Korean J. Math. **17** (2009), No. 1, pp. 83–90

ISOMORPHISMS AND DERIVATIONS IN C^*-TERNARY ALGEBRAS

Jong Su An* and Chunkil Park

Abstract. In this paper, we investigate isomorphisms between C^*-ternary algebras and derivations on C^*-ternary algebras associated with the Cauchy–Jensen functional equation

$$2f(\frac{x+y}{2}+z) \quad = \quad f(x)+f(y)+2f(z),$$

which was introduced and investigated by Baak in [2].

1. Introduction and preliminaries

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their applications in physics (see [17, 18]).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x,y,z) \mapsto [x,y,z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x,y,[z,w,v]] = [x,[w,z,y],v] = [[x,y,z],w,v]$, and satisfies $\|[x,y,z]\| \le \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x,x,x]\| = \|x\|^3$ (see [1, 38]). Every left Hilbert C^*-module is a C^*-ternary algebra via the ternary product $[x,y,z] := \langle x,y\rangle z$.

If a C^*-ternary algebra $(A,[\cdot,\cdot,\cdot])$ has an identity, i.e., an element $e \in A$ such that $x = [x,e,e] = [e,e,x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x,e,y]$ and $x^* := [e,x,e]$, is a unital C^*-algebra. Conversely, if $(A,\circ)$ is a unital C^*-algebra, then $[x,y,z] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra.

Received February 20, 2009. Revised March 10, 2009.
2000 Mathematics Subject Classification: Primary 39B52, 17A40, 46B03, 47Jxx.
Key words and phrases: Cauchy–Jensen functional equation, C^*-ternary algebra isomorphism, C^*-ternary derivation, generalized Hyers–Ulam stability.

*Corresponding author.

A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *C^*-ternary algebra homomorphism* if

$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$. If, in addition, the mapping H is bijective, then the mapping $H : A \to B$ is called a *C^*-ternary algebra isomorphism*. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *C^*-ternary derivation* if

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in A$ (see [1, 4, 21]).

In 1940, S.M. Ulam [37] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

In 1941, D.H. Hyers [11] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

In 1978, Th. M. Rassias [28] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded.

Theorem 1.1. *(Th.M. Rassias) Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$*. If* $p < 0$ *then inequality* (1.1) *holds for* $x, y \neq 0$ *and* (1.2) *for* $x \neq 0$.

In 1990, Th.M. Rassias [29] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \ge 1$. In 1991, Z. Gajda [7] following the same approach as in Th.M. Rassias [28], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [7], as well as by Th.M. Rassias and P. Šemrl [34] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$. The counterexamples of Z. Gajda [7], as well as of Th.M. Rassias and P. Šemrl [34] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [8], S. Jung [15], who among others studied the Hyers–Ulam stability of functional equations. The inequality (1.1) that was introduced for the first time by Th.M. Rassias [28] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as *generalized Hyers–Ulam stability* of functional equations (cf. the books of P. Czerwik [5], D.H. Hyers, G. Isac and Th.M. Rassias [12]).

P. Găvruta [8] provided a further generalization of Th.M. Rassias' Theorem. In 1996, G. Isac and Th.M. Rassias [14] applied the generalized Hyers–Ulam stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. In [13], D.H. Hyers, G. Isac and Th.M. Rassias studied the asymptoticity aspect of Hyers–Ulam stability of mappings. During the several papers have been published on various generalizations and applications of Hyers–Ulam stability and generalized Hyers–Ulam stability to a number of functional equations and mappings, for example : quadratic functional equation, invariant means, multiplicative mappings - superstability, bounded nth differences, convex functions, generalized orthogonality functional equation, Euler–Lagrange functional equation, Navier–Stokes equations. Several mathematician have contributed works on these subjects; we mention a few: S. Jung and B. Chung [16], M. Mirzavaziri and M.S. Moslehian [20], C. Park [22]–[27], Th.M. Rassias [30]–[33], F. Skof [36].

In [9], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.3}$$

then f satisfies the Jordan–von Neumann functional equality

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [35]. Gilányi [10] and Fechner [6] proved the generalized Hyers–Ulam stability of the functional inequality (1.3). In [3], the author proved the generalized Hyers–Ulam stability of functional inequalities associated with Jordan–von Neumann type additive functional equations.

Throughout this paper, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$, and that B is a C^*-ternary algebra with norm $\|\cdot\|_B$.

In Section 2, we investigate isomorphisms between C^*-ternary algebras associated with the Cauchy–Jensen functional equation.

In Section 3, we investigate derivations on C^*-ternary algebras associated with the Cauchy–Jensen functional equation.

2. Isomorphisms between C^*-ternary algebras

In this section, we investigate isomorphisms between C^*-ternary algebras associated with the Cauchy–Jensen functional equation.

Lemma 2.1. ([3]) *Let $f : A \to B$ be a mapping such that*

$$\|f(x) + f(y) + 2f(z)\|_B \leq \|2f(\frac{x+y}{2} + z)\|_B$$

for all $x, y, z \in A$. Then f is Cauchy additive.

Theorem 2.2. *Let $r > 3$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping such that*

$$\|f(\mu x) + \mu f(y) + 2f(z)\|_B \leq \|2f(\frac{x+y}{2} + z)\|_B, \tag{2.1}$$

$$\|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) \tag{2.2}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x, y, z \in A$. Then the mapping $f : A \to B$ is a C^*-ternary algebra isomorphism.

Proof. Let $\mu = 1$ in (2.1). By Lemma 2.1, the mapping $f : A \to B$ is Cauchy additive.

Letting $y = -x$ and $z = 0$, we get

$$\|f(\mu x) + \mu f(-x)\|_B \leq \|2f(0)\|_B = 0$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. So

$$f(\mu x) - \mu f(x) = f(\mu x) + \mu f(-x) = 0$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. Hence $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{T}^1$. By the same reasoning as in the proof of Theorem 2.1 of [24], the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.2) that

$$\begin{aligned}&\|f([x,y,z]) - [f(x), f(y), f(z)]\|_B \\ &\quad = \lim_{n\to\infty} 8^n \|f(\frac{[x,y,z]}{2^n \cdot 2^n \cdot 2^n}) - [f(\frac{x}{2^n}), f(\frac{y}{2^n}), f(\frac{z}{2^n})]\|_B \\ &\quad \le \lim_{n\to\infty} \frac{8^n \theta}{2^{nr}} (\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) = 0\end{aligned}$$

for all $x, y, z \in A$. Thus

$$f([x,y,z]) = [f(x), f(y), f(z)]$$

for all $x, y, z \in A$. Hence the bijective mapping $f : A \to B$ is a C^*-ternary algebra isomorphism. □

THEOREM 2.3. *Let $r < 3$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (2.1) *and* (2.2). *Then the mapping $f : A \to B$ is a C^*-ternary algebra isomorphism.*

Proof. The proof is similar to the proof of Theorem 2.2. □

3. Derivations on C^*-ternary algebras

In this section, we investigate derivations on C^*-ternary algebras associated with the Cauchy–Jensen functional equation.

THEOREM 3.1. *Let $r > 3$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (2.1) *such that*

$$\begin{aligned}\|f([x,y,z]) &- [f(x), y, z] - [x, f(y), z] - [x, y, f(z)]\|_A \\ &\le \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) \qquad (3.1)\end{aligned}$$

for all $x, y, z \in A$. Then the mapping $f : A \to A$ is a C^*-ternary derivation.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (3.1) that

$$\begin{aligned}
&\|f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]\|_A \\
&= \lim_{n\to\infty} 8^n \|f(\frac{[x,y,z]}{8^n}) - [f(\frac{x}{2^n}), \frac{y}{2^n}, \frac{z}{2^n}] \\
&-[\frac{x}{2^n}, f(\frac{y}{2^n}), \frac{z}{2^n}] - [\frac{x}{2^n}, \frac{y}{2^n}, f(\frac{z}{2^n})]\|_A \\
&\le \lim_{n\to\infty} \frac{8^n\theta}{2^{nr}} (\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$f([x,y,z]) = [f(x),y,z] + [x,f(y),z] + [x,y,f(z)]$$

for all $x, y, z \in A$. Thus the mapping $f : A \to A$ is a C^*-ternary derivation. □

THEOREM 3.2. *Let $r < 3$ and θ be positive real numbers, and let $f : A \to A$ be a mapping satisfying* (2.2) *and* (3.1). *Then the mapping $f : A \to A$ is a C^*-ternary derivation.*

Proof. The proof is similar to the proofs of Theorems 2.2 and 3.1. □

References

[1] M. *Amyari and M.S. Moslehian, Approximately ternary semigroup homomorphisms*, Lett. Math. Phys. **77** (2006), 1–9.

[2] C. Baak, *Cauchy–Rassias stability of Cauchy–Jensen additive mappings in Banach spaces*, Acta Math. Sinica (to appear).

[3] C. Baak, Y. Cho, M. Han, M.S. Moslehian, *Stability of functional inequalities associated with Jordan–von Neumann type additive functional equations*, (preprint).

[4] C. Baak, *Isomorphisms between C^*-ternary algebras*, J. Math. Anal. Appl. (to appear).

[5] P. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, New Jersey, Hong Kong, Singapore and London, 2002.

[6] W. Fechner, *Stability of a functional inequalities associated with the Jordan–von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.

[7] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.

[8] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[9] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung* , Aequationes Math. **62** (2001), 303–309.

[10] A. Gilányi, *On a problem by K. Nikodem*, Math. Inequal. Appl. **5** (2002), 707–710.

[11] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[12] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[13] D.H. Hyers, G. Isac and Th.M. Rassias, *On the asymptoticity aspect of Hyers–Ulam stability of mappings*, Proc. Amer. Math. Soc. **126** (1998), 425–430.

[14] G. Isac and Th.M. Rassias, *Stability of ψ-additive mappings : Applications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.

[15] S. Jung, *On the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **204** (1996), 221–226.

[16] S. Jung and B. Chung, *Remarks on Hyers–Ulam stability of Butler–Rassias functional equation*, Dyn. Contin. Discrete Impuls. Syst. Ser. A - Math. Anal. **13** (2006), 193–197.

[17] R. Kerner, *The cubic chessboard: Geometry and physics*, Classical Quantum Gravity **14** (1997), A203–A225.

[18] R. Kerner, *Ternary algebraic structures and their applications in physics*(preprint).

[19] Gy. Maksa and P. Volkmann, *Characterization of group homomorphisms having values in an inner product space*, Publ. Math. Debrecen **56** (2000), 197–200.

[20] M. Mirzavaziri and M.S. Moslehian, *Automatics continuity of σ-derivations on C^*-algebras*, Proc. Amer. Math. Soc. **134** (2006), 3319–3327.

[21] M.S. Moslehian and L. Szekelyhidi, *Stability of ternary homomorphisms via generalized Jensen equation*, Results Math. (to appear).

[22] C. Park, *Multi-quadratic mappings in Banach spaces*, Proc. Amer. Math. Soc. **131** (2003), 2501–2504.

[23] C. Park, *On an approximate automorphism on a C^*-algebra*, Proc. Amer. Math. Soc. **132** (2004), 1739–1745.

[24] C. Park, *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.

[25] C. Park, *A generalized Jensen's mapping and linear mappings between Banach modules*, Bull. Braz. Math. Soc. **36** (2005), 333–362.

[26] C. Park, *Hyers–Ulam–Rassias stability of a generalized Euler–Lagrange type additive mapping and isomorphisms between C^*-algebras*, Bull. Belgian Math. Soc.–Simon Stevin (to appear).

[27] C. Park, *Hyers–Ulam–Rassias stability of a generalized Apollonius–Jensen type additive mapping and isomorphisms between C^*-algebras*, Math. Nachr. (to appear).

[28] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[29] Th.M. Rassias,*Problem 16; 2, Report of the 27^{th} International Symp. on Functional Equations*, Aequationes Math. **39** (1990), 292–293; 309.

[30] Th.M. Rassias,*The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), 352–378.
[31] Th.M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.
[32] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.
[33] Th.M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2003.
[34] Th.M. Rassias and P. Šemrl, *On the behaviour of mappings which do not satisfy Hyers–Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.
[35] J. Rätz, *On inequalities associated with the Jordan–von Neumann functional equation*, Aequationes Math. **66** (2003), 191–200.
[36] F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
[37] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[38] H. Zettl, *A characterization of ternary rings of operators*, Adv. Math. **48** (1983), 117–143.

Department of Mathematics Education
Pusan National University
Pusan 609-735, Republic of Korea
E-mail: jsan63@pusan.ac.kr

Department of Mathematics
Hanyang University
Seoul 133-791, Republic of Korea
E-mail: baak@hanyang.ac.kr