f(x)$, the equality

$$\begin{aligned} f(x) &+ f(y)-f(x+y) \\ &= \frac{3E_3f(x,-x+y)+E_3f(x,-x-y)+E_3f(y,2x-y)}{6}=0 \end{aligned}$$

holds for all $x,y\in X\setminus\{0\}$. From the above equality and $f(0)=0$, we get the equality

$$f(x)+f(y)-f(x+y)=0$$

for all $x,y\in X$. □

Lemma 2.2. *Suppose that the even function $f : X \to Y$ satisfies (2.1) for all $x, y \in X \setminus \{0\}$ and*

$$f(2x) = 4f(x)$$

for all $x \in X$. Then f is a quadratic function.

Proof. Since f is even and $f(2x) = 4f(x)$, the equality

$$\begin{aligned} 2f(x) &+ 2f(y) - f(x+y) - f(x-y) \\ &= \frac{3E_3 f(x, -x+y) + E_3 f(x, -x-y) + E_3 f(y, 2x-y)}{6} = 0 \end{aligned}$$

holds for all $x, y \in X \setminus \{0\}$. From the above equality and $f(0) = 0$, we get the equality

$$2f(x) + 2f(y) - f(x+y) - f(x-y) = 0$$

for all $x, y \in X$. □

From Lemma 2.1 and Lemma 2.2, we get the following lemma.

Lemma 2.3. *The function $f : X \to Y$ satisfies (2.1) for all $x, y \in X \setminus \{0\}$ if and only if there exist a quadratic function $Q : X \to Y$ and an additive function $A : X \to Y$ such that*

$$f(x) = Q(x) + A(x) + f(0)$$

for all $x \in X$. The functions $Q, A : X \to Y$ are given by

$$\begin{aligned} Q(x) &:= \frac{f(x) + f(-x)}{2} - f(0), \\ A(x) &:= \frac{f(x) - f(-x)}{2} \end{aligned}$$

for all $x \in X$.

The following lemma is seen in [6].

Lemma 2.4. *Let a be a positive real number and $\Phi : X \setminus \{0\} \to [0, \infty)$ a map. Suppose that the function $f : X \to Y$ satisfies the inequality*

$$\|f(x) - \frac{f(2x)}{a}\| \le \frac{\Phi(x)}{a} \quad \text{and} \quad f(0) = 0.$$

(i) If $\sum_{l=0}^{\infty} \frac{1}{a^{l+1}} \Phi(2^l x) < \infty$ for all $x \in X \setminus \{0\}$, then there exists a unique function $F : X \to Y$ satisfying

$$\|f(x) - F(x)\| \le \sum_{l=0}^{\infty} \frac{1}{a^{l+1}} \Phi(2^l x)$$

for all $x \in X \setminus \{0\}$ and F is given by $F(x) = \lim_{n\to\infty} \frac{f(2^n x)}{a^n}$ for all $x \in X$.

(ii) If $\sum_{l=0}^{\infty} a^l \Phi(\frac{x}{2^{l+1}}) < \infty$ *for all* $x \in X \setminus \{0\}$*, then there exists a unique function* $F : X \to Y$ *satisfying*

$$\|f(x) - F(x)\| \le \sum_{l=0}^{\infty} a^l \Phi(\frac{x}{2^{l+1}}) < \infty$$

for all $x \in X \setminus \{0\}$ *and* F *is given by* $F(x) = \lim_{n\to\infty} a^n f(\frac{x}{2^n})$ *for all* $x \in X$*.*

THEOREM 2.5. *Let* $\varepsilon > 0$ *and* $p < 1$*. If a function* $f : X \to Y$ *satisfies*

$$\|E_3 f(x, y)\| \le \varepsilon(\|x\|^p + \|y\|^p) \tag{2.2}$$

for all $x, y \in X \setminus \{0\}$*, then there exists a unique generalized polynomial function* $F : X \to Y$ *of degree 2 with* $f(0) = F(0)$ *such that*

$$\|f(x) - F(x)\| \le \big(\frac{2}{|2 - 2^p|} + \frac{2}{|4 - 2^p|}\big)\varepsilon\|x\|^p \tag{2.3}$$

for all $x \in X \setminus \{0\}$*. In particular,* F *is represented by*

$$F(x) = \lim_{n\to\infty} \big(\frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}\big) + f(0)$$

for all $x \in X$*.*

Proof. From (2.2), we get the inequalities

$$\begin{aligned}
\|\frac{f(x) - f(-x)}{2} &- \frac{f(2x) - f(-2x)}{4}\| \\
&= \frac{1}{4}\| - E_3 f(x, -x) + E_3 f(-x, x)\| \le \varepsilon\|x\|^p, &(2.4)\\
\|\frac{f(x) + f(-x)}{2} &- f(0) - \frac{1}{4}\big(\frac{f(2x) + f(-2x)}{2} - f(0)\big)\| \\
&= \frac{1}{8}\|E_3 f(x, -x) + E_3 f(-x, x)\| \le \frac{\varepsilon}{2}\|x\|^p &(2.5)
\end{aligned}$$

for all $x \in X \setminus \{0\}$. By Lemma 2.4, there exist functions $A, Q : X \to Y$ defined by

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{2^{n+1}},$$
$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n}$$

for all $x \in X$ and the functions A, Q satisfy the inequalities

$$\|\frac{f(x) - f(-x)}{2} - A(x)\| \le \frac{2\varepsilon}{|2 - 2^p|}\|x\|^p, \tag{2.6}$$

$$\left\|\frac{f(x)+f(-x)}{2}-f(0)-Q(x)\right\|\leq\frac{2\varepsilon}{|4-2^p|}\|x\|^p \tag{2.7}$$

for all $x\in X\setminus\{0\}$. From (2.2) and $p<1$, we obtain

$$\begin{aligned} E_3A(x,y) &= \lim_{n\to\infty}\tfrac{E_3f(2^nx,2^ny)-E_3f(-2^nx,-2^ny)}{2^{n+1}}=0,\\ E_3Q(x,y) &= \lim_{n\to\infty}\tfrac{E_3f(2^nx,2^ny)+E_3f(-2^nx,-2^ny)}{2^{2n+1}}=0 \end{aligned}$$

for all $x,y\in X\setminus\{0\}$. Since $A(2x)=2A(x)$ and $Q(2x)=4Q(x)$, A is an additive function and Q is a quadratic function by Lemma 2.1 and Lemma 2.2. From (2.6), (2.7), and the inequality

$$\|f(x)-F(x)\|\leq\left\|\frac{f(x)-f(-x)}{2}-A(x)\right\|+\left\|\frac{f(x)+f(-x)}{2}-f(0)-Q(x)\right\|$$

for all $x\in X\setminus\{0\}$, we get the inequality (2.3), where $F(x)=Q(x)+A(x)+f(0)$. Now, let F' be another generalized polynomial function of degree 2 satisfying (2.3) with $F'(0)=f(0)$. Then there exist a quadratic function $Q':X\to Y$ and an additive function $A':X\to Y$ such that $F'(x)=Q'(x)+A'(x)+f(0)$ by Lemma 2.3. Since $Q,Q':X\to Y$ are quadratic functions, we get

$$\begin{aligned} \|Q(x)-Q'(x)\| &= \frac{1}{4^n}\|Q(2^nx)-Q'(2^nx)\|\\ &\leq \frac{1}{4^n}\|f(2^nx)-F(2^nx)\|+\frac{1}{4^n}\|f(2^nx)-F'(2^nx)\|\\ &+ \frac{1}{4^n}\|A(2^nx)-A'(2^nx)\|\\ &\leq \frac{2^{np}}{4^n}\Big(\frac{4}{|2-2^p|}+\frac{4}{|4-2^p|}\Big)\varepsilon\|x\|^p+\frac{1}{2^n}\|A(x)-A'(x)\| \end{aligned}$$

for all $x\in X\setminus\{0\}$ and $n\in\mathbb{N}$. As $n\to\infty$, we may conclude that $Q(x)=Q'(x)$ for all $x\in X$. Since $Q=Q'$, we get

$$\begin{aligned} \|F(x)-F'(x)\| &= \frac{1}{2^n}\|A(2^nx)-A'(2^nx)\|\\ &\leq \frac{1}{2^n}\|f(2^nx)-F(2^nx)\|+\frac{1}{2^n}\|f(2^nx)-F'(2^nx)\|\\ &\leq \frac{2^{np}}{2^n}\Big(\frac{4}{|2-2^p|}+\frac{4}{|4-2^p|}\Big)\varepsilon\|x\|^p \end{aligned}$$

for all $x\in X\setminus\{0\}$ and $n\in\mathbb{N}$. As $n\to\infty$, we may conclude that $F(x)=F'(x)$ for all $x\in X$. □

By the similar method in the proof of Theorem 2.5, we can prove the following theorem.

Theorem 2.6. *Let $p > 2$ and $\varepsilon > 0$. If a function $f : X \to Y$ satisfies (2.2) for all $x, y \in X \setminus \{0\}$, then there exists a unique generalized polynomial function $F : X \to Y$ of degree 2 with $f(0) = F(0)$ such that*

$$\|f(x) - F(x)\| \leq \frac{2\varepsilon}{2^p - 4}\|x\|^p \tag{2.8}$$

for all $x \in X \setminus \{0\}$. In particular, F is represented by

$$\begin{aligned} F(x) &= \lim_{n\to\infty} \big((2^{n-1} + 2^{2n-1})f(\frac{x}{2^n}) \\ &+ (-2^{n-1} + 2^{2n-1})f(\frac{-x}{2^n}) - 4^n f(0)\big) + f(0) \end{aligned}$$

for all $x \in X$.

Proof. By (2.4), (2.5) and Lemma 2.4 ii), there exist functions $A, Q : X \to Y$ defined by

$$\begin{aligned} A(x) &:= \lim_{n\to\infty} 2^{n-1}(f(\frac{x}{2^n}) - f(-\frac{x}{2^n})), \\ Q(x) &= \lim_{n\to\infty} 2^{2n-1}(f(\frac{x}{2^n}) + f(-\frac{x}{2^n}) - 2f(0)) \end{aligned}$$

for all $x \in X$ and the functions A, Q satisfy the inequalities (2.6)-(2.7) for all $x \in X \setminus \{0\}$. From (2.2) and $p > 2$, we obtain $E_3A(x, y) = 0$, $E_3Q(x, y) = 0$ for all $x, y \in X \setminus \{0\}$. From (2.6) and (2.7), we have the inequality

$$\begin{aligned} \|f(x) &- Q(x) - A(x) - f(0)\| \\ &\leq \frac{1}{2}\sum_{i=1}^{n} \|(4^{n-1} + 2^{n-1})E_3 f(\frac{x}{2^n}, -\frac{x}{2^n}) \\ &+ (4^{n-1} - 2^{n-1})E_3 f(-\frac{x}{2^n}, \frac{x}{2^n})\| \\ &+ 4^n \|\frac{1}{2}\big(f(\frac{x}{2^n}) + f(-\frac{x}{2^n})\big) - f(0) - Q(\frac{x}{2^n})\| \\ &+ 2^n \|\frac{1}{2}\big(f(\frac{x}{2^n}) - f(-\frac{x}{2^n})\big) - A(\frac{x}{2^n})\| \\ &\leq \sum_{i=1}^{n} \frac{2\cdot 4^{n-1}\varepsilon}{2^{np}}\|x\|^p + \big(\frac{4^n}{2^{np}}\frac{2\varepsilon}{|4-2^p|} + \frac{2^n}{2^{np}}\frac{2\varepsilon}{|2-2^p|}\big)\|x\|^p \end{aligned}$$

for all $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$. As $n \to \infty$, we get the inequality (2.8), where $F(x) = Q(x) + A(x) + f(0)$. Now, let F' be another generalized polynomial function of degree 2 satisfying (2.3) with $F'(0) = f(0)$. Then there are a quadratic function $Q' : X \to Y$ and an additive function

$A' : X \to Y$ such that $F'(x) = Q'(x)+A'(x)+f(0)$. Since $A, A' : X \to Y$ are additive functions, we get

$$\begin{aligned}
\|A(x) - A'(x)\| &= 2^n \|A(\frac{x}{2^n}) - A'(\frac{x}{2^n})\| \\
&\leq 2^n \big(\|f(\frac{x}{2^n}) - F(\frac{x}{2^n})\| + \|f(\frac{x}{2^n}) - F'(\frac{x}{2^n})\| \\
&+ \|Q(\frac{x}{2^n}) - Q'(\frac{x}{2^n})\|\big) \\
&\leq \frac{2^{n+2}}{2^{np}|4-2^p|}\varepsilon\|x\|^p + \frac{1}{2^n}\|Q(x) - Q'(x)\|
\end{aligned}$$

for all $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $A(x) = A'(x)$ for all $x, y \in X$. Since $A = A'$, we get

$$\begin{aligned}
\|F(x) - F'(x)\| &= 4^n \|Q(\frac{x}{2^n}) - Q'(\frac{x}{2^n})\| \\
&\leq 4^n \big(\|f(\frac{x}{2^n}) - F(\frac{x}{2^n})\| + \|f(\frac{x}{2^n}) - F'(\frac{x}{2^n})\|\big) \\
&\leq \frac{4^{n+1}}{2^{np}|4-2^p|}\varepsilon\|x\|^p
\end{aligned}$$

for all $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x) = F'(x)$ for all $x, y \in X$. □

THEOREM 2.7. *Let $1 < p < 2$ and $\varepsilon > 0$. If a function $f : X \to Y$ satisfies (2.2) for all $x, y \in X \setminus \{0\}$, then there exists a unique generalized polynomial function $F : X \to Y$ of degree 2 with $f(0) = F(0)$ such that (2.3) holds for all $x \in X \setminus \{0\}$. In particular, F is represented by*

$$\begin{aligned}
F(x) &= \lim_{n\to\infty} \big(\frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \\
&+ 2^{2n-1}(f(\frac{x}{2^n}) + f(\frac{-x}{2^n})) - 4^n f(0)\big) + f(0)
\end{aligned}$$

for all $x \in X$.

Proof. Let the function Q be as in the proof of Theorem 2.5 and the function A as in the proof of Theorem 2.6. We easily show that there exists a generalized polynomial function $F : X \to Y$ of degree 2 with $f(0) = F(0)$ satisfying (2.3) for all $x \in X \setminus \{0\}$, where $F = Q+A+f(0)$. Now, let F' be another generalized polynomial function of degree 2 satisfying (2.3) with $F'(0) = f(0)$. Then there are a quadratic function $Q' : X \to Y$ and an additive function $A' : X \to Y$ such that $F'(x) = Q'(x)+A'(x)+f(0)$. Since $A, A' : X \to Y$ are additive functions, Since $A, A' : X \to Y$ are additive functions, we get

$$\begin{aligned}\|A(x)-A'(x)\| &= 2^n\|A(\frac{x}{2^n})-A'(\frac{x}{2^n})\| \\ &\le 2^n\big(\|f(\frac{x}{2^n})-F(\frac{x}{2^n})\|+\|f(\frac{x}{2^n})-F'(\frac{x}{2^n})\| \\ &+ \|Q(\frac{x}{2^n})-Q'(\frac{x}{2^n})\|\big) \\ &\le \frac{2^{n+2}}{2^{np}}\big(\frac{1}{|2-2^p|}+\frac{1}{|4-2^p|}\big)\varepsilon\|x\|^p+\frac{1}{2^n}\|Q(x)-Q'(x)\|\end{aligned}$$

for all $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $A(x) = A'(x)$ for all $x, y \in X$. Since $A = A'$, we get

$$\begin{aligned}\|F(x)-F'(x)\| &= \|\frac{Q(2^nx)-Q'(2^nx)}{4^n}\| \\ &\le \frac{1}{4^n}\big(\|f(2^nx)-F(2^nx)\|+\|f(2^nx)-F'(2^nx)\|\big) \\ &\le \frac{2^{np}}{4^n}\big(\frac{4}{|2-2^p|}+\frac{4}{|4-2^p|}\big)\varepsilon\|x\|^p\end{aligned}$$

for all $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x) = F'(x)$ for all $x, y \in X$. □

3. Superstability of the functional equation $E_3f = 0$

THEOREM 3.1. *Let* $\varepsilon > 0$ *and* $p < 0$. *If a function* $f : X \to Y$ *satisfies (2.2) for all* $x, y \in X \setminus \{0\}$, *then* f *is a generalized polynomial function of degree 2.*

Proof. By Theorem 2.5, there exists a unique generalized polynomial function $F : X \to Y$ of degree 2 with $f(0) = F(0)$ such that the inequality (2.3) holds for all $x \in X \setminus \{0\}$. Hence the inequality

$$\begin{aligned}3\|f(x)-F(x)\| &\le \|E_3f((k+1)x,-kx)-E_3F((k+1)x,-kx)\| \\ &+ \|(f-F)((2k+3)x)\| \\ &+ 3\|(f-F)((k+2)x)\|+\|(f-F)(-kx)\| \\ &\le \Big(\big((2k+3)^p+3(k+2)^p+k^p\big)\big(\frac{2}{|2-2^p|}+\frac{2}{|4-2^p|}\big) \\ &+ \big((k+1)^p+k^p\big)\Big)\varepsilon\|x\|^p\end{aligned}$$

holds for all $x \in X \setminus \{0\}$ and $k \in \mathbb{N}$. Taking as $k \to \infty$, we conclude $f(x) = F(x)$ for all $x \in X \setminus \{0\}$. □

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