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분산 특이변동 시스템의 리아푸노프 행렬 방정식의 해에 대한 단일 경계치

(New Unified bounds for the solution of the Lyapunov matrix equation for Decentralized Singularly Perturbed Unified System)

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요 약

이 논문에서는 델타연산자를 사용하는 단일접근법에 의해 분산 특이변동 시스템에 대한 리아프노프 행렬 방정식에 대한 경계치의 새로운 결과가 제시되었고 기존의 연구결과들 중 결함이 있는 가정에 의해 얻어진 것들에 대한 보편화 작업도 수행되었다.

Abstract

In this paper, new bounds for the solution of the unified Lyapunov matrix equation for decentralized singularly perturbed systemare obtained, and some of the existing works using deficient assumptions are also generalized.

Keywords : Lyapunov matrix inequalities, unified system, bound estimates, singularly perturbed system

I. Introduction

Lyapunov matrix equation has played a fundamental role in various control system analyses and design problems[1]. Thus, finding anexact solution of the Lyapunov matrix equation is important in most applications. However, for some applications such as a system stability analysis, we don't need an exact solution but the reasonable bound estimates since obtaining thesolution itself results in very large computational burden when the dimension of system matrices is increased. Therefore, many researchers have been considerably attracted to this estimation problem for Riccatiand Lyapunov matrix equation[1~7, 8~13]. Also, recently the bound estimates for unified Lyapunov matrix equation is introduced by Mrabti and Hmamed[14~15]. In this paper, the unified bounds are presented based on the unified theory introduced by Middleton and Goodwin[16]. However, unfortunately most results for the bound estimates are based on using the assumption of $\lambda_1(A+A^T)<0$, $\lambda_1(A_\phi+A_\phi^T+\triangle A_\phi A_\phi^T)<0$, or $\lambda_1(A_d A_d^T)<1$. But Fang et al.[1] presented newupper bounds for continuous-time Lyapunov equation, did not use the common assumption that $A+A^T$ is negative definite. Hence, the objective of this paper is to extend this work to unified bounds without the assumption that

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$A_\phi + A_\phi^T + \triangle A_\phi A_\phi^T$ is asymptotically stable. Also, these bounds are compared to those for continuous and discrete-time Lyapunov matrix equations. Bounds for the trace and the largest eigenvalues will be presented.

Ⅱ. Notations and Preliminaries

In this paper, the following notations will be used: $A \in R^{n\times n}$ is a real matrix, A^T denotes the matrix transpose, $tr(A)$ is the trace of A, $(\lambda_i(A))$ are arranged in descending order when they are real, i.e., $\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A)$. $Re\lambda_i(A)$ are arranged in descending order, i.e., $Re\lambda_1(A) \ge Re\lambda_2(A) \ge \cdots \ge Re\lambda_n(A)$. The matrix measure induced by the 2-norm is denoted by $\mu_2(A)$ and $\mu_2(A) = \frac{1}{2}\lambda_1(A+A^T)$.

Lemma 1[17]: For any matrix A and any symmetric matrix B, set $\overline{A} = \frac{(A+A^T)}{2}$, then we have

$$\lambda_n(\overline{A})tr(B) - \lambda_n(B)\big(n\lambda_n(\overline{A}) - tr(A)\big) \le tr(AB) \le \lambda_1(\overline{A})tr(B) - \lambda_1(B)\big(n\lambda_1(\overline{A}) - tr(A)\big)$$

In particular, for any positive semidefinite matrix B, we have

$$\lambda_n(\overline{A})tr(B) \le tr(AB) \le \lambda 1(\overline{A})tr(B)$$

Lemma 2[1]: Let $\mu_E(\cdot)$ denotes the matrix measure induced by the vector norm

$\| T^{-1}x \| = \sqrt{x^T(T^{-1})^T(T^{-1})x} = \sqrt{x^TEx}$ where $E = T^{-T}T^{-1}$ is a positive definite matrix.

Then, the matrix measure $\mu_E(A)$ is defined by

$$\mu_E(A) = \frac{1}{2}\lambda_1(EAE^{-1} + A^T) = \mu_2(T^{-1}AT)$$

where the Euclidean norm-induced matrix measure is given by $\mu_2(A) = \frac{1}{2}\lambda_1(A+A^T)$.

Remark: Fang *et al.*[1] suggested that the similarity transformation matrix T is symmetric and defined by $T = \sqrt{E}$. However, this is not true since T may be not symmetrical with respect to any matrix A. Thus, Fang's definition should be corrected as above.

Lemma 3[14]: For symmetric positive semidefinite matrices A and B, with $1 \le i, j \ge n$,

$$\lambda_{i+j-1}(AB) \le \lambda_i(A)\lambda_j(B) \quad \text{if } i+j \le n+1$$
$$\lambda_{i+j-n}(AB) \ge \lambda_i(A)\lambda_j(B) \quad \text{if } i+j \ge n+1$$

Lemma 4[14]: For real symmetric matrices $A, B \ge 0$,

$$\prod_{i=1}^{k}\lambda_i(AB) \le \prod_{i=1}^{k}\lambda_i(A)\lambda_i(B)$$
$$\prod_{i=1}^{k}\lambda_{n-i+1}(AB) \ge \prod_{i=1}^{k}\lambda_{n-i+1}(A)\lambda_{n-i+1}(B)$$

with equality when $k = n$.

Lemma 5[14]: For symmetric $n \times n$ matrices A and B,

$$\sum_{i=1}^{k}\lambda_i(A+B) \le \sum_{i=1}^{k}\lambda_i(A) + \sum_{i=1}^{k}\lambda_i(B)$$
$$\sum_{i=1}^{k}\lambda_{n-i+1}(A+B) \ge \sum_{i=1}^{k}\lambda_{n-i+1}(A) + \sum_{i=1}^{k}\lambda_{n-i+1}(B)$$

with equality when $k = n$.

Lemma 6[18]: For symmetric $n \times n$ matrices A and B,

$$\sum_{i=1}^{k}\lambda_i(AB) \le \sum_{i=1}^{k}\lambda_i(A)\lambda_i(B)$$

Lemma 7[18]: Let $A \in R^{n\times n}$. Assume $A = T^T\Lambda T$ where T is orthogonal and Λ is diagonal real with $0 \le \lambda_i(\Lambda) < 1$. Then

$$(I-A)^{-1} = I + A + A^2 + \cdots$$

Lemma 8[19]:(Rayleigh-Ritz Inequality): For any $x \in R^{n\times n}$ and $A = A^T \in R^{n\times n}$,

$$\lambda_n(A)x^Tx \le x^TAx \le \lambda_1(A)x^Tx$$

Now, let us consider the unified Lyapnov equation

$$0 = A_\phi^TP + PA_\phi + \triangle A_\phi^TPA_\phi + Q \tag{1}$$

$$(I + \triangle A_\phi)^T\frac{P}{\triangle}(I + \triangle A_\phi) - \frac{P}{\triangle} + Q \tag{2}$$

where A_ϕ, P, $Q \in R^{n\times n}$, $Q>0$ and A_ϕ make an asymptotically stable matrix, $\triangle$ denotes the sampling time. The unified Lyapunov equation has unique positive definite solution P. The explicit solution P of (1) can be given by [16]:

$$P = \mathop{S}_{0}^{\infty} E(A_\phi^T, t) Q E(A_\phi, t) dt \tag{3}$$

where

$$E(A_\phi, t) = \begin{cases} \exp(At) = \lim\limits_{\triangle\to 0} E(A,t) & continuous-like\ time \\ (I+\triangle A)^{t/\triangle} & discrete-like\ time \end{cases}$$

And $S_{t_1}^{t_2} f(t)dt$ is given as follows for any real function f:

$$S_{t_1}^{t_2} f(t)dt = \begin{cases} \int_{t_1}^{t_2} f(t)dt & continuous-like\ time \\ \triangle \sum\limits_{k=t_1/\triangle}^{k=(t_2/\triangle)-1} f(k\triangle) & discrete-like\ time \end{cases}$$

Ⅲ. MAIN RESULTS

1. Discrete-Time Systems

Consider the linear shift-invariant decentralized discrete-time system

$$\begin{aligned} x(k+1) &= A_d x(k) + \sum_{i=1}^{N} B_{di} u_i(k) \\ y_i(k) &= C_{di} x(k) \end{aligned} \tag{4}$$

which for singularly perturbed systems has the following form

$$\begin{aligned} x(k+1) &= A_{d11}x(k) + A_{d12}\zeta(k) + \sum_{i=1}^{N} B_{d1i}u_i(k), \quad x(0)=x_0 \\ \zeta(k+1) &= A_{d21}x(k) + A_{d22}\zeta(k) + \sum_{i=1}^{N} B_{d2i}u_i(k), \quad \zeta(0)=\zeta_0 \\ y_i(k) &= C_{d1}x(k) + C_{d2}\zeta(k) \end{aligned} \tag{5}$$

where A_d, B_{di}, and C_{di} are given as follows:

$$A_d = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}, \ B_{di} = \begin{bmatrix} B_{d1i} \\ B_{d2i} \end{bmatrix}, \ C_{di} = \begin{bmatrix} C_{d1i} & C_{d2i} \end{bmatrix}$$

In this decentralized system, A_{d22} is assumed to be a nonsingular matrix, B_{d1i} and C_{d1i} are supposed to be zero. Dropping the fast dynamics of the full-order system, the reduced-order model of each subsystem is described by

$$\begin{aligned} x_{si}(k+1) &= A_{d0i}x_{si}(k) + B_{d0i}u_{si}(k) \\ y_{si}(k) &= C_{d0i}x_{si}(k) + D_{d0i}u_{si}(k) \end{aligned} \tag{6}$$

where

$$\begin{aligned} A_{d0i} &= A_{d11} + A_{d12}(I-A_{d22})^{-1}A_{d21} \\ B_{d0i} &= B_{d1i} + A_{d12}(I-A_{d22})^{-1}B_{d2i} \\ C_{d0i} &= C_{d1i} + C_{d2i}(I-A_{d22})^{-1}A_{d21} \\ D_{d0i} &= C_{d2i}(I-A_{d22})^{-1}B_{d2i} \end{aligned}$$

2. Estimate Bounds for Lyapunov Matrix Equation

Many researchers[20~24] have obtained the results of the upper bound for discrete Lyapuunov matrix equation. Almost all of the results were based on the assumption of $\lambda_1(A_{d0i}A_{d0i}^T) < 1$. However, Lee *et al.*[25] indicated a drawback of this assumptionfor discrete-time system. They showed that the stability of A_{d0i} in discrete-time systems does not imply that $\lambda_1(A_{d0i}A_{d0i}^T)$ is inside the unit circle.

Now, we extend Lee's work to that of decentralized discret-time system. Consider the algebraic Lyapunov matrix equation for decentralized discret-time system

$$P - A_{d0i}^T P A_{d0i} + Q_0 = 0 \tag{7}$$

Since the previous works for upper bound estimates does not cover the case that $\lambda_1(A_{d0i}A_{d0i}^T)$ is not inside the unit circle, we should make the following modification. Using the similarity transformation, we set $\hat{P} = T^TPT$, $\widehat{Q_0} = T^TQ_0T$, $\widehat{A_{d0i}} = T^TA_{d0i}T$. Then, we obtain the modified Lyapunov equation

$$\begin{aligned} (T^TPT) - (T^TA_{d0i}^T T^{-T})(T^TPT)(T^{-1}A_{d0i}T) \\ + (T^TQ_0T) = 0 \end{aligned} \tag{8}$$

Using (8) and Lemma 8, we can obtain the following theorems.

Theorem 1: For the decentralized discrete Lyapunov equation (8),

$$tr(P) \le \frac{\lambda_1(E)tr(E^{-1}Q_0)}{1-\lambda_1(\widehat{A_{d0i}}\widehat{A_{d0i}^T})} \quad \text{if} \quad \| \widehat{A_{d0i}} \| < 1$$

where $\| \widehat{A_{d0i}} \| = \sqrt{\lambda_1(\widehat{A_{d0i}}\widehat{A_{d0i}^T})}$.

Theorem 2: Let P satisfy the decentralized discrete Lyapunov equation (8). Then we have

$$tr(P) \le \lambda_1(E)\sum_{i=1}^{k}\frac{\lambda_i(E^{-1}Q_0)}{\lambda_{n-i+1}(I-\widehat{A_{d0i}}\widehat{A_{d0i}^T})}, \quad i=1,2,\cdots,k \le n$$

where $\lambda_1(\widehat{A_{d0i}}\widehat{A_{d0i}^T}) < 1$.

Theorem 3: For the decentralized discrete Lyapunov equation (8),

$$\lambda_k(P) \le \lambda_1(E)\left[\frac{1}{k}\sum_{i=1}^{k}\frac{\lambda_i(E^{-1}Q_0)}{\lambda_{n-i+1}(1-\widehat{A_{d0i}}\widehat{A_{d0i}^T})}\right]^k$$

where $i=1,2,\cdots,k \le n$.

Theorem 4: Let the positive definite matrix P be the solution of (8). If $\sigma_1(\widehat{A_{d0i}}) < 1$,

$$\lambda_i(P) \le \lambda_1(E)\left[\lambda_i\left(\frac{\lambda_1(E^{-1}Q_0)\widehat{A_{d0i}^T}\widehat{A_{d0i}}}{[1-\sigma_1^2(\widehat{A_{d0i}})]}\right)+E^{-1}Q_0\right];\ 1 \le i \le n$$

$$tr(P) \le \frac{\lambda_1(E)\lambda_1(E^{-1}Q_0)}{[1-\sigma_1^2(\widehat{A_{d0i}})]}tr(\widehat{A_{d0i}^T}\widehat{A_{d0i}})+tr(E^{-1}Q_0)$$

where $\sigma_1(\widehat{A_{d0i}}) = \sqrt{\lambda_1(\widehat{A_{d0i}}\widehat{A_{d0i}^T})}$

Remark: The theorems presented above are based on[18~19] and modified to cover the case that the common condition $\lambda_1(A_{d0i}A_{d0i}^T) < 1$ is not valid. With applying this modification, more generalized results are obtained.

3. Unified Systems

Let us consider the linear decentralized unified system[26]

$$\begin{aligned} \phi x(\tau) &= A_\phi x(\tau) + \sum_{i=0}^{k} B_{\phi i}u_i(\tau) \\ y_i(\tau) &= C_{\phi i}x(\tau) \end{aligned} \tag{9}$$

To represent a system that possesses a two-time-scale property, system (9) can be rewritten as

$$\begin{aligned} \phi x(\tau) &= A_{\phi 11}x(\tau) + A_{\phi 12}\zeta(\tau) + \sum_{i=0}^{k} B_{\phi 1i}u_i(\tau) \\ \mu\phi\zeta(\tau) &= A_{\phi 21}x(\tau) + A_{\phi 22}\zeta(\tau) + \sum_{i=0}^{k} B_{\phi 2i}u_i(\tau) \\ y_i(\tau) &= C_{\phi 1i}x(\tau) + C_{\phi 2i}\zeta(\tau) \end{aligned} \tag{10}$$

where $x \in R^n$ and $\zeta \in R^m$ are state vectors, $u_i \in R^r$ is a control vector, and A_ϕ, $B_{\phi i}$, $C_{\phi i}$ are the constant matrices of appropriate dimension. In this decentralized unified system, $A_{\phi 22}$ is assumed to be a nonsingular matrix, $B_{\phi 1i}$ and $C_{\phi 1i}$ are supposed to be zero. By setting the parameter $\mu = 0$, we assume that the fast modes ζ have reached the quasi-steady state and we drop the fast dynamics of the full-order system. Then, the behavior of the system can be represented by its slow modes. Then, the reduced-order model of each subsystem is described by

$$\begin{aligned} \phi x_{si}(\tau) &= A_{\phi 0i}x_{si}(\tau) + B_{\phi 0i}u_{si}(\tau) \\ y_{si}(\tau) &= C_{\phi 0i}x_{si}(\tau) + D_{\phi 0i}u_{si}(\tau) \end{aligned} \tag{11}$$

where
$$\begin{aligned} A_{\phi 0i} &= A_{\phi 11} - A_{\phi 12}A_{\phi 22}^{-1}A_{\phi 21} \\ B_{\phi 0i} &= B_{\phi 1i} - A_{\phi 12}A_{\phi 22}^{-1}B_{\phi 2i} \\ C_{\phi 0i} &= C_{\phi 1i} - C_{\phi 2i}A_{\phi 22}^{-1}A_{\phi 21} \\ D_{\phi 0i} &= -C_{\phi 2i}A_{\phi 22}^{-1}B_{\phi 2i} \end{aligned}$$

4. Estimate Bounds for Lyapunov Matrix Equation

Mrabti and Hmamed[14~15] have developed the results of the lower and upper bounds for unified Lyapunov matrix equation. All the results were based on the assumption of $\lambda_1(A_{\phi 0i} + A_{\phi 0i}^T + \triangle A_{\phi 0i}A_{\phi 0i}^T) < 0$. However, since the stability of $A_{\phi 0i}$ in unified systems does not guarantee that $\lambda_1(A_{\phi 0i} + A_{\phi 0i}^T + \triangle A_{\phi 0i}A_{\phi 0i}^T)$ is negative definite. Hence, the objective of this section is to modify and extend the previous works by removing this assumption. Consider the algebraic unified Lyapunov matrix equation

$$0 = A_{\phi 0i}^T P + PA_{\phi 0i} + \triangle A_{\phi 0i}^T PA_{\phi 0i} + Q_0 \tag{12}$$

where $A_{\phi 0i}$, P, $Q_0 \in R^{n\times n}$, $Q_0 > 0$ and $A_{\phi 0i}$ is an asymptotically stable matrix. The unified Lyapunov equation has unique positive definite solution P. Since the previous works for upper bound estimates was not valid for being $\lambda_1(A_{\phi0i}+A^T_{\phi0i}+\Delta A_{\phi0i}A^T_{\phi0i})$ not negative definite, we should make the following modification. Using the similarity transformation, we set

$$\hat{P}=T^TPT,\ \widehat{Q_0}=T^TQ_0T,\ \widehat{A_{\phi0i}}=T^TA_{\phi0i}T \tag{13}$$

Also, we obtain (14) with rewriting (12).

$$\begin{aligned}P^{-1/2}Q_0P^{-1/2}=&-P^{-1/2}A^T_{\phi0i}P^{1/2}-P^{-1/2}A_{\phi0i}P^{-1/2}\\&-\Delta P^{-1/2}A^T_{\phi0i}P^{1/2}P^{1/2}A_{\phi0i}P^{-1/2}\end{aligned} \tag{14}$$

Then, combining (13) and (14), we obtain the modified unified Lyapunov equation

$$\begin{aligned}&P^{-1/2}\widehat{Q_0}\widehat{P^{-1/2}}=-\widehat{P^{-1/2}}\widehat{A^T_{\phi0i}}\widehat{P^{1/2}}-\widehat{P^{1/2}}\widehat{A_{\phi0i}}\widehat{P^{-1/2}}\\&-\Delta(\widehat{P^{-1/2}}\widehat{A^T_{\phi0i}}\widehat{P^{1/2}})(\widehat{P^{1/2}}\widehat{A_{\phi0i}}\widehat{P^{-1/2}})\end{aligned} \tag{15}$$

Using (15) and Lemma 5 and 7, we obtain the following theorems.

Theorem 5: Let the positive definite matrix P be the solution of (15).

If $\lambda_1(A_{\phi0i}+A^T_{\phi0i}+\Delta A_{\phi0i}A^T_{\phi0i})<0$, then

$$tr(P)\le-\lambda_1(E)\sum_{i=1}^{k}\frac{\lambda_i(\widehat{Q_0})}{\lambda_i(\widehat{A_{\phi0i}}+\widehat{A^T_{\phi0i}}+\Delta\widehat{A_{\phi0i}}\widehat{A^T_{\phi0i}})} \tag{16}$$

$i=1,2,\cdots,k\le n$

$$\lambda_1(P)\le-\frac{\lambda_1(E)\lambda_1(\widehat{Q_0})}{\lambda_1(\widehat{A_{\phi0i}}+\widehat{A^T_{\phi0i}}+\Delta\widehat{A_{\phi0i}}\widehat{A^T_{\phi0i}})} \tag{17}$$

Proof: Letting $T^TPT=\hat{P},\ T^{-1}A_{\phi0i}T=\widehat{A_{\phi0i}},\ T^TQ_0T=\widehat{Q_0}$, we have

$$\begin{aligned}&P^{-1/2}\widehat{Q_0}\widehat{P^{-1/2}}=-\widehat{P^{-1/2}}\widehat{A^T_{\phi0i}}\widehat{P^{1/2}}-\widehat{P^{1/2}}\widehat{A_{\phi0i}}\widehat{P^{-1/2}}\\&-\Delta\widehat{P^{-1/2}}\widehat{A^T_{\phi0i}}\widehat{P^{1/2}}\widehat{P^{1/2}}\widehat{A_{\phi0i}}\widehat{P^{-1/2}}\end{aligned} \tag{18}$$

From [15] and [18],

$$\begin{aligned}\hat{P}=\Delta[\widehat{Q_0}+(I+\Delta\widehat{A^T_{\phi0i}})\widehat{Q_0}(I+\Delta\widehat{A_{\phi0i}})+\\(I+\Delta\widehat{A^T_{\phi0i}})^2\widehat{Q_0}(I+\Delta\widehat{A_{\phi0i}})^2+\cdots]\end{aligned} \tag{19}$$

$$\lambda_i(I+\Delta\widehat{A^T_{\phi0i}})^l\widehat{Q_0}(I+\Delta\widehat{A_{\phi0i}})^l=\lambda_i(\widehat{Q_0}\widehat{B^l_{\phi)0i}}) \tag{20}$$

where

$$\lambda_i(\widehat{B_{\phi0i}})=1+\Delta\lambda_i(\widehat{A_{\phi0i}}+\widehat{A^T_{\phi0i}}+\Delta\widehat{A_{\phi0i}}+\widehat{A^T_{\phi0i}}) \tag{21}$$

Using Lemma 5, 6, and (20),

$$\begin{aligned}\sum\lambda_i(P)&\le\Delta[\sum(\lambda_i(\widehat{Q_0})+\lambda_i(\widehat{Q_0}\widehat{B_{\phi0i}})\\&\quad+\lambda_i(\widehat{Q_0}\widehat{B^2_{\phi0i}})+\cdots]\\&\le\Delta\sum\lambda_i(\widehat{Q_0})[1+\lambda_i(\widehat{B_{\phi0i}})+\lambda_i^2(\widehat{B_{\phi0i}})+\cdots]\end{aligned} \tag{22}$$

Using Lemma 7 , (21), and assuming $\lambda1(\widehat{B_{\phi0i}})<1$, we have

$$\begin{aligned}\sum\lambda_i(P)&\le\sum\lambda_i(\widehat{Q_0})[1-\lambda_i(\widehat{B_{\phi0i}})]^{-1}\\&=\Delta\sum\lambda_i(\hat{Q}_0)\left[1-1-\Delta\lambda_i(\hat{A}_{\phi0i}+\hat{A}^T_{\phi0i}+\Delta\hat{A}_{\phi0i}\hat{A}^T_{\phi0i})\right]^{-1}\\&=\Delta\sum\lambda_i(\hat{Q}_0)\left[-\Delta\lambda_i(\hat{A}_{\phi0i}+\hat{A}^T_{\phi0i}+\Delta\hat{A}_{\phi0i}\hat{A}^T_{\phi0i})\right]^{-1}\end{aligned} \tag{23}$$

Using Lemma 6, (23) becomes

$$\begin{aligned}&\sum\lambda_n(E^{-1})\lambda_i(P)\le\sum\lambda_i(\hat{P})=\sum\lambda_i(TT^TP)\\&=\sum\lambda_i(E^{-1}P)\\&\qquad\le\sum\lambda_i(\hat{Q}_0)\left[-\lambda_i(\hat{A}_{\phi0i}+\hat{A}^T_{\phi0i}+\Delta\hat{A}_{\phi0i}\hat{A}^T_{\phi0i})\right]^{-1}\\&\le\sum\lambda_i(\hat{Q}_0)\left[\lambda_{n-i+1}(-\hat{A}_{\phi0i}-\hat{A}^T_{\phi0i}-\Delta\hat{A}_{\phi0i}\hat{A}^T_{\phi0i})\right]^{-1}\end{aligned} \tag{24}$$

Since $\lambda_n(E^{-1})=\lambda_1^{-1}(E)$, we have

$$\begin{aligned}&\sum\lambda_i(P)\le\lambda_1(E)\square\\&\sum\lambda_i(\hat{Q}_0)\left[\lambda_{n-i+1}(-\hat{A}_{\phi0i}-\hat{A}^T_{\phi0i}-\Delta\hat{A}_{\phi0i}\hat{A}^T_{\phi0i})\right]^{-1}\end{aligned} \tag{25}$$

From (24), we have

$$\lambda_1(P)\le\frac{-\lambda_1(E)\lambda_1(\hat{Q}_0)}{\lambda_1(\hat{A}_{\phi0i}+\hat{A}^T_{\phi0i}+\Delta\hat{A}_{\phi0i}\hat{A}^T_{\phi0i})} \tag{26}$$

This completes the proof.

Theorem 6: Let P satisfy the decentralized unified Lyapunov equation (15) where

$\lambda_1(A_{\phi oi}+A^T_{\phi oi}+\Delta A_{\phi oi}A^T_{\phi oi})<0$,

$$tr(P)\le\lambda_1(E)\sum_{i=1}^{k}\frac{\lambda_i(\hat{Q}_0)}{\lambda_n(-\hat{A}_{\phi0i}-\hat{A}^T_{\phi0i}-\Delta\hat{A}_{\phi0i}\hat{A}^T_{\phi0i})} \tag{27}$$

where $i=1,2,\ldots k\le n$

$$tr(P) \le \lambda_1(E)\sum_{i=1}^{k} \frac{\lambda_1(\hat{Q}_0)}{\lambda_{n-i+1}(-\hat{A}_{\phi 0i} - \hat{A}^T_{\phi 0i} - \Delta \hat{A}_{\phi 0i}\hat{A}^T_{\phi 0i})} \quad (28)$$

Proof: From (27),

$$\begin{aligned}&\left[\lambda_{n-i+1}(-\hat{A}_{\phi 0i} - \hat{A}^T_{\phi 0i} - \Delta \hat{A}_{\phi 0i}\hat{A}^T_{\phi 0i})\right]^{-1} \\ &\quad \le \left[\lambda_n(-\hat{A}_{\phi 0i} - \hat{A}^T_{\phi 0i} - \Delta \hat{A}_{\phi 0i}\hat{A}^T_{\phi 0i})\right]^{-1}\end{aligned} \quad (29)$$

Then, we have

$$\sum \lambda_i(P) \le$$

$$\lambda_1(E)\sum \lambda_i(\hat{Q}_0)\left[-\lambda_n(-\hat{A}_{\phi 0i} - \hat{A}^T_{\phi 0i} - \Delta \hat{A}_{\phi 0i}\hat{A}^T_{\phi 0i})\right]^{-1} \quad (30)$$

This completes the proof.

Remark: (27) and (28) are dual bounds. The upper bound for the largest eigenvalue (17) is new. For $\Delta = 0$, (28) is the same as [1, Theorem 3.7].

Remark: The theorems presented above is based on[14~15] and newly develope to cover the case that the common condition $\lambda_1(A_\phi + A_\phi^T + \Delta A_\phi A_\phi^T) < 0$ is not valid. With this modification, more generalized results are obtained.

Ⅳ. EXAMPLE

By the definition of the unified system, when $\Delta = 0$, system (10) becomes a continuous-time system. So, the numerical examples for continuous-like unified system are omitted here.

1. Discrete-Time Systems

Example 1: A discrete-time model is obtained from its continuous-time model[27] by discretizing it using MATLAB function $c2d$ with the sampling period $\Delta = 0.5$. The corresponding discrete-time system matrix is obtained as

$$A_d = \begin{bmatrix} 0.8280 & 0.3147 & -0.0016 & -0.0030 \\ -1.1401 & -0.0872 & 0.0026 & 0.0005 \\ 2.8793 & 0.4003 & -0.0064 & -0.0032 \\ 1.9163 & 1.4194 & -0.0028 & -0.0140 \end{bmatrix}$$

From the system matrix of reduced-order model $A_{d0i} = \begin{bmatrix} 0.8178 & 0.3099 \\ -1.1317 & -0.0855 \end{bmatrix}$, we obtain

$$\lambda_1(P) = 24.4165, \ tr(P) = 31.1169$$

The eigenvalues of $A_{doi}A^T_{doi}$ is given by

$$\lambda_1 = 0.0392, \ \lambda_2 = 2.0138$$

Since $\lambda_1(A_{doi}A^T_{doi})$ is not stable, we should overcome this difficulty. Then, similarity transformation matrix T is introduced. For this example, $T = \begin{bmatrix} 0.5+0.5895i & 0.5-0.5895i \\ -1.4773i & 1.4773i \end{bmatrix}$, and the Jordan-transformed matrix and its eigenvalues are obtained as

$$\hat{A}_{d0i}\hat{A}^T_{d0i} = \begin{bmatrix} 0.2808 & 0 \\ 0 & 0.2808 \end{bmatrix},$$

$$\lambda_1 = \lambda_2 = 0.2808$$

Now, the assumption $\lambda_1(A_{doi}A^T_{doi}) < 1$ is removed. Then, the upper bounds are given by the theorems described in section 4.4.

The bound in Theorem 1, then yields $tr(P) \le 46.3829$.

By Theorem 2, we obtain $tr(P) \le 46.3829$.

From Theorem 3, we have $tr(P) \le 42.8372$.

The bounds in Theorem 4, then yield

$$\lambda_1(P) \le 42.3872, \ tr(P) \le 57.4157$$

The numerical results show that the best values for the trace and the largest eigenvalues are 46.3829 and 42.3872, respectively. As shown above, the common assumption used for bound estimation problem has been removed applying similarity transformation to estimate bounds, and more generalized results can be obtained.

2. Unified Systems

Example 2: Consider a fourth-order example with the system matrix given by [27]

$$A = \begin{bmatrix} 1 & 0 & -0.64 & 0.02 \\ 0 & -0.5 & 0.345 & -1 \\ 200 & -524 & -265 & 0 \\ 500 & 200 & 0 & -100 \end{bmatrix}$$

For unified Lyapunov equation (12), Let $\Delta = 0.5$. Then, the system matrix is given by

$$A_\phi = A_\delta = \begin{bmatrix} -0.3440 & 0.6294 & -0.0032 & -0.0060 \\ -2.2802 & -2.1744 & 0.0052 & 0.0010 \\ 5.7586 & 0.8006 & -2.0128 & -0.0064 \\ 3.8326 & 2.8388 & -0.0056 & -2.0280 \end{bmatrix}$$

We recall the definition of the unified systems. Then, the discrete-like time case of unified system is described as

$$\delta x(k) = \left[\frac{(A_{d0i} - I)}{\Delta}\right] x(k).$$

And the system matrix and transformation matrix of the reduced-order model are given by

$$A_{\delta 0i} = \frac{(A_{d0i} - I)}{\Delta(=0.5)} = \begin{bmatrix} 0.8178 & 0.3099 \\ -1.1317 & -0.0855 \end{bmatrix},$$

$$T = \begin{bmatrix} 0.5 - 0.5895i & 0.5 + 0.5895i \\ 1.4773i & -1.4773i \end{bmatrix}$$

Then, we obtain $\lambda_1(P) = 30.5315$ $tr(P) = 38.6524$.

The eigenvalues of $A_{\delta 0i} + A_{\delta 0i}^T + 0.5 A_{\delta 0i} A_{\delta 0i}^T$ is given by

$$\lambda_1 = -1.9217, \quad \lambda_2 = 2.0276$$

Using similarity transformation matrix T, we have the Jordan-transformed matrix and its eigenvalues

$$\hat{A}_{\delta 0i} + \hat{A}_{\delta 0i}^T + 0.5 \hat{A}_{\delta 0i} \hat{A}_{\delta 0i}^T = \begin{bmatrix} -1.2677 + 0.766i & 0 \\ 0 & -1.2677 - 0.766i \end{bmatrix},$$

$$\lambda_1 = \lambda_2 = -1.4384$$

Then, we obtain the numerical results from the theorems in Main results. The bounds in Theorem 5, then yields $tr(P) \le 38.6524$, $\lambda_1(P) \le 35.6976$. By Theorem 6, the bounds are given by

$$tr(P) \le 41.6524, \ \lambda_1(P) \le 37.3953$$

The numerical results show that the best values for the trace and the largest eigenvalues are 38.6524 and 35.6976, respectively. These results are better than those of bounds of discrete-time systems. As seen above, when applying similarity transformation, we are able to overcome the difficulty when the common assumption of $\lambda_1(A_\phi + A_\phi^T + \Delta A_\phi A_\phi^T) < 0$ is not valid. Hence, more generalized results are obtained.

V. CONCLUSION

Stability analysis using bound estimates for the solution of unified Lyapunov matrix equation is the topic of this paper. This issue is inspired by the work of Fang *et al.*[1] and that of Mrabti and Hmamed[14~15]. Based on the upper bounds developed previously for discrete-time and unified Lyapunov matrix equation[18~20], those bounds are extended and generalized with removing the assumption of $\lambda_1(A_d A_d^T) < 1$ and $\lambda_1(A_\phi + A_\phi^T + \Delta A_\phi A_\phi^T) < 0$. When applying similarity transformation to the theorems for each system, i.e., discrete-time, and unified system, the inequalities for the upper bounds maintain their validity. The upper bound estimates are based on the solution of Lyapunov matrix equation for each system. In addition, the comparison for discrete-time vs. discrete-like time of unified system is provided. The numerical results illustrated by Example 1 and 2 show that the upper bounds for each system hold true.

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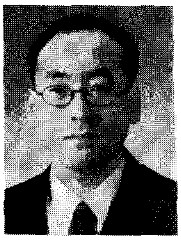

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