JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 2, June 2009

SEMI-HOMOMORPHISMS OF BCK-ALGEBRAS

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Abstract. As a generalization of a homomorphism of BCK-algebras, the notion of a semi-homomorphism of BCK-algebras is introduced, and its characterization is given. A condition for a semi-homomorphism to be a homomorphism is provided.

1. Introduction

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [1], and have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D-posets ($= MV$-algebras). The notion of homomorphisms of BCK-algebras play an important role in studying BCK-algebras and related algebraic structures. As a generalization of a homomorphism of BCK-algebras, we introduce the notion of a semi-homomorphism of BCK-algebras, and give its characterization. We provide a condition for a semi-homomorphism to be a homomorphism.

2. Preliminaries

We first display basic concepts on BCK-algebras. By a *BCK-algebra* we mean an algebra $(X; *, 0)$ of type (2, 0) satisfying the axioms:

(a1) $(\forall x, y, z \in X)$ $(((x * y) * (x * z)) * (z * y) = 0)$,
(a2) $(\forall x, y \in X)$ $((x * (x * y)) * y = 0)$,
(a3) $(\forall x \in X)$ $(x * x = 0, 0 * x = 0)$,
(a4) $(\forall x, y \in X)$ $(x * y = 0, y * x = 0 \Rightarrow x = y)$.

Received September 12, 2008; Accepted November 20, 2008.
2000 Mathematics Subject Classification: Primary 06F35, 06A06; Secondary 03G25.
Key words and phrases: (irreducible) ideal, order system, semi-homomorphism.
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This paper has been supported by the 2009 Hannam University Fund.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. A BCK-algebra X is said to be *bounded* if there exists the bound 1 such that $x \leq 1$ for all $x \in X$.

In any BCK-algebra X, the following hold:

(b1) $(\forall x \in X)\ (x * 0 = x)$,
(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(b3) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$,
(b4) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)$.

A mapping $f : X \to Y$ of BCK-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCK-algebras, then $f(0) = 0$.

A subset A of a BCK-algebra X is called an *ideal* of X if it satisfies:

(c1) $0 \in A$,
(c2) $(\forall x \in A)\ (\forall y \in X)\ (y * x \in A \Rightarrow y \in A)$.

Note that every ideal A of a BCK-algebra X satisfies:

$$(\forall x \in A)\,(\forall y \in X)\,(y \leq x \Rightarrow y \in A). \tag{2.1}$$

The set of all ideals of a BCK-algebra X is denoted by $Id(X)$. It is known that $Id(X)$ is an infinitely distributive lattice (see [7]).

If A is a nonempty subset of a BCK-algebra X, then the ideal of X generated by A, in symbol $\langle A\rangle$, is the set (see [2, Theorem 3])

$$\langle A\rangle = \left\{ x \in X \;\middle|\; \begin{array}{l} (\cdots((x * a_0) * a_1) * \cdots) * a_n = 0 \\ \text{for some } a_0, a_1, \cdots, a_n \in A \end{array} \right\} \tag{2.2}$$

If $A = \{a\}$, then we denote $\langle\{a\}\rangle$ by $\langle a\rangle$ and call it as the ideal of X generated by a.

Definition 2.1. ([3]) An ideal A of a BCK-algebra X is said to be *irreducible* if it satisfies:

$$(\forall B, C \in Id(X))\,(A = B \cap C \Rightarrow A = B \text{ or } A = C). \tag{2.3}$$

Denote by $IId(X)$ the set of all irreducible ideals of X.

Definition 2.2. ([5]) A subset I of a BCK-algebra X is called an *order system* of X if it satisfies:

(c3) I is an upper set, that is, I satisfies:

$$(\forall x \in X)\,(\forall y \in I)\,(y \leq x \Rightarrow x \in I),$$

(c4) $(\forall x, y \in I)\ (\exists z \in I)\ (z \leq x,\ z \leq y)$.

Denote by $Os(X)$ the set of all order systems of X.

3. Semi-homomorphisms

DEFINITION 3.1. A mapping $f : X \to Y$ of BCK-algebras is called a *semi-homomorphism* if it satisfies:

(d1) $f(0) = 0$,
(d2) $(\forall x, y \in X)$ $(f(x) * f(y) \leq f(x * y))$.

Note that every homomorphism of BCK-algebras is also a semi-homomorphism, but the converse is not true in general as seen in the following examples.

EXAMPLE 3.2. *Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:*

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	a	a	0	0
d	d	a	a	a	0

Define a mapping $f : X \to X$ by $f(0) = 0$, $f(a) = b$, $f(b) = c$, $f(c) = d$ and $f(d) = d$. Then f is a semi-homomorphism, but f is not a homomorphism since $f(c * a) = b \neq a = f(c) * f(a)$.

EXAMPLE 3.3. *Let $X_1 = X_2 = \{0, a, b, c\}$ be BCK-algebras with the following Cayley tables respectively:*

$*_1$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

$*_2$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
c	c	a	a	0

Define a mapping $f : X_1 \to X_2$ by $f(0) = 0$, $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then f is a semi-homomorphism, but f is not a homomorphism since $f(a *_1 b) = a \neq 0 = f(a) *_2 f(b)$.

DEFINITION 3.4. ([6]) Let X be a BCK-algebra. For a fixed element $a \in X$, we define a map $R_a : X \to X$ by $R_a(x) = x * a$ for all $x \in X$, and call R_a a *right map* on X. Also, a *left map* on X is defined analogously, and denoted it by L_a.

Obviously, R_0 and L_0 are homomorphisms and so semi-homomorphisms. Generally, both a right map R_a and a left map L_a for $a \neq 0$ on a

BCK-algebra may not be semi-homomorphisms as seen in the following examples.

EXAMPLE 3.5. *Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:*

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	b
c	c	b	a	0	b
d	d	d	d	d	0

For any $x \in X$, let R_x (resp. L_x) be a right (resp. left) map on X. Then every right map on X is a homomorphism, i.e., R_0, R_a, R_b, R_c and R_d are all homomorphisms, and hence these are all semi-homomorphisms. But the left map L_x is not a semi-homomorphism for $x = a, b, c$ and d, since $L_x(0) \neq 0$.

EXAMPLE 3.6. *Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	a
c	c	c	c	0

For any $x \in X$, let R_x (resp. L_x) be a right (resp. left) map on X. Then R_0 and R_b are homomorphisms, and so these are semi-homomorphisms. But R_a and R_c are not semi-homomorphisms since $R_a(b) * R_a(a) = a \nleq 0 = R_a(b * a)$ and $R_c(b) * R_c(c) = a \nleq 0 = R_c(b * c)$, and hence R_a and R_c are not homomorphisms. And the left map L_x is not a semi-homomorphism for $x = a, b$ and c, since $L_x(0) \neq 0$.

PROPOSITION 3.7. *Let R_a be a right map on a BCK-algebra X. Then the following are equivalent:*

(i) *R_a is a semi-homomorphism,*
(ii) *R_a is a homomorphism,*
(iii) *$R_a = R_a^2$.*

Proof. (i) $\Rightarrow$ (ii) Assume that R_a is a semi-homomorphism. Since $0 \leq a$ by (a3), it follows from (b1) and (b4) that $y * a \leq y$ for all $y \in X$. Then

$$R_a(x * y) = (x * y) * a = (x * a) * y \leq (x * a) * (y * a) = R_a(x) * R_a(y)$$

for all $x, y \in X$ by using (b2) and (b4). Hence R_a is a homomorphism.

(ii) $\Rightarrow$ (iii) Assume that R_a is a homomorphism. Then

$$R_a^2(x) = R_a(R_a(x)) = R_a(x * a) = R_a(x) * R_a(a) = R_a(x)$$

for all $x \in X$. Hence $R_a = R_a^2$.

(iii) $\Rightarrow$ (i) Assume that $R_a = R_a^2$. It follows from (b2), (b3) and (b4) that

$$\begin{aligned} R_a(x) * R_a(y) &= R_a^2(x) * R_a(y) = ((x * a) * a) * (y * a) \\ &= ((x * a) * (y * a)) * a \\ &\leq (x * y) * a = R_a(x * y) \end{aligned}$$

for all $x, y \in X$. Hence R_a is a semi-homomorphism. □

Consider a left map L_a on a BCK-algebra X. If L_a is a semi-homomorphism, then $0 = L_a(0) = a * 0 = a$. Hence we have the following proposition.

Proposition 3.8. *For a left map L_a on a BCK-algebra X, the following are equivalent:*

(i) *L_a is a semi-homomorphism,*
(ii) *L_a is a homomorphism,*
(iii) $a = 0$.

Theorem 3.9. *If $f : X \to Y$ is a semi-homomorphism of BCK-algebras, then the set*

$$\ker(f) := \{x \in X \mid f(x) = 0\}$$

is an ideal of X.

Proof. Obviously $0 \in \ker(f)$ by (d1). Let $x, y \in X$ be such that $y \in \ker(f)$ and $x * y \in \ker(f)$. Then $f(y) = 0$ and $f(x * y) = 0$. It follows from (b1) and (d2) that

$$f(x) = f(x) * 0 = f(x) * f(y) \leq f(x * y) = 0$$

so that $f(x) = 0$, i.e., $x \in \ker(f)$. Therefore $\ker(f)$ is an ideal of X. □

Theorem 3.9 is a generalization of [4, Proposition 10].

We give a characterization of a semi-homomorphism of BCK-algebras. We first need the following lemma.

Lemma 3.10. ([3]) *If A is an ideal of a BCK-algebra X and a is not contained in A, then there is an irreducible ideal B of X such that $A \subseteq B$ and $a \notin B$.*

Note that $\langle a\rangle$ is the ideal generated by a in a BCK-algebra X. If $x \not\leq y$ for $x, y \in X$, then $\langle y\rangle$ is an ideal of X which does not contain x. Hence we have the following corollary.

COROLLARY 3.11. *If $x \not\leq y$ in a BCK-algebra X, then there is an irreducible ideal B of X such that $y \in B$ and $x \notin B$.*

THEOREM 3.12. *Let $f : X \to Y$ be a mapping of BCK-algebras. Then f is a semi-homomorphism if and only if it satisfies:*

$$(\forall B \subseteq Y)\,(B \in Id(Y) \;\Rightarrow\; f^{-1}(B) \in Id(X)). \tag{3.1}$$

Proof. Assume that f is a semi-homomorphism. Let $B \in Id(Y)$. Obviously $0 \in f^{-1}(B)$ by (d1). Let $x, y \in X$ be such that $x*y \in f^{-1}(B)$ and $y \in f^{-1}(B)$. Then $f(x * y) \in B$ and $f(y) \in B$. Since $f(x) * f(y) \leq f(x * y)$ by (d2), it follows from (2.1) that $f(x) * f(y) \in B$ so from (c2) that $f(x) \in B$, i.e., $x \in f^{-1}(B)$. Therefore $f^{-1}(B) \in Id(X)$. Conversely suppose that f satisfies (3.1). If $f(0) \neq 0$, then there exists an ideal C of Y such that $f(0) \notin C$, i.e., $0 \notin f^{-1}(C)$. This is a contradiction, and so $f(0) = 0$. Assume that (d2) is not valid. Then $f(x) * f(y) \not\leq f(x * y)$ for some $x, y \in X$. It follows from Corollary 3.11 that there exists an irreducible ideal B of Y such that $f(x * y) \in B$ and $f(x) * f(y) \notin B$. Consider the ideal $\langle B \cup \{f(y)\}\rangle$ of Y generated by $B \cup \{f(y)\}$. Then $f(x) \notin \langle B \cup \{f(y)\}\rangle$ because if not then

$$(\cdots((f(x) * f(y)) * b_1) * \cdots) * b_n = 0 \in B$$

for some $b_1, b_2, \cdots, b_n \in B$. Since B is an ideal of Y, it follows from (c2) that $f(x) * f(y) \in B$, which is a contradiction. Using Lemma 3.10, there exists an irreducible ideal Q of Y such that $\langle B \cup \{f(y)\}\rangle \subseteq Q$ and $f(x) \notin Q$, that is, $B \subseteq Q$, $y \in f^{-1}(Q)$ and $x \notin f^{-1}(Q)$. Since $x*y \in f^{-1}(B) \subseteq f^{-1}(Q)$, it follows from (3.1) and (c2) that $x \in f^{-1}(Q)$. This is a contradiction. Therefore (d2) is valid. □

LEMMA 3.13. *Let $f : X \to Y$ be a homomorphism of BCK-algebras. For any $A \in IId(X)$, let*

$$[f(X \setminus A)) := \{y \in Y \mid f(a) \leq y \text{ for some } a \in X \setminus A\}. \tag{3.2}$$

Then $[f(X \setminus A))$ is an order system of Y.

Proof. Let $y \in Y$ and $w \in [f(X \setminus A))$ be such that $w \leq y$. Then there exists $a \in X \setminus A$ such that $f(a) \leq w$. It follows from the transitivity of $\leq$ that $f(a) \leq y$ so that $y \in [f(X \setminus A))$. Hence (c3) is valid. Now let $x, y \in [f(X \setminus A))$. Then there exist $a, b \in X \setminus A$ such that $f(a) \leq x$ and $f(b) \leq y$. Obviously $f(a), f(b) \in [f(X \setminus A))$. Therefore (c4) is valid. □

LEMMA 3.14. ([5]) *Let* $A \in Id(X)$ *and* $I \in Os(X)$. *If* A *and* I *are disjoint, then there exists an irreducible ideal* B *of* X *such that* $A \subseteq B$ *and* $B \cap I = \emptyset$.

We provide a condition for a semi-homomorphism to be a homomorphism.

THEOREM 3.15. *Let* $f : X \to Y$ *be a semi-homomorphism of BCK-algebras. Then the following assertions are equivalent:*

(i) f *is a homomorphism,*

(ii) *For every* $A \in IId(X)$ *and* $B \in IId(Y)$,

$$f^{-1}(B) \subseteq A \Rightarrow (\exists P \in IId(Y))(B \subseteq P,\ f^{-1}(P) = A). \tag{3.3}$$

Proof. Assume that f is a homomorphism. Let $A \in IId(X)$ and $B \in IId(Y)$ be such that $f^{-1}(B) \subseteq A$. Consider the ideal $\langle B \cup f(A)\rangle$ of Y generated by $B \cup f(A)$. By means of Lemma 3.13, $[f(X \setminus A))$ is an order system of Y. Now we prove that $\langle B \cup f(A)\rangle$ and $[f(X \setminus A))$ are disjoint. Suppose that they are not disjoint and take $w \in \langle B \cup f(A)\rangle \cap [f(X \setminus A))$. Then $f(a) \leq w$ for some $a \in X \setminus A$ and

$$(\cdots((w * f(a_1)) * f(a_2)) * \cdots) * f(a_n) \in B$$

for some $a_1, a_2, \cdots, a_n \in A$. Using (b4), we have

$$\begin{aligned}&(\cdots((f(a) * f(a_1)) * f(a_2)) * \cdots) * f(a_n)\\ &\leq (\cdots((w * f(a_1)) * f(a_2)) * \cdots) * f(a_n).\end{aligned}$$

Since B is an ideal of Y, it follows from (2.1) that

$$(\cdots((f(a) * f(a_1)) * f(a_2)) * \cdots) * f(a_n) \in B.$$

Since f is a homomorphism, we have

$$f((\cdots((a * a_1) * a_2) * \cdots) * a_n) \in B,$$

and so $(\cdots((a * a_1) * a_2) * \cdots) * a_n \in f^{-1}(B) \subseteq A$. It follows from (c2) that $a \in A$. This is a contradiction. By Lemma 3.14, there exists an irreducible ideal P of Y such that $\langle B \cup f(A)\rangle \subseteq P$ and $P \cap [f(X \setminus A)) = \emptyset$. It follows that $B \subseteq P$ and $f(A) \subseteq P$ so that $A \subseteq f^{-1}(P)$. Now if $x \in f^{-1}(P)$, then $f(x) \in P$. Since $P \cap [f(X \setminus A)) = \emptyset$, we have $f(x) \notin [f(X \setminus A))$, and so $x \in A$. Therefore $A = f^{-1}(P)$. Conversely suppose that (3.3) is valid. Let $a, b \in X$ be such that $f(a * b) \not\leq f(a) * f(b)$. Then there exists an irreducible ideal B of Y such that $f(a) * f(b) \in B$ and $a * b \notin f^{-1}(B)$. Since f is a semi-homomorphism, $f^{-1}(B) \in Id(X)$ by (3.1). Consider the ideal $\langle f^{-1}(B) \cup \{b\}\rangle$ of X generated by $f^{-1}(B) \cup \{b\}$. Then $a \notin \langle f^{-1}(B) \cup \{b\}\rangle$. For, if not then $a * b \in f^{-1}(B)$, a contradiction. Using Lemma 3.10, there exists $A \in IId(X)$ such that $\langle f^{-1}(B) \cup \{b\}\rangle \subseteq$

A and $a \notin A$, that is, $f^{-1}(B) \subseteq A$, $b \in A$ and $a \notin A$. It follows from (3.3) that there exists $P \in IId(Y)$ such that $B \subseteq P$ and $f^{-1}(P) = A$. Since $f(a) * f(b) \in B \subseteq P$ and $f(b) \in f(A) \subseteq P$, we have $f(a) \in P$ by (c2), which is a contradiction. Hence f is a homomorphism. □

Let $f : X \to Y$ be a homomorphism of BCK-algebras and let

$$\tau := \{B \in IId(Y) \mid f^{-1}(B) \in IId(X)\}. \tag{3.4}$$

Consider a mapping

$$\Phi : \tau \to IId(X),\ B \mapsto f^{-1}(B). \tag{3.5}$$

Let $A \in IId(X)$ and consider the ideal $\langle f(A)\rangle$ of Y. We will prove that if f is injective, then $\langle f(A)\rangle$ and $[f(X \setminus A))$ are disjoint. Let f be injective. Assume that $\langle f(A)\rangle \cap [f(X \setminus A)) \neq \emptyset$ and take $y \in \langle f(A)\rangle \cap [f(X \setminus A))$. Then $y \in \langle f(A)\rangle$ and $y \in [f(X \setminus A))$, and hence $f(b) \leq y$ for some $b \in X \setminus A$ and

$$(\cdots((y * f(a_1)) * f(a_2)) * \cdots) * f(a_n) = 0$$

for some $a_1, a_2, \cdots, a_n \in A$. It follows from (b4) that

$$\begin{aligned}&(\cdots((f(b) * f(a_1)) * f(a_2)) * \cdots) * f(a_n)\\ &\leq (\cdots((y * f(a_1)) * f(a_2)) * \cdots) * f(a_n) = 0\end{aligned}$$

so that

$$\begin{aligned}&f((\cdots((b * a_1) * a_2) * \cdots) * a_n)\\ &= (\cdots((f(b) * f(a_1)) * f(a_2)) * \cdots) * f(a_n)\\ &= 0 = f(0).\end{aligned}$$

Since f is injective, we get

$$(\cdots((b * a_1) * a_2) * \cdots) * a_n = 0 \in A$$

and hence $b \in A$ by (c2). This is a contradiction. Therefore $\langle f(A)\rangle$ and $[f(X \setminus A))$ are disjoint. Using Lemma 3.14, there exists $B \in IId(Y)$ such that $f(A) \subseteq B$ and $B \cap [f(X \setminus A)) = \emptyset$, that is, $f^{-1}(B) = A$. Hence Φ is surjective. Now suppose that Φ is surjective and let $a, b \in X$ be such that $b \not\leq a$. Then there exists an irreducible ideal A of X such that $a \in A$ and $b \notin A$. Since Φ is surjective,

$$(\exists B \in \tau \subseteq IId(Y))\,(f^{-1}(B) = A).$$

Thus $a \in f^{-1}(B)$ and $b \notin f^{-1}(B)$, i.e., $f(a) \in B$ and $f(b) \notin B$. It follows that $f(b) \not\leq f(a)$, which implies that f is injective. Hence we have the following theorem.

THEOREM 3.16. *Let* $f : X \to Y$ *be a homomorphism of BCK-algebras. For a mapping* Φ *which is given in* (3.5)*, the following are equivalent:*

(i) f *is injective,*
(ii) Φ *is surjective.*

References

[1] Y. Imai and K. Iséki, *On axiom systems of propositional calculi,* Proc. Jpn. Acad. **42** (1966), 19-21.
[2] K. Iséki, *On ideals in BCK-algebras,* Math. Seminar Notes **3** (1975), 1–12.
[3] K. Iséki, *On some ideals in BCK-algebras,* Math. Seminar Notes **3** (1975), 65–70.
[4] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras,* Math. Japonica **23** (1978), no. 1, 1–26.
[5] Y. B. Jun, K. J. Lee and C. H. Park, *A method to make BCK-algebras,* Commun. Korean Math. Soc. **22** (2007), No. 4, 503-508.
[6] J. Meng and Y. B. Jun, *BCK-algebras,* Kyungmoon Sa Co. Korea, 1994.
[7] M. Palasiński, *On ideals and congruence lattices of BCK-algebras,* Math. Seminar Notes **9** (1981), 441–443.

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