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SOME REMARKS ON THE PERIODIC CONTINUED FRACTION

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Abstract. Using the Binet's formula, we show that the quotient related ratio $l_1(x) \neq 0$ for the eventually periodic continued fraction x. Using this ratio, we also show that the derivative of the Minkowski question mark function at the simple periodic continued fraction is infinite or 0. In particular, $l_1([\overline{1}]) = 2\log\gamma$ where γ is the golden mean $(1+\sqrt{5})/2$ and the derivative of the Minkowski question mark function at the simple periodic continued fraction $[\overline{1}]$ is infinite.

1. Introduction

Recently the differentiability and non-differentiability of the Minkowski question mark function has been studied as the investigation of the multifractal properties of the singular function([1, 3, 5]). The non-differentiability of the Minkowski question mark function is closely related to the Stern-Brocot intervals([2]). For the study of Stern-Brocot intervals, they used the ratio([2])

$$l_1(x) = \lim_{n\to\infty} \frac{2\log q_n(x)}{\sum_{i=1}^{n} a_i(x)}$$

for the continued fraction $x \in (0,1)$ and the quotient

$$p_n(x)/q_n(x) = [a_1(x), a_2(x), a_3(x), ..., a_n(x)],$$

where $x = [a_1(x), a_2(x), a_3(x), ...]$. It is well-known that $l_1(x) = 0$ for the Lebesgue measure almost all $x \in (0,1)$([2]). However there has been no study of concrete example dissatisfying the relation $l_1(x) = 0$. In this paper, we find that

$$l_1(x) = \lim_{n\to\infty} \frac{2\log q_n(x)}{\sum_{i=1}^{n} a_i(x)} \neq 0$$

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showing

$$\underline{l_1}(x) = \liminf_{n\to\infty} \frac{2\log q_n(x)}{\sum_{i=1}^{n} a_i(x)} > 0$$

for the eventually periodic continued fraction

$$x = [a_1(x), a_2(x), a_3(x), ...] \in (0, 1).$$

For this, it is essential to show that $l_1([\overline{1}]) = 2\log\gamma$ where γ is the golden mean $(1+\sqrt{5})/2$([4]). We apply the Binet's formula to its justification. Further, for $a = 1, 2, 3, 4$, the derivative of the Minkowski question mark function at the point $[\overline{a}]$ of the simple periodic continued fraction is infinite. In contrast, for the integer $a \geq 5$, the derivative of the Minkowski question mark function at the point $[\overline{a}]$ of the simple periodic continued fraction is 0.

2. Preliminaries

From now on, $\mathbb{N}$ denotes the set of the positive integers. We introduce the continued fraction

$$x = \cfrac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x)+...}}}$$

denoted by $x = [a_1(x), a_2(x), a_3(x), ...]$ where $a_i(x) \in \mathbb{N}$ for $i \in \mathbb{N}$. In this case, we also define the quotient of x for $n \in \mathbb{N}$ by

$$\frac{p_n(x)}{q_n(x)} = [a_1(x), a_2(x), a_3(x), ..., a_n(x)] = \cfrac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x)+\cdots + \frac{1}{a_n(x)}}}}.$$

We note that the quotient is the reduced form. We define the periodic continued fraction $[\overline{a_1, ..., a_n}]$ by

$$[\overline{a_1, ..., a_n}] = [a_1, ..., a_n, a_1, ..., a_n, a_1, ..., a_n, ...],$$

satisfying $a_{kn+i} = a_i$ for every non-negative integer k with $1 \leq i \leq n$, $n \in \mathbb{N}$. In particular, we call the periodic continued fraction $[\overline{a}]$ the simple periodic continued fraction. We also define the eventually periodic continued fraction $[b_1, ..., b_m, \overline{c_1, ..., c_k}]$ by

$$[b_1, ..., b_m, \overline{c_1, ..., c_k}] = [b_1, ..., b_m, c_1, ..., c_k, c_1, ..., c_k, c_1, ..., c_k, ...]$$

for $m, k \in \mathbb{N}$.

For our main result, we need the following Proposition from the Binet's formula derived by Binet in 1843, although the result was known to Euler, Daniel Bernoulli, and de Moivre more than a century earlier.

PROPOSITION 2.1. (Binet's formula) *Let q_n satisfy the recurrence relation $q_n = aq_{n-1} + bq_{n-2}$ for every $n \in \mathbb{N}$ with $q_{-1} = E$, $q_0 = C$ and $q_1 = D$. Assume that the equation $r^2 - ar - b = 0$ has the distinct solutions. Then $q_n = A\lambda_1^n + B\lambda_2^n$ where λ_1, λ_2 are the distinct solutions of the equation $r^2 - ar - b = 0$ and A, B satisfy the initial conditions $q_0 = C$ and $q_1 = D$.*

3. Main results

THEOREM 3.1. *For each $a \in \mathbb{N}$, we have*

$$q_n([\overline{a}]) = \frac{\sqrt{a^2+4}+a}{2\sqrt{a^2+4}}\lambda_1^n + \frac{\sqrt{a^2+4}-a}{2\sqrt{a^2+4}}\lambda_2^n$$

where $\lambda_1 = \frac{a+\sqrt{a^2+4}}{2}, \lambda_2 = \frac{a-\sqrt{a^2+4}}{2}$.

Proof. From the definition of the quotient of $[\overline{a}]$, we clearly see that $q_n = aq_{n-1} + q_{n-2}$ with $q_{-1} = 0$, $q_0 = 1$ and $q_1 = a$(cf. [5]). By the Binet's formula, the closed form of q_n satisfying the recurrence relation $q_n = aq_{n-1}+q_{n-2}$ is $q_n = A\lambda_1^n+B\lambda_2^n$ where λ_1, λ_2 are the solutions of the equation $r^2 - ar - 1 = 0$ and A, B are from the initial conditions $q_0 = 1$ and $q_1 = a$. Clearly $r = \frac{a\pm\sqrt{a^2+4}}{2}$ and $A = \frac{\sqrt{a^2+4}+a}{2\sqrt{a^2+4}}$, $B = \frac{\sqrt{a^2+4}-a}{2\sqrt{a^2+4}}$ from $A + B = 1$ and $A\lambda_1 + B\lambda_2 = a$. □

For $a = 1$, we have the following fundamental fact to show one of our main results.

COROLLARY 3.2. *For each $n \in \mathbb{N}$,*

$$q_n([\overline{1}]) = \frac{\sqrt{5}+1}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{\sqrt{5}-1}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n.$$

Proof. It follows from the above Theorem for $a = 1$. □

THEOREM 3.3. *For each $a \in \mathbb{N}$,*

$$l_1([\overline{a}]) = \frac{2}{a}\log\frac{a+\sqrt{a^2+4}}{2}.$$

Proof. Since $\lim_{n\to\infty}(\frac{a-\sqrt{a^2+4}}{2})^n = 0$,

$$l_1([\overline{a}]) = \lim_{n\to\infty}\frac{2\log q_n([\overline{a}])}{\sum_{i=1}^n a_i([\overline{a}])} = \lim_{n\to\infty}\frac{2\log(\frac{a+\sqrt{a^2+4}}{2})^n}{na} = \frac{2}{a}\log\frac{a+\sqrt{a^2+4}}{2}.$$

□

For $a = 1$, we have the following interesting fact associated with the golden mean, which ensures the infinite derivative of the Minkowski question mark function at the point $[\overline{1}]$.

COROLLARY 3.4.

$$l_1([\overline{1}]) = 2\log\frac{1+\sqrt{5}}{2} > \log 2 > 0.$$

LEMMA 3.5. *For the eventually periodic continued fraction*

$$x = [b_1, ..., b_m, \overline{c_1, ..., c_k}]$$

where $m, k \in \mathbb{N}$, *we have* $q_n(x) \geq q_n([\overline{1}])$ *for all* $n \in \mathbb{N}$. *Further,* $a_i(x) \leq \max\{b_1, ..., b_m, c_1, ..., c_k\}$ *for all* $i \in \mathbb{N}$.

Proof. It follows easily from the definitions of $q_n(x)$ and $a_i(x)$. □

THEOREM 3.6.

$$\underline{l_1}(x) = \liminf_{n\to\infty} \frac{2\log q_n(x)}{\sum_{i=1}^n a_i(x)} > 0$$

for the eventually periodic continued fraction

$$x = [a_1(x), a_2(x), a_3(x), ...] = [b_1, ..., b_m, \overline{c_1, ..., c_k}] \in (0, 1)$$

where $m, k \in \mathbb{N}$.

Proof. Let $\max\{b_1, ..., b_m, c_1, ..., c_k\} = M$. We note that $q_n(x) \geq q_n([\overline{1}])$ and $a_i(x) \leq M$ from the above Lemma.

$$\underline{l_1}(x) = \liminf_{n\to\infty} \frac{2\log q_n(x)}{\sum_{i=1}^n a_i(x)} \geq \liminf_{n\to\infty} \frac{2\log q_n([\overline{1}])}{\sum_{i=1}^n M} = \lim_{n\to\infty} \frac{2\log q_n([\overline{1}])}{\sum_{i=1}^n M}.$$

Therefore

$$\underline{l_1}(x) \geq \lim_{n\to\infty} \frac{2\log q_n([\overline{1}])}{nM} = \frac{2}{M}\log\frac{1+\sqrt{5}}{2} > 0.$$

□

PROPOSITION 3.7. ([3]) *Let* $x \in (0, 1)$. *If* $l_1(x) > \log 2$, *then* $Q'(x) = \infty$, *where* $Q(x)$ *is the Minkowski question mark function. Similarly, if* $0 \leq l_1(x) < \log 2$, *then* $Q'(x) = 0$.

Proof. It follows from the corollary 6.2 of [3]. □

THEOREM 3.8. *For the positive integer* $a \leq 4$, $Q'([\overline{a}]) = \infty$, *where* $Q(x)$ *is the Minkowski question mark function. For the integer* $a \geq 5$, $Q'([\overline{a}]) = 0$.

Proof. Let $f(x) = \frac{2}{x} \log \frac{x+\sqrt{x^2+4}}{2}$. We note that $f(1), f(2), f(3), f(4) > \log 2$ and $f(5), f(6) < \log 2$. Putting $g(x) = \frac{2}{x} \log \frac{x+2x}{2}$, we have $f(x) \leq g(x)$ for $x \geq 2$ and $g'(x) < 0$ for $x \geq 2$, which means that $g(x)$ is decreasing for $x \geq 2$. Observing $g(7) < \log 2$, we easily see that $l_1([\overline{a}]) = f(a) < \log 2$ for the integer $a \geq 5$. It follows from the above Proposition. □

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