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NONTRIVIAL PERIODIC SOLUTION FOR THE SUPERQUADRATIC PARABOLIC PROBLEM

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Abstract. We show the existence of a nontrivial periodic solution for the superquadratic parabolic equation with Dirichlet boundary condition and periodic condition with a superquadratic nonlinear term at infinity which have continuous derivatives. We use the critical point theory on the real Hilbert space $L_2(\Omega \times (0, 2\pi))$. We also use the variational linking theorem which is a generalization of the mountain pass theorem.

1. Introduction

Let Ω be a bounded, connected open subset of R^n with smooth boundary $\partial\Omega$. In this paper we consider the multiplicity of the solutions of the following parabolic boundary value problem

$$\begin{aligned} D_t u &= \Delta u + F_u(x,t,u) \qquad \text{in } \ \Omega \times R, \\ u(x,t) &= 0, \qquad x \in \partial\Omega, \ t \in R, \\ u(x,t) &= u(x,t+T), \qquad \text{in } \ \Omega \times R, \end{aligned} \tag{1.1}$$

where the period T is given and $F : \Omega \times R \times R \to R$ is a superquadratic function at infinity which has a continuous derivative $F_u(x,t,u)$ for almost any $x \in \Omega$. Moreover we assume that F satisfies the following conditions:

(F1) $F(x,t,0) = F_x(x,t,0) = F_t(x,t,0) = F_{xx}(x,t,0) = F_{tt}(x,t,0) = F_{xt}(x,t,0) = 0$, $F(x,t,r) > 0$ if $r \neq 0$, $\inf_{\substack{(x,t)\in\Omega\times R \\ |r|=R}} F(x,t,r) > 0$;

(F2) $|F_r(x,t,r)| \leq c(|r|^\nu)$ $\forall x,\ t,\ r$;

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(F3) $rF_r(x,t,r) \geq \mu F(x,t,r)$ $\forall x,\ t,\ r$;
(F4) $|F_r(x,t,r)| \leq dF(x,t,r)^{\delta}$
where $c \geq 0$, $d > 0$, $R > 0$, $\mu \in]2, 2^*[$, $\nu \leq 2^* - 1 - (2^* - \mu)(1 - \frac{2^{*'}}{2^*})$ and $\frac{1}{2} < \delta \leq \frac{1}{2^{*'}}$.
In this paper we consider the case $T = 2\pi$. That is

$$D_t u = \Delta u + F_u(x,t,u) \qquad \text{in } \Omega \times R, \tag{1.2}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t \in R,$$

$$u(x,t) = u(x,t+2\pi), \qquad \text{in } \Omega \times R,$$

The physical phenomena for this kind of parabolic boundary value problem occur in the heat flow dynamics with superquadratic nonlinearity. We observe that $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \to \infty$ are the eigenvalues of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, $u|_{\partial\Omega} = 0$ and ϕ_k is the eigenfunction corresponding to the eigenvalue λ_k for each k. We note that the first eigenfunction $\phi_1(x) > 0$.

The purpose of this paper is to find the number of weak solutions of (1.2) under the assumptions $(F1) - (F4)$ on the nonlinear term F.

The steady-state case of (1.1) is the elliptic problem

$$\Delta w + F(x,w) = 0 \qquad \text{in } \Omega, \tag{1.3}$$

$$w = 0 \qquad \text{on } \partial\Omega.$$

For the multiplicity results of (1.3) the readers refer to [9].
The main result is the following:

THEOREM 1.1. *Assume that F satisfies the conditions $(F1) - (F4)$. Then (1.2) has a nontrivial periodic solution.*

In section 2 we introduce the Hilbert space H whose elements are expressed by the square integrable Fourier series expansions on $\Omega \times (0, 2\pi)$, consider the parabolic problem (1.2) on H and obtain some results on the operator $D_t - \Delta$ and F. In section 3 we introduce the variational linking theorem which is a crucial role for the proof of Theorem 1.1 and show that I satisfies the linking geometry. In section 4 we prove Theorem 1.1.

2. Parabolic problem on H

Let Q be the space $\Omega \times (0, 2\pi)$. The space $L_2(\Omega \times (0, 2\pi))$ is a Hilbert space equipped with the usual inner product

$$< v, w >= \int_0^{2\pi} \int_\Omega v(x,t)\bar{w}(x,t)dxdt$$

and a norm

$$\|v\|_{L^2(Q)} = \sqrt{< v, v >}.$$

We shall work first in the complex space $L_2(\Omega \times (0, 2\pi))$ but shall later switch to the real space. The functions

$$\Phi_{jk}(x,t) = \phi_k \frac{e^{ijt}}{\sqrt{2\pi}}, \qquad j = 0, \pm 1, \pm 2, \ldots, \ k = 1, 2, 3, \ldots$$

form a complete orthonormal basis in $L_2(\Omega \times (0, 2\pi))$. Every elements $v \in L_2(\Omega \times (0, 2\pi))$ has a Fourier expansion

$$v = \sum_{jk} v_{jk}\Phi_{jk}$$

with $\sum |v_{jk}|^2 < \infty$ and $v_{jk} =< v, \Phi_{jk} >$. Let us define a subspace H of $L_2(\Omega \times (0, 2\pi))$ as

$$H = \{u \in L_2(\Omega \times (0, 2\pi))| \ \sum_{jk} (j^2 + \lambda_k^2)^{\frac{1}{2}} u_{jk}^2 < \infty\}. \tag{2.1}$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum_{jk} (j^2 + \lambda_k^2)^{\frac{1}{2}} u_{jk}^2]^{\frac{1}{2}}.$$

A weak solution of problem (1.2) is of the form $u = \sum u_{jk}\Phi_{jk}$ satisfying $\sum |u_{jk}|^2 (j^2 + \lambda_k^2)^{\frac{1}{2}} < \infty$, which implies $u \in H$. Thus we have that if u is a weak solution of (1.2), then $u_t = D_t u = \sum_{j\ k} iju_{jk}\Phi_{jk}$ belong to H and $-\Delta u = \sum \lambda_k u_{jk}\Phi_{jk}$ belong to H.

We have some properties on $\|\cdot\|$ and $D_t - \Delta$. Since $|j + \lambda_k| \to \infty$ for all j, k, we have that:

LEMMA 2.1. *(i)* $\|u\| \geq \|u(x,0)\| \geq \|u(x,0)\|_{L_2(\Omega)}$.
(ii) $\|u\|_{L_2(Q)} = 0$ *if and only if* $\|u\| = 0$.
(iii) $u_t - \Delta u \in H$ *implies* $u \in H$.

Proof. (i) Let $u = \sum_{j\ k} u_{jk}\Phi_{jk}$. Then

$$\begin{aligned} \|u\|^2 &= \sum (j^2+\lambda_k^2)^{\frac{1}{2}} u_{jk}^2 \\ &\geq \sum \lambda_k^2 u_{jk}^2(x.0) = \|u(x.0)\|^2 \\ &\geq \sum u_{jk}^2(x,0) = \|u(x,0)\|_{L_2(\Omega)}^2. \end{aligned}$$

(ii) Let $u = \sum_{j\ k} u_{jk}\Phi_{jk}$.

$$\|u\| = 0 \Leftrightarrow \sum_{j\ k} (j^2+\lambda_k^2)^{\frac{1}{2}} u_{jk}^2 = 0 \Leftrightarrow \sum_{j\ k} u_{jk}^2 = 0 \Leftrightarrow \|u\|_{L_2(Q)} = 0.$$

(iii) Let $u_t - \Delta u = f \in H$. Then f can be expressed by

$$f = \sum f_{jk}\Phi_{jk}, \qquad \sum_{j\ k} (j^2+\lambda_k^2)^{\frac{1}{2}} f_{jk}^2 < \infty$$

Then we have

$$\|(D_t-\Delta)^{-1} f\|^2 = \sum_{j\ k} \frac{(j^2+\lambda_k^2)^{\frac{1}{2}}}{j^2+\lambda_k^2} f_{jk}^2 < C \sum_{j\ k} f_{jk}^2 < \infty$$

for some $C > 0$. □

LEMMA 2.2. *For any real $\alpha \neq \lambda_k$, the operator $(D_t - \Delta - \alpha)^{-1}$ is linear, self-adjoint, and a compact operator from $L_2(\Omega \times (0, 2\pi))$ to H with the operator norm $\frac{1}{|\alpha-\lambda_k|}$, where λ_k is an eigenvalue of $-\Delta$ closest to α.*

Proof. Suppose that $\alpha \neq \lambda_k$. Since $\lambda_k \to +\infty$, the number of elements in the set $\{\lambda_k |\ \lambda_k < \alpha\}$ is finite, where λ_k is an eigenvalue of $-\Delta$. Let $h = \sum_{j\ k} h_{jk}\Phi_{jk}$, where $\Phi_{jk} = \phi_k \frac{e^{ijt}}{\sqrt{2\pi}}$. Then

$$(D_t - \Delta - \alpha)^{-1} h = \sum_{j\ k} \frac{1}{ij + \lambda_k - \alpha} h_{jk}\Phi_{jk}.$$

Hence

$$\|(D_t-\Delta-\alpha)^{-1}h\|^2 = \sum_{j\ k} \frac{1}{j^2+(\lambda_k-\alpha)^2} (j^2+(\lambda_k-\alpha)^2)^{\frac{1}{2}} h_{jk}^2 \leq \sum_{j\ k} C h_{jk}^2 < \infty$$

for some $C > 0$. Thus $(D_t - \Delta - \alpha)^{-1}$ is a bounded operator from $L_2(\Omega \times (0, 2\pi))$ to H and also send bounded subset of $L_2(\Omega \times (0, 2\pi))$ to a compact subset of H, hence $(D_t - \Delta - \alpha)^{-1}$ is a compact operator. □

From Lemma 2.2 we obtain the following lemma:

LEMMA 2.3. *Let $F(x,t,u) \in L_2(\Omega \times (0,2\pi))$. Then all the solutions of*

$$u_t - \Delta u = F(x,t,u) \qquad \text{in } L_2(\Omega \times (0,2\pi))$$

belong to H.

From now on we restrict ourselves to the real L_2-space and observe that this is an invariant space for R. So $L_2(\Omega \times (0,2\pi))$ denotes the real square-integrable functions on $\Omega \times (0,2\pi)$ and H the subspace of $L_2(\Omega \times (0,2\pi))$ satisfying (2.1). The functions

$$\Phi_{0k} = \frac{1}{\sqrt{2\pi}}\phi_k(x),$$

$$\Phi^c_{jk} = \frac{1}{\sqrt{\pi}}\cos jt\phi_k(x),$$

$$\Phi^s_{jk} = \frac{1}{\sqrt{\pi}}\sin jt\phi_k(x) \qquad j,k = 1,2,3,\ldots.$$

form a real orthonormal basis in the real space $L_2(\Omega \times (0,2\pi))$, and the Fourier coefficients of a real valued function $u \in L_2(\Omega \times (0,2\pi))$ are given by $u_{0k} =< u, \Phi_{0k} >$, $u^c_{jk} =< u, \Phi^c_{jk} >$, $u^s_{jk} =< u, \Phi^s_{jk} >$. We also have that

$$u^c_{jk} = \sqrt{2}Re\ u_{jk}, \qquad u^s_{jk} = -\sqrt{2}Im\ u_{jk}, \text{ for } j,k = 1,2,3,\ldots,$$

$$\sqrt{2}u_{jk} = u^c_{jk} - iu^s_{jk}, \qquad \sqrt{2}u_{-j,k} = u^c_{jk} + iu^s_{jk}, \qquad \bar{u}_{jk} = u_{-j,k}.$$

The function $w = Au = (D_t - \Delta)^{-1}u$ is given, in terms of its Fourier coefficients, by

$$w^c_{jk} - iw^s_{jk} = \sqrt{2}w_{jk} = \frac{\sqrt{2}}{j^2 + \lambda_k^2}u_{jk}(ij + \lambda_k).$$

(2.2) can be expressed by matrix notation

$$\begin{pmatrix} w^c_{jk} \\ w^s_{jk} \end{pmatrix} = A_{jk}\begin{pmatrix} u^c_{jk} \\ u^s_{jk} \end{pmatrix}, \qquad A_{jk} = \frac{1}{j^2 + \lambda_k^2}\begin{pmatrix} -\lambda_k & j \\ -j & -\lambda_k \end{pmatrix}.$$

We also have that

$$(w^c_{jk})^2 + (w^s_{jk})^2 = |w_{jk}|^2 + |w_{-j,k}|^2 = \frac{1}{j^2 + \lambda_k^2}(|u_{jk}|^2 + |u_{-j,k}|^2). \quad (2.2)$$

Note that, for $u \in H$,

$$\int_Q (D_t u - \Delta u) u dx dt = \int_Q \sum_{mn} \lambda_n (u_{mn}^{c^2} \Phi_{mn}^{c^2} + u_{mn}^{s^2} \Phi_{mn}^{s^2}) dx dt = \|u\|^2. \quad (2.3)$$

Let us define the functional on H,

$$I(u) = \int_0^{2\pi} \int_\Omega [\frac{1}{2} D_t u \cdot u + \frac{1}{2} |\nabla u|^2 - F(x,t,u)] dt dx. \quad (2.4)$$

We note that I is well defined. By the following Lemma 2.4, the solutions of (1.2) coincide with the critical points of $I(u)$.

LEMMA 2.4. *The functional $I(u)$ is continuous and Fréchet differentiable in H with Fréchet derivative*

$$DI(u)v = \int_0^{2\pi} \int_\Omega [D_t u - \Delta u - F_u(x,t,u)u] dx dt. \quad (2.5)$$

Proof. Let $u \in H$. To prove the continuity of $I(u)$ we consider

$$\begin{aligned}
& |I(u+v) - I(u)| \\
= \; & |\frac{1}{2} \int_0^{2\pi} \int_\Omega [(D_t u + D_t v)(u+v) + (-\Delta u - \Delta v)(u+v)] dx dt \\
& - \int_0^{2\pi} \int_\Omega F(x,t,u+v) dx dt - \frac{1}{2} \int_0^{2\pi} \int_\Omega [(D_t u)u + (-\Delta u)u] dx dt \\
& + \int_0^{2\pi} \int_\Omega F(x,t,u) dx dt| \\
= \; & |\frac{1}{2} \int_0^{2\pi} \int_\Omega [(D_t u - \Delta u)v + (D_t v - \Delta v)u + (D_t v - \Delta v)v] dx dt \\
& - \int_0^{2\pi} \int_\Omega [F(x,t,u+v) - F(x,t,u)] dx dt|.
\end{aligned}$$

Let $u = \sum(\frac{1}{2} u_{jk}^c \Phi_{jk}^c + \frac{1}{2} u_{jk}^s \Phi_{jk}^s)$, $v = \sum(\frac{1}{2} v_{jk}^c \Phi_{jk}^c + \frac{1}{2} v_{jk}^s \Phi_{jk}^s)$, $j = 0, 1, \ldots$, $k = 1, 2, \ldots$. Then we have

$$|\int_0^{2\pi} \int_\Omega (D_t u - \Delta u)v| = \frac{1}{2} |\sum (j + \lambda_k)(u_{jk}^c v_{jk}^c + u_{jk}^s v_{jk}^s)| \le \|u\| \cdot \|v\|,$$

$$|\int_0^{2\pi} \int_\Omega (D_t u - \Delta u)v| = \frac{1}{2} |\sum (j + \lambda_k)(u_{jk}^c v_{jk}^c + u_{jk}^s v_{jk}^s)| \le \|u\| \cdot \|v\|,$$

$$|\int_0^{2\pi} \int_\Omega (D_t v - \Delta v)v| = \frac{1}{2} |\sum (j + \lambda_k)(u_{jk}^c v_{jk}^c + u_{jk}^s v_{jk}^s)| \le \|u\| \cdot \|v\|.$$

On the other hand, it follows from the differentiability of F that

$$|F(x,t,u+v) - F(x,t,u)| = O(\|v\|).$$

Thus it follows that $I(u)$ is continuous at u. To prove the *Fréchet* differentiability at $u \in H$, with (2.5), of $I(u)$ we consider

$$\begin{aligned} & |I(u+v) - I(u) - DI(u)v| \\ = \ & |\int_0^{2\pi}\int_\Omega \frac{1}{2}v(D_t v - \Delta v)dxdt \\ - \ & \int_0^{2\pi}\int_\Omega [F(x,t,u+v) - F(x,t,u) - F_u(x,t,u)v]dxdt \\ \leq \ & \frac{1}{2}\|v\|^2 + o(\|v\|), \end{aligned}$$

since F is differentiable at $u \in H$. It follows that $I(u)$ is *Fréchet* differentiable at $u \in H$. □

By (F1) and (F3), we obtain the lower bound for $F(x,t,u)$ in the term of $|u|^\mu$.

Lemma 2.5. *Assume that F satisfies the conditions (F1) and (F3). Then there exist a_0, $b_0 \in R$ with $a_0 > 0$ such that*

$$F(x,t,r) \geq a_0(|r|^\mu) - b_0, \qquad \forall x,t,r. \tag{2.6}$$

Proof. Let r be such that $|r| \geq R$. Let us set $\varphi(\xi) = F(x,t,\xi r)$ for $\xi \geq 1$. Then

$$\varphi(\xi)' = rF_r(x,t,\xi r) \geq \frac{\mu}{\xi}\varphi(\xi).$$

Multiplying by $\xi^{-\mu}$, we get

$$(\xi^{-\mu}\varphi(\xi))' \geq 0,$$

hence $\varphi(\xi) \geq \varphi(1)\xi^\mu$ for $\xi \geq 1$. Thus we have

$$F(x,t,r) \geq F\big(x,t,\frac{Rr}{|r|}\big)\big(\frac{|r|}{R}\big)^\mu$$

$$\geq c_0\big(\frac{|r|}{R}\big)^\mu \geq a_0(|r|^\mu) - b_0, \text{ for some } a_0 > 0, b_0,$$

where $c_0 = \inf\{F(x,t,r)|\ (x,t) \in Q,\ |r| = R\}$. □

LEMMA 2.6. *Assume that F satisfies the conditions $(F1)$, $(F2)$ and $(F3)$. Then*
(i) $\int_Q F(x,t,0)dxdt = 0$, $\int_Q F(x,t,u)dxdt > 0$ if $u \neq 0$,
$grad(\int_Q F(x,t,u))dxdt = o(\|u\|)$ as $u \to 0$;
(ii) there exist $a_0 > 0$, $\mu > 2$ and $b_1 \in R$ such that

$$\int_Q F(x,t,u)dxdt \geq a_0\|u\|^{\mu}_{L^{\mu}} - b_1 \qquad \forall u \in H;$$

(iii) $u \mapsto grad(\int_Q F(x,t,u))dxdt$ is a compact map;
(iv) if $\int_Q uF_u(x,t,u)dxdt - 2\int_Q F(x,t,u)dxdt = 0$,
then $grad(\int_Q F(x,t,u)dxdt) = 0$;
(v) if $\|u_n\| \to +\infty$ and $\frac{\int_Q u_n F_u(x,t,u_n)dxdt - 2\int_Q F(x,t,u_n)dxdt}{\|u_n\|} \to 0$,
then there exist $(u_{h_n})_n$ and $w \in H$ such that

$$\frac{grad(\int_Q F(x,t,u_n)dxdt)}{\|u_{h_n}\|} \to w \text{ and } \frac{u_{h_n}}{\|u_{h_n}\|} \rightharpoonup 0.$$

Proof. (i) (i) follows from (F1) and (F2), since $1 < \nu$.
(ii) By Lemma 2.5, for $u \in H$,

$$\int_Q F(x,t,u)dxdt \geq a_0\|u\|^{\mu}_{L^{\mu}} - b_1,$$

where $b_1 \in R$. Thus (ii) holds.
(iii) (iii) is easily obtained with standard arguments.
(iv) (iv) is implied by (F3) and the fact that $F(x,t,u) > 0$ for $u \neq 0$.
(v) By Lemma 2.5 and (F3), for u,

$$\int_Q uF_u(x,t,u)dxdt - 2\int_Q F(x,t,u)dxdt \geq$$

$$(\mu - 2)\int_Q F(x,t,u)dxdt \geq (\mu - 2)(a_0\|u\|^{\mu}_{L^{\mu}} - b_1).$$

By (F2),

$$\|\text{grad}(\int_Q F(x,t,u)dxdt)\| \leq C'\|F_u(x,t,u)\|_{L^{2^{*'}}} \leq C''\||u|^{\nu}\|_{L^{2^{*'}}},$$

for suitable constants C', C''. To get the conclusion it suffices to estimate $\|\frac{|u|^{\nu}}{\|u\|}\|_{L^{2^{*'}}}$ in terms of $\frac{\|u\|^{\mu}_{L^{\mu}}}{\|u\|}$. If $\mu \geq 2^{*'}\nu$, then this is a consequence of *Hölder* inequality. If $\mu < 2^{*'}\nu$, by the standard interpolation arguments,

it follows that $\|\frac{|u|^\nu}{\|u\|}\|_{L^{2^{*\prime}}} \le C\big(\frac{\|u\|_{L^\mu}^\mu}{\|u\|}\big)^{\frac{\nu\alpha}{\mu}}\|u\|^\beta$, where α is such that $\frac{\alpha}{\mu} + \frac{1-\alpha}{2^*} = \frac{1}{2^{*\prime}\nu}$ $(\alpha > 0)$ and $\beta = (1-\alpha)\nu - 1 - \frac{\nu\alpha}{\mu}$. Notice that, the assumptions on μ and ν imply that $\nu \le 2^* - 1 - (2^* - \mu)(1 - \frac{2^{*\prime}}{2^*})$. Thus we prove (v). □

Lemma 2.7. *Assume that F satisfies the conditions $(F1) - (F4)$. Then there exist φ, $\psi : [0, +\infty] \to R$ continuous and such that $\frac{\psi(s)}{s} \to 0$ as $s \to 0$, $\varphi(s) > 0$ if $s > 0$,*
(i) $\|grad \int_Q F(x,t,u)dxdt\|^2 \le \psi(\int_Q F(x,t,u)dxdt)$, $\forall u \in H$,
(ii) $\int_Q [uF_u(x,t,u)]dxdt - 2\int_Q F(x,t,u)dxdt \ge \varphi(u)$, $\forall u \in H$.

Proof. (i) By (F4), for all $u \in H$,

$$\begin{aligned}
\|\mathrm{grad}\Big(\int_Q F(x,t,u)dxdt\Big)\| &\le \|F_u(x,t,u)\|_{L^{2^{*\prime}}} \\
&\le C_1\|F(x,t,u)^\delta\|_{L^{2^{*\prime}}} \\
&\le C_2\|F(x,t,u)^\delta\|_{L^{2^{*\prime}}}\|_{L^{2^{*\prime}}} \\
&\le C_3\|F(x,t,u)^\delta\|_{L^{\frac{1}{\delta}}} \\
&\le C_4\|F(x,t,u)\|_{L^1}^\delta \\
&= C_5\Big(\int_Q F(x,t,u)dxdt\Big)^\delta,
\end{aligned}$$

where C_1, C_2, C_3, C_4 and C_5 are constants. Since $\delta > \frac{1}{2}$, (i) follows.
(ii) By (F3),

$$\int_Q F_u(x,t,u)dxdt - 2\int_Q F(x,t,u)dxdt \ge$$

$$(\mu - 2)\int_Q F(x,t,u)dxdt \ge (\mu-2)(a_0\|u\|_{L^\mu}^\mu - b_1).$$

Thus (ii) follows. □

3. Linking geometry

Now we are looking for the nontrivial periodic weak solution of 1.2). By Lemma 2.4, the weak solutions of (1.2) coincide with the critical points of the corresponding functional $I(u)$. To find the critical points

of $I(u)$ we shall use the variational linking theorem. Now we recall the variational linking theorem (cf. [9]).

LEMMA 3.1. *(Variational Linking Theorem)*
Let H be a real Hilbert space with $H = H_1 \oplus H_2$ and $H_2 = H_1^{\perp}$. We suppose that
(I1) $I \in C^1(H, R)$, satisfies $(P.S.)^*$ condition, and
(I2) $I(u) = \frac{1}{2}(Lu, u) + bu$, where $Lu = L_1P_1u + L_2P_2u$ and $L_i : H_i \to H_i$ is bounded and selfadjoint, $i = 1, 2$,
(I3) b' is compact, and
(I4) there exists a subspace $\tilde{H} \subset H$ and sets $S \subset H$, $T \subset \tilde{H}$ and constants $\alpha > w$ such that,
(i) $S \subset H_1$ and $I|_S \geq \alpha$,
(ii) T is bounded and $I|_{\partial T} \leq w$,
(iii) S and ∂T link.
Then I possesses a critical value $c \geq \alpha$.

Let $H^+ = \text{span}\{\Phi^c_{jk}, \Phi^s_{jk} |\ j \geq 1,\ k \geq 1\}$, $H^- = \text{span}\{\Phi^c_{jk}, \Phi^s_{jk} |\ j \leq -1,\ k \geq 1\}$ and $H_{0k} = \text{span}\{\Phi_{0k} |\ k \geq 1\}$. Then H^+, H^- and H_{0k} are mutually orthogonal and $H = H^+ \oplus H_{0k} \oplus H^-$. Let

$$H_{nn} = \text{span}\{\Phi^c_{jk}, \Phi^s_{jk} |\ -n \leq j \leq n,\ 1 \leq k \leq n\},$$

$$H^+_{nn} = \text{span}\{\Phi^c_{jk}, \Phi^s_{jk} |\ 1 \leq j \leq n,\ 1 \leq k \leq n\},$$

$$H^-_{nn} = \text{span}\{\Phi^c_{jk}, \Phi^s_{jk} |\ -n \leq j \leq -1,\ 1 \leq k \leq n\}.$$

Then $(H_{nn})_n$ is a sequence of closed subspaces of H with the conditions:

$$H_{nn} = H^-_{nn} \oplus H_{0n} \oplus H^+_{nn},\ \text{where } H^+_{nn} \subset H^+,\ H^-_{nn} \subset H^- \text{ for all } n, \quad (3.1)$$

$$(H^+_{nn} \text{ and } H^-_{nn} \text{ are subspaces of } H), \dim H_{nn} < +\infty,$$

$$H_{nn} \subset H_{n+1\ n+1}, \cup_{n \in N} H_{nn} \text{ is dense in } H. \quad (3.2)$$

Let $P_{H_{nn}}$ be the orthogonal projection from H onto H_{nn}.
Let us set

$$H_m = \text{span}\{\Phi^c_{jk}, \Phi^s_{jk} |\ j \in Z,\ 1 \leq k \leq m\}.$$

Then

$$H_m = \cup_{j \in Z} H_{jm}, \qquad H = \cup_{m \in N} H_m.$$

Let us prove that the functional I satisfies the linking geometry.

LEMMA 3.2. *Assume that F satisfies the conditions $(F1)-(F4)$,*
(i) there exist a small number $\rho_m > 0$ and a small ball $B_{\rho_m} \subset H_m^{\perp}$ with radius ρ_m such that if $u \in \partial B_{\rho_m}$, then

$$\alpha_m = \inf I(u) > 0,$$

(ii) there are an $e \in H_m^{\perp} \cap B_{\rho_m}$, $R_m > \rho_m$ and a large ball B_{R_m} with radius $R_m > 0$ such that if

$$W_m = (\bar{B_{R_m}} \cap H_m) \oplus \{re|\ 0 < r < R_m\},$$

then

$$\sup_{u \in \partial W_m} I(u) \leq 0.$$

Proof. (i) By (i) of Lemma 2.6, we have that, for $u \in H_m^{\perp}$,

$$I(u) = \frac{1}{2}\int_Q (D_t u - \Delta u)dxdt - \int_Q F(x,t,u)dxdt$$

$$\geq \frac{1}{2}\lambda_{m+1}\|u\|^2 - 0(\|u\|).$$

Then there exists a small number $\rho_m > 0$ and a small ball $B_{\rho_m} \subset H_m^{\perp}$ with radius ρ_m such that if $u \in \partial B_{\rho_m}$, then $\inf I(u) > 0$. Thus the assertion (1) hold.
(ii) We note that

$$\text{if } \ u \in H_m, \text{ then } \int_Q (D_t u - \Delta u)dxdt \leq \lambda_m \|u\|_{L_2(Q)}, \tag{3.3}$$

$$\text{if } \ u \in H_m^{\perp}, \text{ then } \int_Q (D_t u - \Delta u)dxdt \geq \lambda_{m+1}\|u\|_{L_2(Q)}. \tag{3.4}$$

Let B_{ρ_m} be a ball in (i). Let us choose an element $e \in H_m^{\perp} \cap B_{\rho_m}$ with $\|e\| = 1$. Let us choose $u \neq 0 \in H_m \oplus \{re|\ r > 0\}$. By Lemma 2.5, we have that

$$I(u) \leq \frac{1}{2}\lambda_m \|u\|^2_{L_2(Q)} + \frac{1}{2}r^2 - a_0\|u\|^{\mu}_{L^{\mu}} + b_1$$

for some $a_0 > 0$ and b_1. Since $\mu > 2$, there exists $R_m > 0$ and a ball B_{R_m} with radius R_m such that if $u \in (\bar{B_{R_m}} \cap H_m) \oplus \{re|\ 0 < r < R_m\}$, then $\sup I(u) < 0$. So the assertion (ii) hold. So the lemma is proved. □

We shall prove that the functional I satisfies the $(P.S.)^*_c$ condition with respect to $(H_{nn})_n$ for any $c \in R$.

LEMMA 3.3. *Assume that F satisfies the conditions $(F1)-(F4)$. Then the functional I satisfies the $(P.S.)^*_c$ condition with respect to $(H_{nn})_n$ for any real number c.*

Proof. Let $c \in R$ and (h_n) be a sequence in N such that $h_n \to +\infty$, $(u_n)_n$ be a sequence in $H_{h_n h_n}$ such that

$$I(u_n) \to c, P_{H_{h_n h_n}} \nabla I(u_n) \to 0.$$

We claim that $(u_n)_n$ is bounded. By contradiction we suppose that $\|u_n\| \to +\infty$ and set $\hat{u_n} = \frac{u_n}{\|u_n\|}$. Then

$$\langle P_{H_{h_n h_n}} \nabla I(u_n), \hat{u_n} \rangle = \langle \nabla I(u_n), \hat{u_n} \rangle = 2\frac{I(u_n)}{\|u_n\|} -$$

$$\frac{\int_Q \nabla F(x,t,u_n) \cdot u_n dxdt - 2\int_Q F(x,t,u_n) dxdt}{\|u_n\|} \longrightarrow 0.$$

Hence

$$\frac{\int_Q \nabla F(x,t,u_n) \cdot u_n dxdt - 2\int_Q F(x,t,u_n) dxdt}{\|u_n\|} \longrightarrow 0.$$

By (v) of Lemma 2.6,

$$\frac{\text{grad} \int_Q F(x,t,u_n) dxdt}{\|u_n\|} \qquad \text{converges}$$

and $\hat{u_n} \rightharpoonup 0$. We get

$$\begin{aligned} & \frac{P_{H_{h_n h_n}} \nabla I(u_n)}{\|u_n\|} \\ = \; & P_{H_{h_n h_n}}(D_t - \Delta)\hat{u_n} - \frac{P_{H_{h_n h_n}} \text{grad}(\int_Q F(x,t,u_n) dxdt)}{\|u_n\|} \longrightarrow 0 \end{aligned}$$

so $(P_{H_{h_n h_n}}(D_t - \Delta)\hat{u_n})$ converges. Since $(\hat{u_n})_n$ is bounded and $(D_t - \Delta)^{-1}$ is a compact mapping, up to subsequence, $(\hat{u_n})_n$ has a limit. Since $\hat{u_n} \rightharpoonup 0$, we get $\hat{u_n} \to 0$, which is a contradiction to the fact that $\|\hat{u_n}\|_E = 1$. Thus $(u_n)_n$ is bounded. We can now suppose that $u_n \rightharpoonup u$ for some $u \in H$. Since the mapping $u \mapsto \text{grad}(\int_Q F(x,t,u) dxdt)$ is a compact mapping, $\text{grad}(\int_Q F(x,t,u_n) dxdt) \longrightarrow \text{grad}(\int_Q F(x,t,u) dxdt)$. Thus $(P_{H_{h_n h_n}}(D_t - \Delta)u_n)_n$ converges. Since $(D_t - \Delta)^{-1}$ is a compact operator and $(u_n)_n$ is bounded, we deduce that, up to a subsequence,

$(u_n)_n$ converges to some u strongly with $\nabla I(u) = \lim \nabla I(u_n) = 0$. Thus we prove the lemma. □

4. Proof of theorem 1.1

Assume that the nonlinear term F satisfies (F1), (F2), (F3) and (F4). We note that $I(0,0) = 0$ and $H = H_m \oplus H_m^{\perp}$. By (iii) of Lemma 2.6, $u \mapsto \text{grad}(\int_Q F(x,t,u)dxdt)$ is a compact mapping. By Lemma 3.2, there exist a small number $\rho_m > 0$ and a small ball $B_{\rho_m} \subset H_m^{\perp}$ with radius ρ_m such that if $u \in \partial B_{\rho_m}$, then $\alpha_m = \inf I(u) > 0$, and there is an $e \in H_m^{\perp} \cap B_{\rho_m}$, $R_m > \rho_m > 0$ and a large ball B_{R_m} with radius $R_m > 0$ such that if

$$W_m = (\bar{B_{R_m}} \cap H_m) \oplus \{re |\ 0 < r < R_m\},$$

then

$$\sup_{u \in \partial W_m} I(u) \leq 0.$$

Let us set $\beta_m = \sup_{W_m} I$. We note that $\beta_m < +\infty$. We note that ∂B_{ρ_m} and ∂W_m link. Moreover, by Lemma 3.3, $I_m = I|_{H_{mm}}$ satisfies the $(P.S.)_c^*$ condition for any $c \in R$. Thus by Lemma 3.1 (Linking Theorem), there exists a critical point u_m for I_m with

$$\alpha_m \leq \inf_{\partial B_{\rho_m} \cap H_{mm}} I \leq I(u_m) \leq \sup_{W_m \cap H_{mm}} I \leq \beta_m.$$

Since I_m satisfies the $(P.S.)_c^*$ condition, we obtain that, up to a subsequence, $u_m \to u$, with u a critical point for I such that $\alpha_m \leq I(u) \leq \beta_m$. Hence $u \neq (0,0)$. Thus system (1.2) has a nontrivial solution. Thus Theorem 1.1 is proved.

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