

## SMARANDACHE GT-ALGEBRAS

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ABSTRACT. We introduce the notion of Smarandache GT-algebras, and the notion of Smarandache GT-filters of the Smarandache GT-algebra related to the Tarski algebra, and related some properties are investigated.

### 1. Introduction

The variety of Tarski algebras was introduced by J. C. Abbott in [2]. These algebras are an algebraic counterpart of the  $\{\vee, \rightarrow\}$ -fragment of the propositional classical calculus. S. A. Celani ([5]) introduced Tarski algebras with a modal operator as a generalization of the concept of Boolean algebra with a modal operator which he researched into these fragments of the algebraic viewpoint. Properties of filters in Tarski algebras were treated by S. A. Celani ([5]) and the authors ([7]). Recently, J. Kim, Y. Kim and E. H. Roh ([7]) considered decompositions and expansions of filters in Tarski algebras, and also they have shown that there is no non-trivial quadratic Tarski algebras on a field  $X$  with  $|X| \geq 3$ . However, we feel that the concept of Tarski algebra is relatively too strong for filters. Kim et al. ([8]) established a new algebra, called a GT-algebra, which is a generalization of Tarski algebra, and gave a method to construct a GT-algebra from a quasi-ordered set. Generally, a Smarandache Structure on a set  $A$  means a weak structure  $W$  on  $A$  such that there exists a proper subset  $B$  of  $A$  which is embedded with a strong structure  $S$ . In this paper, we introduce the notion of  $\mathcal{S}^{\mathcal{F}}$ GT-algebras and  $\mathcal{S}_{\Omega}^{\mathcal{F}}$ GT-filters, and investigate some related properties. It's interesting to study the Smarandache Structure in GT-algebras.

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Let us review some definitions and results. By a *Tarski algebra* we mean an algebra  $(X; \rightarrow, 1)$  of type  $(2, 0)$  satisfying the following conditions:

- (T1)  $(\forall a \in X)(1 \rightarrow a = a)$ .
- (T2)  $(\forall a \in X)(a \rightarrow a = 1)$ .
- (T3)  $(\forall a, b, c \in X)(a \rightarrow (b \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c))$ .
- (T4)  $(\forall a, b \in X)((a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a)$ .

DEFINITION 1.1. [8] By a *generalized Tarski algebra* (*GT-algebra*, for short) we mean an algebra  $(X; \rightarrow, 1)$  of type  $(2, 0)$  satisfying the following conditions: (T1), (T2), and (T3).

A reflexive and transitive relation  $\mathfrak{R}$  on a set  $X$  is called a *quasi-ordering* of  $X$ , and the couple  $(X, \mathfrak{R})$  is called a *quasi-ordered set* ([4]). Note that if  $X$  is a GT-algebra, then the relation  $\leq$  by setting  $x \leq y$  if and only if  $x \rightarrow y = 1$  for any  $a, b \in X$  is a quasi-ordering of  $X$ ; with respect to this quasi-ordering  $1$  is the greatest element of  $X$ .

EXAMPLE 1.2. Let  $X := \{a, b, c, 1\}$  be a set with the following Cayley table:

$\rightarrow$	$a$	$b$	$c$	$1$
$a$	1	1	$c$	1
$b$	1	1	$c$	1
$c$	1	1	1	1
$1$	$a$	$b$	$c$	1

Then  $(X; \rightarrow, 1)$  is a GT-algebra ([8]), and the relation  $\mathfrak{R} := \{(a, a), (a, b), (a, 1), (b, a), (b, b), (b, 1), (c, a), (c, b), (c, c), (c, 1), (1, 1)\}$  is a quasi-ordering of  $X$ , which is not an anti-symmetric relation of  $X$ .

LEMMA 1.3. [8] *Let  $X$  be a GT-algebra. Then*

- (p1)  $(\forall a \in X)(a \leq 1)$ .
- (p2)  $(\forall a, b \in X)(a \leq b \rightarrow a)$ .
- (p3)  $(\forall a, b \in X)(a \rightarrow (a \rightarrow b) = a \rightarrow b)$ .
- (p4)  $(\forall a, b \in X)(a \leq (a \rightarrow b) \rightarrow b)$ .
- (p5)  $(\forall a, b, c \in X)(a \leq b \Rightarrow c \rightarrow a \leq c \rightarrow b)$ .

DEFINITION 1.4. [8] Let  $X$  be a GT-algebra. A nonempty subset  $F$  of  $X$  is called a *generalized Tarski-filter* (*GT-filter*, for short) of  $X$  if it satisfies the following conditions:

- (F1)  $(\forall a, b \in X)(b \in F \Rightarrow a \rightarrow b \in F)$ .
- (F2)  $(\forall a, b \in X)(a \rightarrow b \in F, a \in F \Rightarrow b \in F)$ .

Note that every GT-filter contains the element 1 by (T2) and (F1).

**THEOREM 1.5.** [8] *Let  $F$  be a nonempty subset of a GT-algebra  $X$ . Then  $F$  is a GT-filter of  $X$  if and only if it satisfies  $1 \in F$  and (F2).*

**2. Main Theorem**

**LEMMA 2.1.** *Let  $(X; \rightarrow, 1)$  be a nontrivial GT-algebra. For every  $a (\neq 1) \in X$ , the set  $\{a, 1\}$  is a Tarski algebra under the operation on  $X$ .*

*Proof.* Straightforward. □

Lemma 2.1 shows that every nontrivial GT-algebra  $(X; \rightarrow, 1)$  has a Tarski algebra of order 2. The following example shows that there is a GT-algebra in which there are no proper Tarski algebra of order more than equal to 3.

**EXAMPLE 2.2.** Let  $X := \{a, b, c, 1\}$  be a set with the following Cayley table:

$\rightarrow$	$a$	$b$	$c$	$1$
$a$	1	1	1	1
$b$	$a$	1	1	1
$c$	$a$	$b$	1	1
$1$	$a$	$b$	$c$	1

It is routine to check that  $(X; \rightarrow, 1)$  is a GT-algebra which is not a Tarski algebra, and the sets  $\{a, b, 1\}$ ,  $\{a, c, 1\}$ ,  $\{b, c, 1\}$  are not Tarski algebras.

**DEFINITION 2.3.** A *Smarandache GT-algebra* (briefly,  $\mathcal{S}^{\mathfrak{X}}$ GT-algebra) is defined to be a GT-algebra  $X$  in which there exists a proper subset  $\Omega$  of  $X$  such that

- (i)  $1 \in \Omega$  and  $|\Omega| \geq 3$ ,
- (ii)  $\Omega$  is a Tarski algebra with respect to the same operation on  $X$ .

Note that any GT-algebra of order 3 cannot be an  $\mathcal{S}^{\mathfrak{X}}$ GT-algebra. Hence, if  $X$  is an  $\mathcal{S}^{\mathfrak{X}}$ GT-algebra, then  $|X| \geq 4$ . Notice that the GT-algebra  $X$  in Example 2.2 is not an  $\mathcal{S}^{\mathfrak{X}}$ GT-algebra.

**EXAMPLE 2.4.** Let  $X := \{a, b, c, 1\}$  be a set with the following Cayley table:

$\rightarrow$	$a$	$b$	$c$	$1$
$a$	1	$b$	1	1
$b$	$a$	1	1	1
$c$	$a$	$b$	1	1
$1$	$a$	$b$	$c$	1

It is easy to check that  $(X; \rightarrow, 1)$  is an  $\mathcal{S}^{\mathfrak{X}}\text{GT}$ -algebra since  $\Omega := \{a, b, 1\}$  is a Tarski algebra which is properly contained in  $X$ .

In what follows, let  $X$  and  $\Omega$  denote an  $\mathcal{S}^{\mathfrak{X}}\text{GT}$ -algebra and a nontrivial proper Tarski algebra of order more than 2, respectively, unless specified.

DEFINITION 2.5. A nonempty subset  $F$  of  $X$  is called a *Smarandache GT-filter of  $X$  related to  $\Omega$*  (briefly,  $\mathcal{S}_{\Omega}^{\mathfrak{X}}\text{GT}$ -filter of  $X$ ) if it satisfies the following conditions:

- (SF1)  $1 \in F$ ,  
 (SF2)  $(\forall x \in \Omega)(\forall a \in F)(a \rightarrow x \in F \Rightarrow x \in F)$ .

EXAMPLE 2.6. Let  $X := \{a, b, c, 1\}$  be the  $\mathcal{S}^{\mathfrak{X}}\text{GT}$ -algebra with  $\Omega := \{a, b, 1\}$  in Example 2.4. Then the sets  $F_1 := \{a, 1\}$ ,  $F_2 := \{c, 1\}$ ,  $F_3 := \{a, c, 1\}$ ,  $F_4 := \{b, c, 1\}$  are  $\mathcal{S}_{\Omega}^{\mathfrak{X}}\text{GT}$ -filters of  $X$ .

EXAMPLE 2.7. Let  $X := \{a, b, c, d, 1\}$  be a set with the following Cayley table:

$\rightarrow$	$a$	$b$	$c$	$d$	$1$
$a$	1	1	1	$d$	1
$b$	1	1	1	$d$	1
$c$	1	1	1	$d$	1
$d$	$a$	$b$	$c$	1	1
1	$a$	$b$	$c$	$d$	1

It can be readily check that  $(X; \rightarrow, 1)$  is an  $\mathcal{S}^{\mathfrak{X}}\text{GT}$ -algebra with  $\Omega := \{a, d, 1\}$ . Then the set  $F_1 := \{b, d, 1\}$  is an  $\mathcal{S}_{\Omega}^{\mathfrak{X}}\text{GT}$ -filters of  $X$ . But  $F_2 := \{c, 1\}$  is not an  $\mathcal{S}_{\Omega}^{\mathfrak{X}}\text{GT}$ -filter of  $X$  since  $c \rightarrow a = 1 \in F_2$  and  $a \notin F_2$ .

THEOREM 2.8. Let  $\Omega_1$  and  $\Omega_2$  be Tarski algebras contained in a Smarandache GT-algebra  $X$  and  $\Omega_1 \subset \Omega_2$ . Then every  $\mathcal{S}_{\Omega_2}^{\mathfrak{X}}\text{GT}$ -filter of  $X$  is an  $\mathcal{S}_{\Omega_1}^{\mathfrak{X}}\text{GT}$ -filter of  $X$ , but the converse is not true.

*Proof.* Straightforward. □

EXAMPLE 2.9. Let  $X := \{a, b, c, d, e, 1\}$  be a set with the following Cayley table:

$\rightarrow$	$a$	$b$	$c$	$d$	$e$	$1$
$a$	1	1	1	$d$	1	1
$b$	$a$	1	$c$	$d$	1	1
$c$	$a$	$b$	1	$d$	1	1
$d$	$a$	$b$	$c$	1	1	1
$e$	$a$	$b$	$c$	$d$	1	1
1	$a$	$b$	$c$	$d$	$e$	1

Then  $(X; \rightarrow, 1)$  is a GT-algebra,  $\Omega_1 := \{a, d, 1\}$  and  $\Omega_2 := \{a, b, c, d, 1\}$  are Tarski algebras. Hence we know that  $X$  is a Smarandache GT-algebra, and the subset  $F := \{a, c, 1\}$  is an  $\mathcal{S}_{\Omega_1}^{\rightarrow}$  GT-filter of  $X$ , but not an  $\mathcal{S}_{\Omega_2}^{\rightarrow}$  GT-filter of  $X$  since  $a \rightarrow b = 1 \in F$  and  $a \in F$  but  $b \notin F$ .

Example 2.9 shows that there exists a Tarski algebra  $\Omega$  contained in a Smarandache GT-algebra  $X$  such that an  $\mathcal{S}_{\Omega}^{\rightarrow}$  GT-filter of  $X$  is not a GT-filter of  $X$ .

THEOREM 2.10. For any  $a \in X$ , the set  $[a] := \{x \in X | a \leq x\}$  is an  $\mathcal{S}_{\Omega}^{\rightarrow}$  GT-filter of  $X$ .

*Proof.* Obviously,  $1 \in [a]$ . Let  $z \in \Omega$  and  $x \in [a]$  and  $x \rightarrow z \in [a]$ . Then we have

$$a \rightarrow z = 1 \rightarrow (a \rightarrow z) = a \rightarrow (x \rightarrow z) = 1.$$

Hence  $z \in [a]$ . Therefore,  $[a]$  is an  $\mathcal{S}_{\Omega}^{\rightarrow}$  GT-filter of  $X$ . □

LEMMA 2.11. Every  $\mathcal{S}_{\Omega}^{\rightarrow}$  GT-filter  $F$  of  $X$  satisfies the following inclusion:

$$\Omega \rightarrow F \subseteq F,$$

where  $\Omega \rightarrow F := \{x \rightarrow a | x \in \Omega, a \in F\}$ .

*Proof.* Let  $z \in \Omega \rightarrow F$ . Then  $z = x \rightarrow a$  for some  $x \in \Omega$  and  $a \in F$ . Thus we have  $z \in F$  since  $a \rightarrow z = a \rightarrow (x \rightarrow a) = 1 \in F$ . □

Lemma 2.11 shows that every  $\mathcal{S}_{\Omega}^{\rightarrow}$  GT-filter  $F$  of  $X$  satisfies the conditions

$$\Omega \rightarrow F \subseteq F, \text{ and (SF2).}$$

The following example shows that the converse is not true in general.

EXAMPLE 2.12. Let  $X := \{a, b, c, 1\}$  be a set with the following Cayley table:

$\rightarrow$	$a$	$b$	$c$	$1$
$a$	$1$	$b$	$c$	$1$
$b$	$a$	$1$	$c$	$1$
$c$	$a$	$b$	$1$	$1$
$1$	$a$	$b$	$c$	$1$

It is ready to check that  $(X; \rightarrow, 1)$  is an  $\mathcal{S}_\Omega^{\mathfrak{X}}$ GT-algebra with  $\Omega := \{a, c, 1\}$ . Let  $F := \{b\}$ . Then  $F$  satisfies the conditions  $\Omega \rightarrow F \subseteq F$  and (SF2). But  $1 \notin F$ .

If  $F$  is an  $\mathcal{S}_\Omega^{\mathfrak{X}}$ GT-filter of  $X$  satisfies  $\Omega \cap F \neq \emptyset$  and  $\Omega \rightarrow F \subseteq F$ , then there exists  $a \in \Omega \cap F$ , and so we have  $1 = a \rightarrow a \in F$ . Hence we obtain the following theorem.

THEOREM 2.13. *Let  $F$  be a nonempty subset of  $X$  that satisfies  $\Omega \cap F \neq \emptyset$ . Then  $F$  is an  $\mathcal{S}_\Omega^{\mathfrak{X}}$ GT-filter of  $X$  if and only if  $\Omega \rightarrow F \subseteq F$  and (SF2).*

For any GT-algebra  $X$  and  $x, y \in X$ , we denote

$$A(x, y) := \{z \in X \mid x \leq y \rightarrow z\}.$$

THEOREM 2.14. *For any  $x, y \in X$ , the set  $A(x, y)$  is an  $\mathcal{S}_\Omega^{\mathfrak{X}}$ GT-filter of  $X$ .*

*Proof.* Straightforward. □

Now, we give a characterization of  $\mathcal{S}_\Omega^{\mathfrak{X}}$ GT-filters.

THEOREM 2.15. *Let  $F$  be a nonempty subset of  $X$ . Then  $F$  is an  $\mathcal{S}_\Omega^{\mathfrak{X}}$ GT-filter of  $X$  if and only if for any  $x, y \in F$ , either  $A(x, y) \subseteq F$  or  $A(y, x) \subseteq F$ .*

*Proof.* The necessity is straightforward. Suppose that either  $A(x, y) \subseteq F$  or  $A(y, x) \subseteq F$  for every  $x, y \in F$ . Then we have  $1 \in A(x, x) \subseteq F$ . Let  $x \in \Omega$  and  $y \in F$  satisfy  $y \rightarrow x \in F$ . Then we have  $x \in A(y \rightarrow x, y) \subseteq F$ . Hence  $F$  is an  $\mathcal{S}_\Omega^{\mathfrak{X}}$ GT-filter of  $X$ . □

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