JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 2, June 2009

# SOME ALGORITHMS OF THE BEST SIMULTANEOUS APPROXIMATION

Hyang J. Rhee*

Abstract. We consider various algorithms calculating best one-sided simultaneous approximations. We assume that $X$ is a compact subset of $\mathbb{R}^m$ satisfying $X = \overline{\text{int}X}$, $S$ is an n-dimensional subspace of $C(X)$, and $\mu$ is any 'admissible' measure on $X$. For any $l-$tuple $f_1, \cdots, f_\ell$ in $C(X)$, we present various ideas for best approximation to $F$ from $S(F)$. The problem of best (both one and two-sided) approximation is a linear programming problem.

## 1. Introduction

We assume that $X$ is a compact subset of $\mathbb{R}^m$ satisfying $X = \overline{\text{int}X}$, $S$ is an n-dimensional subspace of $C(X)$, and $\mu$ is any 'admissible' measure on $X$, i.e., $\mu$ is non-atomic, positive and finite and $\mu(U) > 0$ for every open set $U$. We assume that we are given $l-$tuple $F = \{f_1, \cdots, f_\ell\}$ in $C(X)$ with

$$S(F) = \bigcap_{i=1}^{\ell} \{s \in S| \ s \leq f_i\}$$

is non-empty. Since $S(F)$ is closed and convex, we have that $S(f) \neq \phi$ for all $f \in C(X)$ if and only if $S$ contains a strictly positive function. The problem we shall discuss is

$$\sup\{\int_X s \ d\mu| \ s \in S(F)\}. \tag{1.1}$$

Find a best one-sided simultaneous approximation to $f_1, \cdots, f_\ell$ from $S(F)$ is equivalent to finding a $s \in S(F)$ satisfying (1.1). We assume

Received November 20, 2008; Accepted March 05, 2009.
2000 Mathematics Subject Classification: Primary 41A28, 41A65.
Key words and phrases: one-sided simultaneous approximation, linear programming.
This work was completed with the support by a fund of Duksung Women's University.

that $s_0 \in S(F)$ is a solution to (1.1) and

$$\sigma_0 = \int_X s_0 \mathrm{d}\mu. \tag{1.2}$$

## 2. A convergent sequence $\{\sigma_m\}$

For each $m \in N$, let $x_1^m, \cdots, x_m^m \in X$, and assume that the sequence $\{x_i^m\}_{i=1}^m$ becomes dense in $X$. Any given basis for $S$, $s^1, \cdots, s^n$, set

$$p_j = \int_X s^j d\mu, \quad j = 1, \cdots, n.$$

For each m, we set

$$\sigma_m = \max\{\sum_{j=1}^n a_j p_j | \sum_{j=1}^n a_j s^j(x_i^m) \le f_k(x_i^m),\ i = 1, \cdots, m,\ k = 1, \cdots, \ell\}.$$

If for some $m$ there exists a solution $s_m$ of $\sigma_m$ with $s_m \in S(F)$ then the $s_m$ is a best one-sided simultaneous approximation to $f_1, \cdots, f_\ell$ from $S(F)$. Before proving the convergence of the algorithm, we need a fact.

Remark 2.1. There exists an $M$ such that the sequence $\{\sigma_m\}_{m \ge M}$ is bounded. Moreover, if $s_m = \sum_{j=1}^n a_j^m s^j$ is a solution of $\sigma_m$ then $\{s_m\}_{m \ge M}$ is uniformly bounded.

Its proof is totally analogous to the proof of Remark 3.0.5. [6] We now prove the convergence result.

Theorem 2.2. *Every convergent subsequence of the set of solutions $\{s_m\}$ converges to a best one-sided simultaneous approximation $s_0$ in (1.2). Thus the sequence $\{\sigma_m\}$ converges to $\sigma_0$ in (1.2).*

*Proof.* Let $\{s_{m_k}\}$ be a subsequence of $\{s_m\}$ with converges to $s_*$. Since $S$ is n-dimensional, this convergence is uniformly convergent to $s_*$ on $X$. Set

$$\sigma_* = \int_X s_* d\mu.$$

Then $\lim_{m_k \to \infty} \sigma_{m_k} = \lim \lim \int_X s_{m_k} d\mu = \int_X \lim s_{m_k} d\mu = \int_X s_* d\mu = \sigma_*$. By definition, $\sigma_m \ge \sigma_0$ for all $m$. Thus $\sigma_* \ge \sigma_0$. In the theorem 3.0.6.[6], it follows that $s_* \in S(F)$. Thus $\sigma_* \le \sigma_0$. So $\sigma_* = \sigma_0$ and $s_*$ is a solution of (1.1). Since $\lim \sigma_{m_k} = \sigma_0$ for every subsequence $\{s_{m_k}\}$ on which converges, and the $\{s_m\}$ are uniformly bounded for $m$ sufficiently large, we have

$$\lim \sigma_m = \sigma_0.$$

$\square$

## 3. The convergence result

Any given basis for $S$, $s^1, \cdots, s^n$, set

$$A = \{a \mid a = (a_1, \cdots, a_n), \sum_j a_j s^j \le f_i,\ i = 1, \cdots, \ell\}.$$

For any $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$, we also set

$$g_i(a, x) = f_i(x) - \sum_{j=1}^{n} a_j s^j(x)$$

and

$$G_i(a) = \min_{x \in X} g_i(a, x).$$

And denoted by

$$G(a) = \min_{1 \le i \le \ell} G_i(a).$$

For any $a^* \in A$ then $g_i(a^*, x) \ge 0$ for all $x \in X$, $i = 1, \cdots, \ell$. So $G_i(a^*) \ge 0$, for all $i = 1, \cdots, \ell$. By definition, $G(a^*) \ge 0$. Conversely, if $G(a^*) \ge 0$, then $g_i(a^*, x) \ge 0$ for all $x \in X$, $i = 1, \cdots, \ell$. For all $i$, $f_i - \sum_j a_j^* s^j \ge 0$. Thus $a^* \in A$. That is, $a \in A$ if and only if $G(a) \ge 0$.

Moreover, finding a best one-sided simultaneous approximation to $F$ from $S(F)$ is equivalent to finding a $a^* \in A$, $\sum_j a_j^* s^j$ satisfying (1.1).

We claim that the best one-sided simultaneous approximation problem is an almost totally general form of a linear programming problem. To demonstrate this fact, consider any linear programming problem of the form

$$\max \sum_{j=1}^{n} a_j p_j$$

subject to:

$$\sum_{j=1}^{n} a_j s^j \le f_i, \quad i = 1, \cdots, \ell.$$

The equivalence holds under certain minor restrictions. These restrictions are:

(1) There exist $\{a_j\}_{j=1}^n$ satisfying $\sum_{j=1}^n a_j s^j \le f_i, \quad i = 1, \cdots, \ell$.
(2) The maximum is in fact attained, i.e., the solution is not $\infty$.
(3) The solution set is bounded.

To verify this equivalence, note that if there exists a $d = (d_1, \cdots, d_n) \neq 0$ satisfying

$$(a) \sum_{j=1}^{n} d_j s^j \leq 0,$$

$$(b) \sum_{j=1}^{n} d_j p_j \geq 0,$$

then either condition (2) or (3) is violated. Thus there exists no $d \neq 0$ satisfying (a) and (b).

In this algorithm, we start with a set $B_M = \{x_1, \cdots, x_M\}$ of points in $X$, where we assume that the points are chosen so that there exists no $d \neq 0$ satisfying

$(a') \sum_{j=1}^{n} d_j s^j(x_k) \leq 0,\ k = 1, \cdots, M,$
$(b') \sum_{j=1}^{n} d_j p_j \geq 0.$

Equivalently, there exists no $s \in S \setminus \{0\}$ satisfying $s(x_k) \leq 0$, $k = 1, \cdots, M$, and $\int_X s d\mu \geq 0$. Thus the problem

$$\max \sum_{j=1}^{n} a_j p_j$$

subject to:

$$\sum_{j=1}^{n} a_j s^j(x_k) \leq f_i(x_k),\ k = 1, \cdots, M\ i = 1, \cdots, \ell$$

has a finite maximum and the solution set is bounded. We shall need some more.

LEMMA 3.1. *Assume that the* $\{x_1, \cdots, x_M\}$ *are given such that there exists no* $d \in \mathbb{R}^n \backslash \{0\}$ *satisfying* $(a')$ *and* $(b')$*. Let* $C_1 < C_2$ *be any fixed constants. Then the set of* $a \in \mathbb{R}^n$ *satisfying*

$$a) \sum_{j=1}^{n} a_j s^j(x_k) \leq f_i(x_k),\ k = 1, \cdots, M\ i = 1, \cdots, \ell$$

$$b) C_1 \leq \sum_{j=1}^{n} a_j p_j \leq C_2$$

*is bounded.*

*Proof.* Suppose that the set of $a \in \mathbb{R}^n$ satisfying $a)$ and $b)$ is unbounded. Thus there exists a sequence of $\{a_r\}_{r=1}^{\infty}$ in $\mathbb{R}^n$ satisfying $a)$ and $b)$, and an index $t \in \{1, \cdots, n\}$ such that

$$(1)\ |a_t^r| = \max\{|a_j^r| : j = 1, \cdots, n\}$$

$$(2)\ \lim_{r\to\infty} \varepsilon a_t^r = \infty, \text{for some } \varepsilon \in \{-1, 1\}.$$

Let $d_j^r = a_j^r / a_t^r$, $j = 1, \cdots, n$. On a subsequence, again denoted by $\{r\}$, we have

$$\lim_{r\to\infty} d_j^r = d_j,\ j = 1, \cdots, n,$$

i.e., the limits exist. Thus $|d_j| \le 1,\ j = 1, \cdots, n$, and $d_t = 1$. Since the $a_r$ satisfy $a)$ and $b)$, it follows after dividing by $a_t^r$ and letting $r \to \infty$, that

$$\varepsilon \sum_{j=1}^{n} d_j s^j(x_i) \le 0,\ i = 1, \cdots, M,$$

$$\sum_{j=1}^{n} d_j p_j = 0.$$

However this contradicts our assumption with respect to $a')$ and $b')$. This proves the lemma. □

We now describe the algorithm. Assume that we are given $B_m = \{x_1, \cdots, x_m\}$ for some $m \ge M$. Then $B_{m+1}$ is obtained as follows.

We first solve the finite problem

$$\sigma_m = \max\{\sum_{j=1}^{n} a_j p_j | \sum_{j=1}^{n} a_j s^j(x_i) \le f_k(x_i),\ i = 1, \cdots, m,\ k = 1, \cdots, \ell\}.$$

Since $m \ge M$, $\{x_1, \cdots, x_M\} \subseteq B_m$. By Lemma 3.1, this problem has a solution $a^m = (a_1^m, \cdots, a_n^m)$. If $G(a^m) \ge 0$, then $\sum_{j=1}^{n} a_j^m s^j \in S(F)$. Set

$$A_m = \{a : a = (a_1, \cdots, a_n), \sum_{j=1}^{n} a_j s^j(x_i) \le f_k(x_i),$$

$$i = 1, \cdots, m,\ k = 1, \cdots, \ell\}.$$

If $\sum_{j=1}^{n} a_j s^j \le f_k$ on $\{x_1, \cdots, x_{m+1}\}$ for all $i \in \{1, \cdots, \ell\}$ then $\sum_{j=1}^{n} a_j s^j \le f_k$ on $\{x_1, \cdots, x_m\}$ for all $i \in \{1, \cdots, \ell\}$, so

$$A_M \supset A_{M+1} \supset \cdots A.$$

Thus $\sigma_m \ge \sigma_{m+1}$, that is, $\{\sigma_{m+1}\}$ is a non-increasing sequence bounded below by $\sigma_0$, $\sum_{j=1}^{n} a_j^m p_j \ge \sigma_0$, i.e., $\sum_{j=1}^{n} a_j^m s^j$ satisfy (1.1), so we have found a best one-sided simultaneous approximation to our original problem. We are finished.

We therefore assume that $G(a^m) < 0$. Then there exists $x_{m+1} \in X \backslash B_m$ and for some $i_0 \in \{1, \cdots, \ell\}$, satisfy

$$f_{i_0}(x_{m+1}) < \sum_{j=1}^{n} a_j^m s^j(x_{m+1})$$

and $G(a^m) = g_{i_0}(a^m, x_{m+1})$. Set $B_{m+1} = B_m \cup \{x_{m+1}\}$.

This is the algorithm. In what follows we assume that the algorithm does not terminate after a finite number of steps.

THEOREM 3.2. *In the above algorithm*

$$\lim_{m\to\infty} \sigma_m = \sigma_0.$$

*And the solution set* $\{a^m\}$ *is a bounded sequence, moreover if* $a^*$ *is any cluster point of this sequence then* $\sum_{i=1}^{n} a_i^* s^i$ *is a solution of* (1.1).

*Proof.* Since $\{\sigma_m\}$ is a non-increasing sequence bounded below by $\sigma_0$, for each $m \geq M$,

$$\sum_{j=1}^{n} a_j^m s^j(x_i) \leq f_k(x_i), \ i = 1, \cdots, M, \ k = 1, \cdots, \ell,$$

and

$$\sigma_0 \leq \sum_{j=1}^{n} a_j^m p_j \leq \sigma_M.$$

From Lemma 3.1, the $\{a^m\}$ form a bounded sequence.

Let $a^* = (a_1^*, \cdots, a_n^*)$ be any cluster point of $\{a^m\}$, and $\sigma_* = \sum_{j=1}^{n} a_j^* p_j$. Then

$$\lim_{m\to\infty} \sigma_m = \sigma_* \geq \sigma_0.$$

If $a^* \in A$, i.e., $\sum_{j=1}^{n} a_j^* s^j \leq f_k$, $k \in \{1, \cdots, \ell\}$, then $\sigma_* \leq \sigma_0$ and the theorem is proved. We shall prove that $a^* \in A$.

Assume that $a^* \notin A$, i.e., $G(a^*) < 0$. Since $a^*$ is a cluster point of $\{a^m\}$, and

$$A_M \supset A_{M+1} \supset \cdots A,$$

$a^* \in \bigcap_{m=M}^{\infty} A_m$. We can choose a subsequence $\{a^{m_r}\}$, $\lim_{r\to\infty} a^{m_r} = a^*$ and $S$ is finite-dimensional, the functions $\sum_{j=1}^{n} a_j^{m_r} s^j$ uniformly converge to $\sum_{j=1}^{n} a_j^* s^j$ on $X$. Thus there exists an $M_1$ such that for all

$m \geq M_1$,

$$||\sum_{j=1}^{n} a_j^* s^j - \sum_{j=1}^{n} a_j^m s^j||_\infty < -\frac{1}{2}G(a^*).$$

Let $m_r \geq \max\{M, M_1\}$. Then

$$G(a^{m_r}) = g_{i_0}(a^{m_r}, x_{m_r+1}) = f_{i_0}(x_{m_r+1}) - \sum_{j=1}^{n} a_j^{m_r} s^j(x_{m_r+1})$$

for some $i_0 \in \{1, \cdots, \ell\}$. Since $a^* \in \bigcap_{m=M}^{\infty} A_m$, we have $a^* \in A_{m_r+1}$, and therefore

$$g_{i_0}(a^*, x_{m_r+1}) = f_{i_0}(x_{m_r+1}) - \sum_{j=1}^{n} a_j^* s^j(x_{m_r+1}) \geq 0.$$

Thus

$$\begin{aligned} G(a^{m_r}) &= g_{i_0}(a^{m_r}, x_{m_r+1}) \\ &= g_{i_0}(a^*, x_{m_r+1}) + \sum_{j=1}^{n}(a_j^* - a_j^{m_r}) s^j(x_{m_r+1}) \\ &\geq \sum_{j=1}^{n}(a_j^* - a_j^{m_r}) s^j(x_{m_r+1}) \\ &> \frac{1}{2}G(a^*). \end{aligned}$$

In other words $G(a^{m_r}) > \frac{1}{2}G(a^*)$ for all $m_r \geq M_1$. But $G$ is continuous on $\mathbb{R}^n$, and $\lim_{r\to\infty} a^{m_r} = a^*$. Thus $G(a^*) \geq \frac{1}{2}G(a^*)$. Since $G(a^*) < 0$, this is a contradiction. Thus $a^* \in A$. $\square$

For example, suppose that $X = [0, \pi]$ and $S = \mathbb{R}$. If $F = \{\sin(x)\}$ and $A_m = \{1/m, \cdots, (m-1)/m\}$, then $\sigma_m = \sin(1/m)\cdot\pi$ and $\lim_{m\to\infty} \sigma_m = 0$. So $\sin(x)$ has a best one-sided approximation 0 from $\mathbb{R}$ on $[0, \pi]$.

This paper is concerned with algorithms for calculating best one-sided simultaneous approximations, a partial discretization, a partial discretization with optimization, respectively. The problem of best two-sided simultaneous approximation can also be shown to be a linear programming problem. This fact is almost as straightforward as in the one-sided approximation. So this algorithms will expand the algorithms for calculating best two-sided simultaneous approximations.

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College of Natural Sciences
Duksung Women's University
Seoul 132-714, Republic of Korea
*E-mail*: rhj@duksung.ac.kr