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AN IMPROVED LOWER BOUNDS OF UNIVARIATE BONFERRONI-TYPE INEQUALITY

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Abstract. Let A_1, A_2, $\cdots$, A_n be a sequence of events on a given probability space. Let m_n be the number of those A_i's which occur. We establish an improved lower bounds of Univariate Bonferroni-Type inequality by using the linearity of binomial moments S_1, S_2, S_3, S_4 and S_5.

1. Introduction

Let A_1, A_2, $\cdots$, A_n be a sequence of events on a given probability space, and let m_n be the number of those A_i's which occur. Put $S_0 = S_{0,n}$ and

$$S_k = S_{k,n} = \sum P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}), 1 \leq k \leq n, \tag{1.1}$$

where the summation is over all subscripts satisfying $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

For convenience in some formulae we adopt the convention $S_{k,n} = 0$ if $k > n$. For the formulation of the method we introduce some notations. $I(A)$ will denote the indicator variable of event A, that is, $I(A) = 1$ or 0 according as occurs or fails to occur, respectively. For the basic events A_j we put $I_j = I(A)$ and $m_n = I_1 + I_2 + \cdots + I_n$.

Note that (1.1) becomes $S_k = \sum P(I_{i_1} = I_{i_2} = \cdots = I_{i_k} = 1)$, $k \geq 1$, where the summation is over all subscripts satisfying $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Note that we can rewrite S_k by means of expectation. Since $I_{i_1} I_{i_2} \cdots I_{i_k} = 1$ if $I_{i_1} = I_{i_2} = \cdots = I_{i_k} = 1$ or 0 otherwise, we also get that $S_k = E[\sum I_{i_1} I_{i_2} \cdots I_{i_k}]$, where the summation is as before. By

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turning to indicator variables we immediately finds that

$$S_k = E\left[\binom{m_n}{k}\right],\ 0 \le k \le n. \tag{1.2}$$

For Bonferroni-type inequalities we require that they be valid for an arbitrary choice of the events on an arbitrary probability space. The best known such inequalities are the method of inclusion and exclusion

$$\sum_{k=0}^{2j+1}(-1)^k S_k \le P(m_n = 0) \le \sum_{k=0}^{2j}(-1)^k S_k, \tag{1.3}$$

where $j \ge 0$ is an arbitrary integer.

There is an interest in improved Bonferroni-type inequalities due to a number of interesting statistical applications. For instance, since $1 - P(m_n = 0) = P(m_n \ge 1)$, (1.3) in its simplest form becomes $S_{1,n} - S_{2,n} \le P(m_n \ge 1) \le S_{1,n}$ which is the most frequently applied form in statistics in determining confidence intervals.

Galambos and Xu([1]) has proved that

$$\frac{2}{t+1}S_1 - \frac{2}{t(t+1)}S_2 \le P(m_n \ge 1) \tag{1.4}$$

where $t \ge 1$ is an arbitrary integer. That is the uniformly best lower bound in the terms of S_1 and S_2.

Margolin and Maurer([3]) has proved that

$$S_{1,n} - S_{2,n} + max_r \sum_{(i \ne j \ne r, i<j)} P(A_i \cap A_j \cap A_r) \le P(m_n \ge 1) \tag{1.5}$$

where r is fixed integer such that $1 \le r \le n$.

Galambos and Xu([1]) has proved that

$$S_1 - \frac{t^2 - t + 1}{\binom{t+1}{2}}S_2 + \frac{3(2t-3)}{\binom{t+1}{2}}S_3 - \frac{12}{\binom{t+1}{2}}S_4 \le P(m_n \ge 1) \tag{1.6}$$

where only relatively large values of t are of interest.

Seneta([4]) has proved that

$$\begin{aligned}\sum_{i=1}^{n}P(A_i) - \sum_{i=2}^{n}\sum_{s=1}^{i-1}P(A_iA_s) + \sum_{i=3}^{n}\sum_{s=2}^{i-1}max_{1 \le j \le s-1}P(A_iA_sA_j)\\ \le P(\cup_{i=1}^{n}A_i).\end{aligned} \tag{1.7}$$

In this direction, we obtain the inequalities of the theorems that follow using the binomial moments S_1, S_2, S_3, S_4 and S_5.

2. Main result

The lower bounds are improved by the following result.

THEOREM 2.1. *For positive integers $n \ge 5$,*

$$\begin{aligned} P(m_n \ge 1) \ge S_1 &- \frac{n^2-4n+6}{\binom{n}{2}} S_2 + \frac{3n^2-18n+28}{\binom{n}{3}} S_3 \\ &- \frac{(n-3)(3n-11)}{\binom{n}{4}} S_4 + \frac{(n-3)(n-4)}{\binom{n}{5}} S_5. \end{aligned} \tag{2.1}$$

Proof. Let $A_1, A_2, \cdots, A_n$ be a sequence of events on a given probability space, and let $x = m_n$ be the number of those A_j's which occur. By the binomial moments of (1.2), the right hand side of (2.1) becomes

$$\begin{aligned} &\binom{x}{1} - \frac{n^2-4n+6}{\binom{n}{2}} \binom{x}{2} + \frac{3n^2-18n+28}{\binom{n}{3}} \binom{x}{3} \\ &- \frac{(n-3)(3n-11)}{\binom{n}{4}} \binom{x}{4} + \frac{(n-3)(n-4)}{\binom{n}{5}} \binom{x}{5}. \end{aligned} \tag{2.2}$$

We thus have to prove that

$$\begin{aligned} f(x) = \binom{x}{1} &- \frac{n^2-4n+6}{\binom{n}{2}} \binom{x}{2} + \frac{3n^2-18n+28}{\binom{n}{3}} \binom{x}{3} \\ &- \frac{(n-3)(3n-11)}{\binom{n}{4}} \binom{x}{4} + \frac{(n-3)(n-4)}{\binom{n}{5}} \binom{x}{5} \le 1 \end{aligned} \tag{2.3}$$

if $x \ge 1$ and (2.2) is less than zero or equal to zero if $x = 0$. The latter case is evident, having zero on both sides. Also, if $x = 1$, both sides of (2.3) equal 1 and if $x = 2$, the right hand side of (2.3) is $2 - \frac{n^2-4n+6}{\binom{n}{2}} = \frac{6(n-2)}{n(n-1)} \le 1$ for $n \ge 2$. If $x = 3$, we have to show that $3 - \frac{3n^2-12n+18}{\binom{n}{2}} + \frac{3n^2-18n+28}{\binom{n}{3}} \le 1$. Multiplying $n(n-1)(n-2)$ on both sides and simplifying, we get $g(n) = 5n^3 - 24n^2 + 94n - 120 \ge 0$. Since $g(n)$ is an increasing function and $g(3) = 81 > 0$, $g(n)$ is greater than zero for $n \ge 3$. If $x = 4$, we have to show that

$$4 - \frac{6n^2-24n+36}{\binom{n}{2}} + \frac{12n^2-72n+112}{\binom{n}{3}} - \frac{(n-3)(3n-11)}{\binom{n}{4}} \le 1. \tag{2.4}$$

Multiplying $n(n-1)(n-2)$ on both sides of (2.4) and simplifying, we get $k(n) = (n-3)(n-4)(n-5)(n-6) \ge 0$. Hence $k(n)$ is greater than zero or equal to zero for positive integers $n \ge 4$. Thus, for the sequel we

may assume $x \geq 5$. Let $h(x) = f(x) - 1$. Then we have to show that for any integers $x \geq 5$,
(2.5)

$$\begin{aligned} h(x) = \binom{x}{1} - \frac{n^2-4n+6}{\binom{n}{2}}\binom{x}{2} + \frac{3n^2-18n+28}{\binom{n}{3}}\binom{x}{3} \\ - \frac{(n-3)(3n-11)}{\binom{n}{4}}\binom{x}{4} + \frac{(n-3)(n-4)}{\binom{n}{5}}\binom{x}{5} - 1 \leq 0. \end{aligned}$$

Multiplying $n(n-1)(n-2)$ on both sides of (2.5) and simplifying, we have $l(x) = (x-1)^2(x-(n-2))(x-(n-1))(x-n) \leq 0$. Note that for integers x with $5 \leq x \leq n$, $l(x)$ obtains its maximum value 0 at $x = 1$, $n-2$, $n-1$, n. Hence $h(x)$ is less than zero or equal to zero for positive integers $x \geq 5$. This completes the proof. □

3. Numerical example

In 1988, Seneta([4]) Consider a numerical example of 4 event A_1 = fail mathematics, A_2 = fail Physics, A_3 = fail Chemistry, A_4 = fail Biology with the same data set of University of Sydney examinations for Science students as in Recsei and Seneta(1987) used. Here we extend extend Seneta's example to the case of $n = 5$ by adding one more event A_5 = fail Economics. Details of the data are presented below:

$P(A_1) = 0.14$, $P(A_2) = 0.26$, $P(A_3) = 0.33$, $P(A_4) = 0.21$, $P(A_5) = 0.24$, $P(A_1A_2) = 0.12$, $P(A_1A_3) = 0.12$, $P(A_1A_4) = 0.07$, $P(A_2A_3) = 0.20$, $P(A_2A_4) = 0.12$, $P(A_3A_4) = 0.16$, $P(A_iA_5) = 0.07$, $P(A_1A_2A_3) = 0.11$, $P(A_1A_2A_4) = 0.06$, $P(A_1A_3A_4) = 0.06$, $P(A_2A_3A_4) = 0.11$, $P(A_iA_jA_5) = 0.065$, $P(A_1A_2A_3A_4) = 0.06$, $P(A_iA_jA_kA_5) = 0.045$, $P(A_1A_2A_3A_4A_5) = 0.03$, where i, j, k are integers such that $1 \leq i < j < k < 5$.

We find that $S_{1,5} = 1.18$, $S_{2,5} = 1.07$, $S_{3,5} = 0.73$, $S_{4,5} = 0.24$. Then (2.1) gives $0.628 \leq P(m_n \geq 1)$.

Inequality	**Value**	**Note**
(1.4)	0.43	t=2
(1.5)	0.585	
(1.6)	0.596	t=4
(1.7)	0.585	
(2.1)	0.628	

In the above table, we see that (2.1) is the best lower bound of Bonferroni-type inequality.

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